EJAM
www.ejamjournal.com

Electronic Journal of Applied Mathematics

Research article

Mersin University Publishing
DOI: 10.61383/ejam.202534125
2025, Vol. 3, No. 4, 67–84

Well-posedness of a conformable parabolic equation with nonhomogeneous Neumann boundary

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Abstract

We study a one dimensional parabolic equation involving a conformable time derivative and nonhomogeneous Neumann boundary data. A weak formulation is introduced in weighted Bochner spaces associated with the natural measure $d_\beta t = t^{\beta-1}\,dt$, and the conformable derivative is interpreted in a weak sense through a suitable transformation of the time variable. A priori energy estimates are derived for the solution in terms of the initial datum, the source term, and the prescribed boundary flux. Using the change of variables $\tau = t^\beta/\beta$, the problem is transformed into a classical parabolic problem, which yields the existence of weak solutions. Uniqueness and continuous dependence on the data are then established by energy methods, leading to well-posedness in the sense of Hadamard. Under additional regularity assumptions on the initial and boundary data, higher spatial and temporal regularity of the solution is also obtained.

Keywords: conformable derivative; parabolic equation; Neumann boundary condition; weak solution; a priori estimate; well-posedness; regularity.
MSC (2020): 35K20; 35D30; 35B30; 35B65
Article history: Received 10 Apr 2025; Accepted 18 Nov 2025; Online 28 Dec 2025

1 Introduction

Parabolic equations provide a classical framework for the mathematical description of diffusion, heat transfer, and related time-dependent processes. A fundamental issue in their analysis is the well-posedness of the associated initial-boundary value problem, namely, the existence and uniqueness of solutions together with their continuous dependence on the prescribed data. The variational theory of linear parabolic equations is well established; see, for instance, [7, 9, 10].

The conformable derivative was introduced by Khalil et al. [8] as a local extension of the ordinary derivative. Further properties of conformable differentiation and integration were developed in [3]. For a differentiable function v and an order $\beta \in (0,1]$, the conformable derivative satisfies $\mathcal{T}_t^\beta v(t) = t^{1-\beta}v'(t)$. Thus, the classical first-order derivative is recovered when $\beta = 1$.

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Conformable derivatives have been considered in several classes of evolution equations. A conformable heat equation was studied in [4]. Well-posedness and regularity properties for sub-diffusion equations involving a conformable time derivative were investigated in [12], while parabolic equations with variable-order conformable derivatives in Hilbert scales were considered in [6]. These studies show that methods from classical evolution equations can be adapted to problems involving conformable time differentiation.

The mathematical nature of the conformable derivative has also been discussed in the literature. In particular, its local character and its close relation to ordinary differentiation have been emphasized in [1, 2, 5, 11]. Accordingly, throughout this paper the conformable derivative is treated as a local time-differential operator rather than as a nonlocal fractional derivative. This viewpoint naturally leads to a weighted temporal structure. Indeed, the measure $d_\beta t = t^{\beta-1}\,dt$ is compatible with the conformable derivative in the sense that $\mathcal{T}_t^\beta v(t)\,d_\beta t = v'(t)\,dt$ whenever v is differentiable.

Let

$$\Omega = (0, l), \qquad Q = \Omega \times (0, T),$$

where $l > 0$ and $T > 0$. We consider the conformable parabolic problem

$$\mathcal{T}_t^\beta u(x,t) - k u_{xx}(x,t) = h(x,t), \qquad (x,t) \in Q,$$

supplemented with the initial condition

$$u(x,0) = u_0(x), \qquad x \in \Omega,$$

and the nonhomogeneous Neumann boundary conditions

$$u_x(0,t) = 0, \qquad u_x(l,t) = g(t), \qquad t \in (0,T).$$

Here $k > 0$, $\beta \in (0,1]$, u_0 is the initial datum, h is the source term, and g denotes the prescribed boundary flux. Related classical parabolic problems with Neumann boundary data occur naturally in heat transfer and boundary-control models; see, for example, [13].

The purpose of this paper is to study the above problem in a weak variational framework adapted to conformable time differentiation. The nonhomogeneous Neumann datum is incorporated into the weak formulation as a boundary functional, while the temporal variable is treated in weighted Bochner spaces associated with the measure $d_\beta t$. Since the bilinear form corresponding to the Neumann Laplacian is not coercive on $H^1(\Omega)$, a suitable Gårding inequality is used in the variational analysis.

A central part of the analysis is the derivation of an a priori energy estimate in terms of the initial datum, the source term, and the boundary flux. The existence of weak solutions is obtained by means of the time transformation $\tau = t^\beta/\beta$, under which

$$\mathcal{T}_t^\beta u = \frac{\partial u}{\partial \tau}, \qquad d\tau = d_\beta t.$$

The conformable problem can therefore be transformed into a classical parabolic problem in the new time variable, allowing standard variational results to be applied [9, 10].

The a priori estimates are subsequently used to prove uniqueness and continuous dependence of weak solutions on the initial datum, the source term, and the Neumann boundary datum. In particular, a quantitative stability estimate is obtained for solutions corresponding to different sets of data. These results yield well-posedness in the sense of Hadamard. Under

additional assumptions on the initial and boundary data, higher regularity of the weak solution is also established.

The paper is organized as follows. Section 2 recalls the basic properties of the conformable derivative and introduces the weighted functional setting. Section 3 presents the initial-boundary value problem and the notion of weak solution. The main a priori estimates are derived in Section 4. Existence, uniqueness, and continuous dependence are established in Section 5. Higher regularity is studied in Section 6. Finally, Section **??** contains the concluding remarks.

2 Preliminaries

In this section, we recall the basic properties of the conformable derivative and introduce the weighted functional framework used throughout the paper. Particular attention is given to the interpretation of the conformable derivative for functions in evolution spaces.

Let $0 < \beta \leq 1$. Following Khalil et al. [8], the conformable derivative of order β of a function $v : [0, T] \to \mathbb{R}$ is defined, for $t > 0$, by

$$\mathcal{T}_t^\beta v(t) := \lim_{\varepsilon \to 0} \frac{v\big(t + \varepsilon t^{1-\beta}\big) - v(t)}{\varepsilon}, \tag{1}$$

whenever the limit exists. If v is differentiable at $t > 0$, then

$$\mathcal{T}_t^\beta v(t) = t^{1-\beta} v'(t). \tag{2}$$

In particular,

$$\mathcal{T}_t^1 v(t) = v'(t).$$

Further properties of conformable differentiation and integration can be found in [3].

