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A strategy for inverse source term and the initial data for the time-fractional diffusion equation for the Hyper Bessel operator

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Abstract

In this paper, we investigate the inverse problem of simultaneously identifying the source term and the initial value for a time-fractional diffusion equation involving a Caputo-like hyper-Bessel operator. We first establish the ill-posedness of the problem and derive a conditional stability estimate. To overcome this difficulty, we apply the modified Fractional Landweber regularization method and provide error estimates under a priori parameter choice rules.

Keywords: Inverse source problem, parabolic equation, regularization method, error estimate

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1 Introduction

Let $\mathcal{D} \in \mathbb{R}$ be bounded domain with smooth boundary $\partial\Omega$,

$$\begin{cases} \mathcal{C}\left(t^{1-\alpha}\dfrac{\partial}{\partial t}\right)^{\gamma} v(x,t) - \Delta v(x,t) = f(x), & x \in \mathcal{D},\ t \in (0,T], \\ v(x