Associated with the conformable derivative, we introduce the weighted measure

$$d_\beta t := t^{\beta-1}\, dt. \tag{3}$$

For a Hilbert space X, define

$$L^2_\beta(0, T; X) := L^2\big((0, T), d_\beta t; X\big), \tag{4}$$

with norm

$$\|v\|^2_{L^2_\beta(0,T;X)} := \int_0^T \|v(t)\|_X^2\, d_\beta t = \int_0^T t^{\beta-1} \|v(t)\|_X^2\, dt. \tag{5}$$

Since $\beta > 0$, the weight $t^{\beta-1}$ is integrable near $t = 0$.

For sufficiently regular scalar-valued functions v and w, the conformable integration-by-parts formula is

$$\int_0^T v(t) \mathcal{T}_t^\beta w(t)\, d_\beta t = \big[v(t)w(t)\big]_0^T - \int_0^T w(t) \mathcal{T}_t^\beta v(t)\, d_\beta t. \tag{6}$$

Indeed, by (2) and (3), $\mathcal{T}_t^\beta w(t)\, d_\beta t = w'(t)\, dt$, so that (6) follows from the usual integration-by-parts formula.

A key observation is that conformable differentiation can be reduced to ordinary differentiation by a change of the time variable. Define

$$\tau = \Phi_\beta(t) := \frac{t^\beta}{\beta}, \qquad S_\beta := \Phi_\beta(T) = \frac{T^\beta}{\beta}. \tag{7}$$

Then

$$\frac{d\tau}{dt} = t^{\beta-1}, \qquad d\tau = d_\beta t, \tag{8}$$

and

$$t = \Phi_\beta^{-1}(\tau) = (\beta\tau)^{1/\beta}. \tag{9}$$

For a function $v : (0, T) \to X$, define its transformed function by

$$\widetilde{v}(\tau) := v\big(\Phi_\beta^{-1}(\tau)\big), \qquad 0 < \tau < S_\beta. \tag{10}$$

If v is differentiable, then

$$\partial_\tau \widetilde{v}(\tau) = \mathcal{T}_t^\beta v(t), \qquad \tau = \Phi_\beta(t). \tag{11}$$

The change of variables also preserves the corresponding L^2 norms.

Lemma 2.1. *Let X be a Hilbert space and let $v \in L^2_\beta(0, T; X)$. If $\widetilde{v}$ is defined by (10), then*

$$\int_0^{S_\beta} \|\widetilde{v}(\tau)\|_X^2 \, d\tau = \int_0^T \|v(t)\|_X^2 \, d_\beta t. \tag{12}$$

Consequently, the transformation $v \mapsto \widetilde{v}$ is an isometry from $L^2_\beta(0, T; X)$ onto $L^2(0, S_\beta; X)$.

Proof. By (8), $d\tau = t^{\beta-1}\, dt = d_\beta t$. Therefore,

$$\int_0^{S_\beta} \|\widetilde{v}(\tau)\|_X^2 \, d\tau = \int_0^T \|v(t)\|_X^2 t^{\beta-1} \, dt,$$

which is exactly (12). □

We next introduce a weak interpretation of the conformable time derivative. This is needed because the solutions considered below are not assumed to be classically differentiable in time.

Definition 2.2. Let X be a Hilbert space and let $v \in L^2_\beta(0, T; X)$. We say that $z \in L^2_\beta(0, T; X)$ is the *weak conformable derivative* of v if the transformed function

$$\widetilde{v}(\tau) = v\big(\Phi_\beta^{-1}(\tau)\big)$$

belongs to $H^1(0, S_\beta; X)$ and

$$\partial_\tau \widetilde{v}(\tau) = z\big(\Phi_\beta^{-1}(\tau)\big) \quad \text{for a.e. } \tau \in (0, S_\beta). \tag{13}$$

In this case, we write $z = \mathcal{T}_t^\beta v$.

Remark 2.3. If v is classically differentiable, then Definition 2.2 agrees with (2). Thus, the weak conformable derivative extends the classical conformable derivative used in (1).

We now introduce the spatial functional setting. Let $\Omega = (0, l)$, and set

$$V := H^1(\Omega), \qquad H := L^2(\Omega), \qquad V' := H^{-1}(\Omega). \tag{14}$$

We identify H with its dual and use the Gelfand triple

$$V \hookrightarrow H \equiv H' \hookrightarrow V', \tag{15}$$

where all embeddings are continuous and dense. We denote by $(\cdot,\cdot)_H$ the inner product in H and by $\langle\cdot,\cdot\rangle_{V',V}$ the duality pairing between V' and V.

The natural evolution space for the problem is

$$\mathcal{W}_\beta(0,T) := \left\{ v \in L^2_\beta(0,T;V) : \mathcal{T}_t^\beta v \in L^2_\beta(0,T;V') \right\}, \tag{16}$$

equipped with the norm

$$\|v\|^2_{\mathcal{W}_\beta(0,T)} := \|v\|^2_{L^2_\beta(0,T;V)} + \|\mathcal{T}_t^\beta v\|^2_{L^2_\beta(0,T;V')}. \tag{17}$$

Here $\mathcal{T}_t^\beta v$ is understood in the weak sense of Definition 2.2, with values in V'.

The transformation (7) identifies this space with the classical evolution space

$$W(0,S_\beta) := \left\{ \widetilde{v} \in L^2(0,S_\beta;V) : \partial_\tau \widetilde{v} \in L^2(0,S_\beta;V') \right\}.$$

The standard theory of evolution equations therefore gives the following embedding.

Lemma 2.4. *The space $\mathcal{W}_\beta(0,T)$ is continuously embedded into $C([0,T];H)$, that is,*

$$\mathcal{W}_\beta(0,T) \hookrightarrow C([0,T];H). \tag{18}$$

In particular, the values $v(0)$ and $v(T)$ are well defined as elements of H.

Proof. Let $v \in \mathcal{W}_\beta(0,T)$ and define $\widetilde{v}$ by (10). By Lemma 2.1 and Definition 2.2,

$$\widetilde{v} \in L^2(0,S_\beta;V), \qquad \partial_\tau \widetilde{v} \in L^2(0,S_\beta;V').$$

Hence, by the standard Lions–Magenes embedding theorem [10], $\widetilde{v} \in C([0,S_\beta];H)$. Since $\Phi_\beta : [0,T] \to [0,S_\beta]$ is continuous and bijective, $v(t) = \widetilde{v}(\Phi_\beta(t))$ belongs to $C([0,T];H)$. □

The following energy identity will be used repeatedly in Sections 4 and 5.

Lemma 2.5. *Let $v \in \mathcal{W}_\beta(0,T)$. Then $t \mapsto \|v(t)\|_H^2$ is continuous on $[0,T]$ and satisfies*

$$\frac{1}{2}\mathcal{T}_t^\beta \|v(t)\|_H^2 = \left\langle \mathcal{T}_t^\beta v(t), v(t) \right\rangle_{V',V} \tag{19}$$

for almost every $t \in (0,T)$. Moreover, for every $0 \le s \le t \le T$,

$$\frac{1}{2}\|v(t)\|_H^2 - \frac{1}{2}\|v(s)\|_H^2 = \int_s^t \left\langle \mathcal{T}_r^\beta v(r), v(r) \right\rangle_{V',V} d_\beta r. \tag{20}$$

Proof. Let

$$\widetilde{v}(\tau) = v\big(\Phi_\beta^{-1}(\tau)\big).$$

Then

$$\widetilde{v} \in L^2(0,S_\beta;V), \qquad \partial_\tau \widetilde{v} \in L^2(0,S_\beta;V').$$

The classical energy identity in the Gelfand triple $V \hookrightarrow H \hookrightarrow V'$ gives

$$\frac{1}{2}\frac{d}{d\tau}\|\widetilde{v}(\tau)\|_H^2 = \langle \partial_\tau \widetilde{v}(\tau), \widetilde{v}(\tau) \rangle_{V',V}$$

for almost every $\tau \in (0,S_\beta)$. Using (11) and $d\tau = d_\beta t$ yields (19) and (20). □

We shall also use the trace inequality on $H^1(\Omega)$. Since $\Omega = (0, l)$, the trace operator from $H^1(\Omega)$ to $\partial\Omega$ is continuous. Hence there exists $C_{\mathrm{tr}} > 0$ such that

$$|v(l)| \le C_{\mathrm{tr}} \|v\|_V, \qquad v \in V. \tag{21}$$

A form more suitable for energy estimates is given below.

Lemma 2.6. *For every $\varepsilon > 0$, there exists a constant $C_\varepsilon > 0$ such that*

$$|v(l)|^2 \le \varepsilon \|v_x\|_H^2 + C_\varepsilon \|v\|_H^2, \qquad v \in V. \tag{22}$$

Proof. Since $\Omega = (0, l)$, the one-dimensional interpolation and trace inequalities imply $|v(l)|^2 \le C\|v\|_H \|v\|_V$. Applying Young's inequality and using $\|v\|_V^2 = \|v\|_H^2 + \|v_x\|_H^2$ gives (22). □

For the diffusion term, define the bilinear form

$$a(v, w) := k \int_\Omega v_x(x) w_x(x)\, dx, \qquad v, w \in V, \tag{23}$$

where $k > 0$. The form is continuous:

$$|a(v, w)| \le k \|v\|_V \|w\|_V, \qquad v, w \in V. \tag{24}$$

Since constant functions belong to the kernel of the Neumann Laplacian, a is not coercive on V. It satisfies, however, the following Gårding inequality.

Lemma 2.7. *For every $v \in V$,*

$$a(v, v) + k\|v\|_H^2 = k\|v\|_V^2. \tag{25}$$

Consequently, the shifted bilinear form

$$a_k(v, w) := a(v, w) + k(v, w)_H \tag{26}$$

is coercive on V.

Proof. By the definition of the $H^1(\Omega)$ norm, $\|v\|_V^2 = \|v\|_H^2 + \|v_x\|_H^2$. Therefore,

$$a(v, v) + k\|v\|_H^2 = k\|v_x\|_H^2 + k\|v\|_H^2 = k\|v\|_V^2.$$

□

Finally, we recall Young's inequality in the form

$$ab \le \varepsilon a^2 + \frac{1}{4\varepsilon} b^2, \qquad a, b \ge 0, \quad \varepsilon > 0. \tag{27}$$

Together with Lemma 2.6, this inequality will be used to estimate the nonhomogeneous boundary term.

We shall also use Gronwall's inequality in the transformed time variable. More precisely, if a nonnegative function E satisfies

$$E(t) \le A + B \int_0^t E(s)\, d_\beta s, \qquad 0 \le t \le T, \tag{28}$$

where $A, B \ge 0$, then the change of variables $\tau = \Phi_\beta(t)$ and the classical Gronwall inequality give

$$E(t) \le A \exp\left(\frac{B}{\beta} t^\beta\right), \qquad 0 \le t \le T. \tag{29}$$

3 Problem Formulation

Let

$$\Omega = (0, l), \qquad Q = \Omega \times (0, T),$$

where $l > 0$ and $T > 0$. Let $0 < \beta \leq 1$ and $k > 0$. We consider the initial-boundary value problem

$$\mathcal{T}_t^\beta u(x,t) - k u_{xx}(x,t) = h(x,t), \qquad (x,t) \in Q, \tag{30}$$

subject to the initial condition

$$u(x,0) = u_0(x), \qquad x \in \Omega, \tag{31}$$

and the Neumann boundary conditions

$$u_x(0,t) = 0, \qquad u_x(l,t) = g(t), \qquad t \in (0,T). \tag{32}$$

Throughout Sections 3–5, we assume that

$$u_0 \in H, \qquad h \in L^2_\beta(0,T;H), \qquad g \in L^2_\beta(0,T), \tag{33}$$

where

$$V = H^1(\Omega), \qquad H = L^2(\Omega), \qquad V' = H^{-1}(\Omega),$$

as introduced in Section 2. No compatibility condition between u_0 and g is imposed at the level of weak solutions. Additional assumptions will be introduced in Section 6 when higher regularity is considered.

To obtain the variational formulation, let $v \in V$. Formally multiplying (30) by v and integrating over Ω gives

$$\left\langle \mathcal{T}_t^\beta u(t), v \right\rangle_{V',V} - k \int_\Omega u_{xx}(x,t) v(x)\, dx = (h(t), v)_H.$$

Integration by parts in the spatial variable yields

$$-k \int_\Omega u_{xx} v\, dx = k \int_\Omega u_x v_x\, dx - k u_x(l,t) v(l) + k u_x(0,t) v(0).$$

Using (32), we obtain

$$\left\langle \mathcal{T}_t^\beta u(t), v \right\rangle_{V',V} + k \int_\Omega u_x(x,t) v_x(x)\, dx = (h(t), v)_H + k g(t) v(l). \tag{34}$$

Recall the bilinear form

$$a(w,v) := k \int_\Omega w_x(x) v_x(x)\, dx, \qquad w, v \in V, \tag{35}$$

and define, for almost every $t \in (0,T)$,

$$F(t;v) := (h(t), v)_H + k g(t) v(l), \qquad v \in V. \tag{36}$$

Then (34) can be written in the abstract form

$$\left\langle \mathcal{T}_t^\beta u(t), v \right\rangle_{V',V} + a(u(t), v) = F(t;v), \qquad v \in V. \tag{37}$$

The right-hand side defines a bounded functional on V. Indeed, by the Cauchy–Schwarz inequality and the trace estimate (21),

$$\begin{aligned} |F(t;v)| &\le \|h(t)\|_H \|v\|_H + k|g(t)|\,|v(l)| \\ &\le (\|h(t)\|_H + kC_{\mathrm{tr}}|g(t)|)\,\|v\|_V. \end{aligned} \tag{38}$$

Consequently,

$$F \in L^2_\beta(0,T;V'). \tag{39}$$

We now introduce the notion of solution used in the sequel.

Definition 3.1. A function $u \in \mathcal{W}_\beta(0,T)$ is called a weak solution of (30)–(32) if

$$u(0) = u_0 \qquad \text{in } H, \tag{40}$$

and

$$\left\langle \mathcal{T}_t^\beta u(t), v \right\rangle_{V',V} + a(u(t),v) = F(t;v) \tag{41}$$

for every $v \in V$ and for almost every $t \in (0,T)$.

The initial condition in Definition 3.1 is meaningful because the embedding established in Section 2 gives

$$\mathcal{W}_\beta(0,T) \hookrightarrow C([0,T];H).$$

Remark 3.2. For a weak solution in $\mathcal{W}_\beta(0,T)$, the trace of u_x on $\partial\Omega$ is not required to exist pointwise. Therefore, the Neumann condition (32) is understood in the variational sense through the boundary term $kg(t)v(l)$ in (41).

Remark 3.3. The assumptions (33) are sufficient for the weak formulation and do not require compatibility between the initial datum and the boundary flux. Conditions such as

$$u_0'(0) = 0, \qquad u_0'(l) = g(0),$$

are relevant only when stronger spatial regularity or classical solutions are considered; see Section 6.

4 A Priori Estimates

In this section, we derive the basic a priori estimates for weak solutions of (30)–(32). These estimates will be used in Section 5 to establish uniqueness and continuous dependence on the data.

Let $u \in \mathcal{W}_\beta(0,T)$ be a weak solution in the sense of Definition 3.1. Taking $v = u(t)$ in (41), we obtain

$$\left\langle \mathcal{T}_t^\beta u(t), u(t) \right\rangle_{V',V} + k\|u_x(t)\|_H^2 = (h(t),u(t))_H + kg(t)u(l,t) \tag{42}$$

for almost every $t \in (0,T)$.

By Lemma 2.5,

$$\left\langle \mathcal{T}_t^\beta u(t), u(t) \right\rangle_{V',V} = \frac{1}{2}\mathcal{T}_t^\beta \|u(t)\|_H^2.$$

Hence (42) becomes

$$\frac{1}{2}\mathcal{T}_t^\beta \|u(t)\|_H^2 + k\|u_x(t)\|_H^2 = (h(t),u(t))_H + kg(t)u(l,t). \tag{43}$$

The source term is estimated by the Cauchy–Schwarz and Young inequalities:

$$|(h(t),u(t))_H| \le \frac{1}{2}\|h(t)\|_H^2 + \frac{1}{2}\|u(t)\|_H^2. \tag{44}$$

For the boundary term, let $\varepsilon > 0$. By Young's inequality,

$$k|g(t)u(l,t)| \le \frac{k}{2\varepsilon}|g(t)|^2 + \frac{k\varepsilon}{2}|u(l,t)|^2.$$

Using Lemma 2.6, for any $\delta > 0$ there exists $C_\delta > 0$ such that

$$|u(l,t)|^2 \le \delta\|u_x(t)\|_H^2 + C_\delta\|u(t)\|_H^2.$$

Therefore,

$$k|g(t)u(l,t)| \le \frac{k\varepsilon\delta}{2}\|u_x(t)\|_H^2 + \frac{k\varepsilon C_\delta}{2}\|u(t)\|_H^2 + \frac{k}{2\varepsilon}|g(t)|^2. \tag{45}$$

Choosing ε and δ such that

$$\frac{k\varepsilon\delta}{2} \le \frac{k}{2},$$

and combining (43)–(45), we obtain constants $c_0 > 0$ and $C_0 > 0$, depending only on k and l, such that

$$\mathcal{T}_t^\beta\|u(t)\|_H^2 + c_0\|u_x(t)\|_H^2 \le C_0\|u(t)\|_H^2 + C_0\|h(t)\|_H^2 + C_0|g(t)|^2 \tag{46}$$

for almost every $t \in (0,T)$.

We can now derive the principal energy estimate.

Theorem 4.1. *Let u be a weak solution of (30)–(32) under the assumptions (33). Then there exists a constant $C > 0$, depending only on β, T, k, and l, such that*

$$\sup_{0\le t\le T}\|u(t)\|_H^2 + \int_0^T \|u_x(t)\|_H^2\, d_\beta t \le C\left(\|u_0\|_H^2 + \|h\|_{L^2_\beta(0,T;H)}^2 + \|g\|_{L^2_\beta(0,T)}^2\right). \tag{47}$$

Proof. Integrating (46) over $(0,t)$ with respect to $d_\beta s$ and using Lemma 2.5, we obtain

$$\begin{aligned}\|u(t)\|_H^2 + c_0\int_0^t \|u_x(s)\|_H^2\, d_\beta s &\\ \le \|u_0\|_H^2 + C_0\int_0^t \|u(s)\|_H^2\, d_\beta s + C_0\int_0^t \left(\|h(s)\|_H^2 + |g(s)|^2\right) d_\beta s.&\end{aligned} \tag{48}$$

Set $E(t) := \|u(t)\|_H^2$ and

$$A := \|u_0\|_H^2 + C_0\left(\|h\|_{L^2_\beta(0,T;H)}^2 + \|g\|_{L^2_\beta(0,T)}^2\right).$$

Dropping the nonnegative gradient term in (48) gives

$$E(t) \le A + C_0\int_0^t E(s)\, d_\beta s. \tag{49}$$

Applying the weighted Gronwall estimate (29), we obtain

$$E(t) \le A\exp\left(\frac{C_0}{\beta}t^\beta\right), \qquad 0 \le t \le T. \tag{50}$$

Consequently,

$$\sup_{0\le t\le T}\|u(t)\|_H^2 \le C\left(\|u_0\|_H^2+\|h\|^2_{L^2_\beta(0,T;H)}+\|g\|^2_{L^2_\beta(0,T)}\right). \tag{51}$$

Returning to (48) with $t=T$ and using (51), we obtain

$$\int_0^T \|u_x(t)\|_H^2\, d_\beta t \le C\left(\|u_0\|_H^2+\|h\|^2_{L^2_\beta(0,T;H)}+\|g\|^2_{L^2_\beta(0,T)}\right). \tag{52}$$

Combining (51) and (52) proves (47). □

The previous estimate also controls the solution in the weighted $H^1(\Omega)$-space.

Corollary 4.2. *Under the assumptions of Theorem 4.1,*

$$\|u\|^2_{L^2_\beta(0,T;V)} \le C\left(\|u_0\|_H^2+\|h\|^2_{L^2_\beta(0,T;H)}+\|g\|^2_{L^2_\beta(0,T)}\right). \tag{53}$$

Proof. Since

$$\|u(t)\|_V^2 = \|u(t)\|_H^2+\|u_x(t)\|_H^2,$$

we have

$$\int_0^T \|u(t)\|_V^2\, d_\beta t = \int_0^T \|u(t)\|_H^2\, d_\beta t + \int_0^T \|u_x(t)\|_H^2\, d_\beta t.$$

Moreover,

$$\int_0^T \|u(t)\|_H^2\, d_\beta t \le \frac{T^\beta}{\beta}\sup_{0\le t\le T}\|u(t)\|_H^2.$$

The result therefore follows from Theorem 4.1. □

We next estimate the conformable time derivative.

Corollary 4.3. *Under the assumptions of Theorem 4.1, there exists $C>0$ such that*

$$\|\mathcal{T}_t^\beta u\|^2_{L^2_\beta(0,T;V')} \le C\left(\|u_0\|_H^2+\|h\|^2_{L^2_\beta(0,T;H)}+\|g\|^2_{L^2_\beta(0,T)}\right). \tag{54}$$

Proof. From the weak formulation (37),

$$\left\langle \mathcal{T}_t^\beta u(t), v\right\rangle_{V',V} = F(t;v)-a(u(t),v), \qquad v\in V.$$

Using (24) and (38), we obtain

$$\left|\left\langle \mathcal{T}_t^\beta u(t), v\right\rangle_{V',V}\right| \le k\|u(t)\|_V\|v\|_V + \left(\|h(t)\|_H + kC_{\mathrm{tr}}|g(t)|\right)\|v\|_V.$$

Taking the supremum over all $v\in V$ with $\|v\|_V=1$ gives

$$\|\mathcal{T}_t^\beta u(t)\|_{V'} \le C\left(\|u(t)\|_V+\|h(t)\|_H+|g(t)|\right). \tag{55}$$

Squaring (55), integrating over $(0,T)$ with respect to $d_\beta t$, and applying Corollary 4.2 yields (54). □

Combining Corollaries 4.2 and 4.3, we obtain the following estimate in the full solution space.

Corollary 4.4. *Under the assumptions of Theorem 4.1,*

$$\|u\|^2_{\mathcal{W}_\beta(0,T)} \le C\left(\|u_0\|^2_H + \|h\|^2_{L^2_\beta(0,T;H)} + \|g\|^2_{L^2_\beta(0,T)}\right). \tag{56}$$

Estimate (56) will be used in Section 5 to establish the continuous dependence of weak solutions on the prescribed data.

5 Well-Posedness

In this section, we establish the existence, uniqueness, and continuous dependence of weak solutions to (30)–(32). Existence is obtained by transforming the conformable problem into a classical parabolic problem, while uniqueness and stability follow from the energy estimates derived in Section 4.

We begin with the existence result.

Theorem 5.1. *Assume that*

$$u_0 \in H, \qquad h \in L^2_\beta(0,T;H), \qquad g \in L^2_\beta(0,T). \tag{57}$$

Then problem (30)–(32) *admits a weak solution* $u \in \mathcal{W}_\beta(0,T)$.

Proof. Consider the time transformation

$$\tau = \Phi_\beta(t) = \frac{t^\beta}{\beta}, \qquad 0 \le \tau \le S_\beta, \qquad S_\beta = \frac{T^\beta}{\beta}. \tag{58}$$

Define

$$\widetilde{h}(x,\tau) := h\big(x, \Phi_\beta^{-1}(\tau)\big), \qquad \widetilde{g}(\tau) := g\big(\Phi_\beta^{-1}(\tau)\big). \tag{59}$$

By Lemma 2.1,

$$\|\widetilde{h}\|_{L^2(0,S_\beta;H)} = \|h\|_{L^2_\beta(0,T;H)}, \tag{60}$$

and

$$\|\widetilde{g}\|_{L^2(0,S_\beta)} = \|g\|_{L^2_\beta(0,T)}. \tag{61}$$

Hence

$$\widetilde{h} \in L^2(0,S_\beta;H), \qquad \widetilde{g} \in L^2(0,S_\beta).$$

For $\tau \in (0,S_\beta)$, define the functional $\widetilde{F}(\tau) \in V'$ by

$$\widetilde{F}(\tau;v) := (\widetilde{h}(\tau),v)_H + k\widetilde{g}(\tau)v(l), \qquad v \in V. \tag{62}$$

Using the trace estimate (21), we obtain

$$|\widetilde{F}(\tau;v)| \le \left(\|\widetilde{h}(\tau)\|_H + kC_{\mathrm{tr}}|\widetilde{g}(\tau)|\right)\|v\|_V.$$

Therefore,

$$\widetilde{F} \in L^2(0,S_\beta;V'). \tag{63}$$

We now consider the classical variational problem

$$\langle \partial_\tau \widetilde{u}(\tau), v\rangle_{V',V} + a(\widetilde{u}(\tau),v) = \widetilde{F}(\tau;v), \qquad v \in V, \tag{64}$$

with

$$\widetilde{u}(0) = u_0. \tag{65}$$

The bilinear form a is continuous on $V \times V$ by (24). Although a is not coercive on V, it satisfies the Gårding inequality (25). To reduce the problem to a coercive one, set

$$\widetilde{u}(\tau) = e^{k\tau} z(\tau). \tag{66}$$

Substitution into (64) gives

$$\langle \partial_\tau z(\tau), v \rangle_{V',V} + a(z(\tau), v) + k(z(\tau), v)_H = e^{-k\tau} \widetilde{F}(\tau; v). \tag{67}$$

By Lemma 2.7, the shifted bilinear form $a_k(z, v) = a(z, v) + k(z, v)_H$ is coercive on V. Furthermore, $e^{-k\tau}\widetilde{F} \in L^2(0, S_\beta; V')$. Hence the standard variational theory for linear parabolic equations [9, 10] yields a function

$$z \in L^2(0, S_\beta; V), \qquad \partial_\tau z \in L^2(0, S_\beta; V'),$$

satisfying (67). Consequently,

$$\widetilde{u} \in L^2(0, S_\beta; V), \qquad \partial_\tau \widetilde{u} \in L^2(0, S_\beta; V'),$$

and $\widetilde{u}$ satisfies (64)–(65).

Define

$$u(x, t) := \widetilde{u}\left(x, \frac{t^\beta}{\beta}\right). \tag{68}$$

By Lemma 2.1, $u \in L^2_\beta(0, T; V)$. Moreover, by Definition 2.2, $\mathcal{T}_t^\beta u \in L^2_\beta(0, T; V')$, and $\mathcal{T}_t^\beta u(t) = \partial_\tau \widetilde{u}\big(\Phi_\beta(t)\big)$ for almost every $t \in (0, T)$. Thus $u \in \mathcal{W}_\beta(0, T)$.

Finally, changing variables back from τ to t in (64) gives

$$\left\langle \mathcal{T}_t^\beta u(t), v \right\rangle_{V',V} + a(u(t), v) = F(t; v)$$

for every $v \in V$ and for almost every $t \in (0, T)$. Since $\widetilde{u}(0) = u_0$, we also have $u(0) = u_0$. Therefore u is a weak solution in the sense of Definition 3.1. □

We next establish uniqueness.

Theorem 5.2. *Under the assumptions of Theorem 5.1, the weak solution of* (30)–(32) *is unique.*

Proof. Let $u_1, u_2 \in \mathcal{W}_\beta(0, T)$ be two weak solutions corresponding to the same data u_0, h, and g. Set $w := u_1 - u_2$. Then

$$w \in \mathcal{W}_\beta(0, T), \qquad w(0) = 0,$$

and subtraction of the two weak formulations gives

$$\left\langle \mathcal{T}_t^\beta w(t), v \right\rangle_{V',V} + a(w(t), v) = 0 \tag{69}$$

for every $v \in V$ and for almost every $t \in (0, T)$.

Taking $v = w(t)$ and using Lemma 2.5, we obtain

$$\frac{1}{2} \mathcal{T}_t^\beta \|w(t)\|_H^2 + k\|w_x(t)\|_H^2 = 0. \tag{70}$$

Integrating over $(0,t)$ with respect to $d_\beta s$ yields

$$\frac{1}{2}\|w(t)\|_H^2 + k\int_0^t \|w_x(s)\|_H^2\, d_\beta s = \frac{1}{2}\|w(0)\|_H^2. \tag{71}$$

Since $w(0)=0$, it follows that

$$\|w(t)\|_H^2 = 0 \qquad \text{for every } t\in[0,T].$$

Therefore $u_1 = u_2$, and uniqueness follows. □

We now study the dependence of solutions on the prescribed data.

Let

$$(u_{0,1}, h_1, g_1) \qquad \text{and} \qquad (u_{0,2}, h_2, g_2)$$

be two sets of data satisfying (33), and let u_1 and u_2 be the corresponding weak solutions. Define

$$w := u_1 - u_2, \qquad \delta u_0 := u_{0,1} - u_{0,2}, \qquad \delta h := h_1 - h_2, \qquad \delta g := g_1 - g_2. \tag{72}$$

Then w is the weak solution corresponding to the data δu_0, δh, and δg.

Theorem 5.3. *Let u_1 and u_2 be the weak solutions associated with*

$$u_{0,i}\in H, \qquad h_i \in L^2_\beta(0,T;H), \qquad g_i\in L^2_\beta(0,T), \qquad i=1,2.$$

Then there exists a constant $C>0$, depending only on β, T, k, and l, such that

$$\begin{aligned}\sup_{0\le t\le T}\|u_1(t)-u_2(t)\|_H^2 &+ \int_0^T \|(u_1-u_2)_x(t)\|_H^2\, d_\beta t \\ &\le C\Big(\|u_{0,1}-u_{0,2}\|_H^2 + \|h_1-h_2\|^2_{L^2_\beta(0,T;H)} + \|g_1-g_2\|^2_{L^2_\beta(0,T)}\Big).\end{aligned} \tag{73}$$

Moreover,

$$\|u_1-u_2\|^2_{\mathcal{W}_\beta(0,T)} \le C\Big(\|u_{0,1}-u_{0,2}\|_H^2 + \|h_1-h_2\|^2_{L^2_\beta(0,T;H)} + \|g_1-g_2\|^2_{L^2_\beta(0,T)}\Big). \tag{74}$$

Proof. The function $w = u_1 - u_2$ satisfies $w(0) = \delta u_0$ and

$$\Big\langle \mathcal{T}_t^\beta w(t), v\Big\rangle_{V',V} + a(w(t),v) = (\delta h(t), v)_H + k\delta g(t) v(l) \tag{75}$$

for every $v\in V$ and for almost every $t\in(0,T)$.

Applying Theorem 4.1 to w gives (73). Applying Corollary 4.4 to the same problem yields (74). □

The preceding theorem immediately gives convergence of solutions when the data converge.

Corollary 5.4. *Suppose that*

$$u_{0,n}\to u_0 \quad \text{in } H,$$
$$h_n \to h \quad \text{in } L^2_\beta(0,T;H),$$

and

$$g_n \to g \quad \text{in } L^2_\beta(0,T).$$

Let u_n and u be the weak solutions associated with $(u_{0,n},h_n,g_n)$ and (u_0,h,g), respectively. Then

$$u_n \to u \quad \text{in } \mathcal{W}_\beta(0,T), \tag{76}$$

and

$$u_n \to u \quad \text{in } C([0,T];H). \tag{77}$$

Proof. The convergence in $\mathcal{W}_\beta(0,T)$ follows directly from (74). The convergence in $C([0,T];H)$ follows from (73). □

We summarize the results of this section in the following theorem.

Theorem 5.5. *Let $0 < \beta \le 1$ and $k > 0$. For every*

$$u_0 \in H, \qquad h \in L^2_\beta(0,T;H), \qquad g \in L^2_\beta(0,T),$$

the initial-boundary value problem (30)–(32) admits a unique weak solution $u \in \mathcal{W}_\beta(0,T)$. Furthermore, there exists a constant $C > 0$, depending only on β, T, k, and l, such that

$$\|u\|^2_{\mathcal{W}_\beta(0,T)} \le C\left(\|u_0\|^2_H + \|h\|^2_{L^2_\beta(0,T;H)} + \|g\|^2_{L^2_\beta(0,T)}\right). \tag{78}$$

The solution depends continuously on the initial datum, the source term, and the Neumann boundary datum. Therefore, (30)–(32) is well posed in the sense of Hadamard.

Remark 5.6. When $\beta = 1$,

$$\mathcal{T}^1_t u = u_t, \qquad d_1 t = dt,$$

and

$$\mathcal{W}_1(0,T) = \left\{u \in L^2(0,T;V) : u_t \in L^2(0,T;V')\right\}.$$

Hence Theorem 5.5 reduces to the standard well-posedness result for the classical parabolic equation with nonhomogeneous Neumann boundary data.

6 Higher Regularity

In this section, we establish higher regularity of the weak solution obtained in Theorem 5.5. In order to handle the nonhomogeneous Neumann condition, we first remove the boundary datum by a suitable lifting.

We introduce the weighted first-order space

$$\mathcal{H}^1_\beta(0,T) := \left\{q \in L^2_\beta(0,T) : \mathcal{T}^\beta_t q \in L^2_\beta(0,T)\right\}, \tag{79}$$

equipped with the norm

$$\|q\|^2_{\mathcal{H}^1_\beta(0,T)} := \|q\|^2_{L^2_\beta(0,T)} + \|\mathcal{T}^\beta_t q\|^2_{L^2_\beta(0,T)}. \tag{80}$$

Here the conformable derivative is understood in the weak sense introduced in Definition 2.2.

Under the change of variables

$$\tau = \Phi_\beta(t) = \frac{t^\beta}{\beta},$$

the space $\mathcal{H}^1_\beta(0,T)$ is identified with $H^1(0,S_\beta)$. In particular,

$$\mathcal{H}^1_\beta(0,T) \hookrightarrow C([0,T]). \tag{81}$$

Hence $g(0)$ is well defined whenever $g \in \mathcal{H}^1_\beta(0,T)$.

To homogenize the Neumann boundary conditions, define

$$\psi(x) := \frac{x^2}{2l}, \qquad x \in \Omega. \tag{82}$$

Then

$$\psi'(0) = 0, \qquad \psi'(l) = 1, \qquad \psi''(x) = \frac{1}{l}. \tag{83}$$

Set

$$u(x,t) = v(x,t) + \psi(x)g(t). \tag{84}$$

Since $u_x = v_x + \psi' g$, the boundary conditions (32) imply

$$v_x(0,t) = v_x(l,t) = 0 \qquad \text{for a.e. } t \in (0,T). \tag{85}$$

Substituting (84) into (30) gives

$$\mathcal{T}_t^{\beta} v - k v_{xx} = f, \tag{86}$$

where

$$f(x,t) := h(x,t) - \psi(x)\mathcal{T}_t^{\beta} g(t) + \frac{k}{l} g(t). \tag{87}$$

The initial datum for v is

$$v_0(x) := v(x,0) = u_0(x) - \psi(x)g(0). \tag{88}$$

If

$$h \in L^2_{\beta}(0,T;H), \qquad g \in \mathcal{H}^1_{\beta}(0,T),$$

then

$$f \in L^2_{\beta}(0,T;H), \tag{89}$$

and there exists a constant $C > 0$ such that

$$\|f\|_{L^2_{\beta}(0,T;H)} \le C \left(\|h\|_{L^2_{\beta}(0,T;H)} + \|g\|_{\mathcal{H}^1_{\beta}(0,T)} \right). \tag{90}$$

We now state the main regularity result.

Theorem 6.1. *Assume that*

$$u_0 \in H^1(\Omega), \qquad h \in L^2_{\beta}(0,T;H), \qquad g \in \mathcal{H}^1_{\beta}(0,T). \tag{91}$$

Then the unique weak solution u of (30)–(32) satisfies

$$u \in C([0,T]; H^1(\Omega)) \cap L^2_{\beta}(0,T;H^2(\Omega)), \tag{92}$$

and

$$\mathcal{T}_t^{\beta} u \in L^2_{\beta}(0,T;H). \tag{93}$$

Moreover, there exists a constant $C > 0$, depending only on β, T, k, and l, such that

$$\begin{aligned} &\sup_{0\le t\le T} \|u(t)\|^2_{H^1(\Omega)} + \|u\|^2_{L^2_{\beta}(0,T;H^2(\Omega))} + \|\mathcal{T}_t^{\beta} u\|^2_{L^2_{\beta}(0,T;H)} \\ &\qquad \le C \left(\|u_0\|^2_{H^1(\Omega)} + \|h\|^2_{L^2_{\beta}(0,T;H)} + \|g\|^2_{\mathcal{H}^1_{\beta}(0,T)} \right). \end{aligned} \tag{94}$$

Proof. Let v be defined by (84). Then v satisfies

$$\begin{cases} \mathcal{T}_t^{\beta} v - k v_{xx} = f, & (x,t) \in Q, \\ v_x(0,t) = v_x(l,t) = 0, & t \in (0,T), \\ v(x,0) = v_0(x), & x \in \Omega. \end{cases} \tag{95}$$

By (91), (81), and (88), $v_0 \in H^1(\Omega)$. Furthermore,

$$\|v_0\|_{H^1(\Omega)} \le C \left(\|u_0\|_{H^1(\Omega)} + \|g\|_{\mathcal{H}^1_{\beta}(0,T)} \right). \tag{96}$$

We now introduce the transformed functions

$$\widetilde{v}(x,\tau) := v\big(x, \Phi_{\beta}^{-1}(\tau)\big), \qquad \widetilde{f}(x,\tau) := f\big(x, \Phi_{\beta}^{-1}(\tau)\big). \tag{97}$$

By Definition 2.2, problem (95) becomes

$$\begin{cases} \partial_\tau \widetilde{v} - k \widetilde{v}_{xx} = \widetilde{f}, & (x,\tau) \in \Omega \times (0, S_\beta), \\ \widetilde{v}_x(0,\tau) = \widetilde{v}_x(l,\tau) = 0, & \tau \in (0, S_\beta), \\ \widetilde{v}(x,0) = v_0(x), & x \in \Omega. \end{cases} \tag{98}$$

For clarity, consider the Neumann operator

$$A_N w := -k w_{xx}, \tag{99}$$

with domain

$$D(A_N) := \left\{ w \in H^2(\Omega) : w_x(0) = w_x(l) = 0 \right\}. \tag{100}$$

The operator A_N is nonnegative and self-adjoint in H, and its associated form domain is $H^1(\Omega)$. In particular,

$$D(A_N^{1/2}) = H^1(\Omega). \tag{101}$$

Problem (98) can therefore be written as

$$\partial_\tau \widetilde{v} + A_N \widetilde{v} = \widetilde{f}, \qquad \widetilde{v}(0) = v_0. \tag{102}$$

By Lemma 2.1,

$$\widetilde{f} \in L^2(0, S_\beta; H),$$

and

$$\|\widetilde{f}\|_{L^2(0,S_\beta;H)} = \|f\|_{L^2_\beta(0,T;H)}. \tag{103}$$

Since

$$v_0 \in D(A_N^{1/2}) \qquad \text{and} \qquad \widetilde{f} \in L^2(0, S_\beta; H),$$

the classical L^2-regularity theory for linear parabolic equations with homogeneous Neumann boundary conditions [9, 10] gives

$$\widetilde{v} \in L^2(0, S_\beta; D(A_N)), \qquad \partial_\tau \widetilde{v} \in L^2(0, S_\beta; H), \tag{104}$$

and

$$\widetilde{v} \in C([0, S_\beta]; H^1(\Omega)). \tag{105}$$

Moreover,

$$\sup_{0\le\tau\le S_\beta} \|\widetilde{v}(\tau)\|^2_{H^1(\Omega)} + \|\widetilde{v}\|^2_{L^2(0,S_\beta;H^2(\Omega))} + \|\partial_\tau \widetilde{v}\|^2_{L^2(0,S_\beta;H)} \le C\left(\|v_0\|^2_{H^1(\Omega)} + \|\widetilde{f}\|^2_{L^2(0,S_\beta;H)}\right). \tag{106}$$

Returning to the original time variable and using Lemma 2.1, we obtain

$$v \in C([0,T];H^1(\Omega)) \cap L^2_\beta(0,T;H^2(\Omega)), \tag{107}$$

and

$$\mathcal{T}_t^\beta v \in L^2_\beta(0,T;H). \tag{108}$$

Finally, from $u = v + \psi g$, with

$$\psi \in H^2(\Omega) \qquad \text{and} \qquad g \in \mathcal{H}^1_\beta(0,T),$$

we have

$$\psi g \in C([0,T];H^2(\Omega)) \cap L^2_\beta(0,T;H^2(\Omega)),$$

and

$$\mathcal{T}_t^\beta(\psi g) = \psi \mathcal{T}_t^\beta g \in L^2_\beta(0,T;H).$$

Therefore, (92) and (93) follow.

Combining (90), (96), and (106) gives (94). □

The previous theorem shows that the weak solution is a strong solution in the following sense.

Corollary 6.2. *Under the assumptions of Theorem 6.1,*

$$\mathcal{T}_t^\beta u, \quad u_{xx} \in L^2_\beta(0,T;H), \tag{109}$$

and

$$\mathcal{T}_t^\beta u - k u_{xx} = h \quad \text{in } L^2_\beta(0,T;H). \tag{110}$$

Moreover, since $u(t) \in H^2(\Omega)$ for almost every $t \in (0,T)$, the Neumann conditions satisfy

$$u_x(0,t) = 0, \qquad u_x(l,t) = g(t) \tag{111}$$

for almost every $t \in (0,T)$ in the trace sense.

Proof. The assertions follow directly from Theorem 6.1 and the lifting (84). □

Remark 6.3. The regularity result of Theorem 6.1 does not require compatibility between u_0 and g at $t = 0$.

If stronger regularity up to the initial time is required, one may assume in addition that $u_0 \in H^2(\Omega)$ and impose the compatibility conditions

$$u_0'(0) = 0, \qquad u_0'(l) = g(0). \tag{112}$$

In this case, the lifted initial datum $v_0 = u_0 - \psi g(0)$ satisfies $v_0 \in D(A_N)$. Such compatibility conditions are not needed for the weak well-posedness result of Theorem 5.5.

Remark 6.4. When $\beta = 1$, $\mathcal{H}^1_1(0,T) = H^1(0,T)$, and Theorem 6.1 reduces to the corresponding regularity result for the classical parabolic equation with nonhomogeneous Neumann boundary data.

Declarations

Funding

This research received no external funding.

Competing Interests

The author declares that they have no competing interests.

Ethical Approval

Not applicable.

Authors's Contributions

The author contributed to all aspects of this work and read and approved the final manuscript.

Availability Data and Materials

Not applicable.

Acknowledgements

We would like to thank the editors and reviewers for their time, effort, and valuable commentswhich led to improvements in our manuscript.

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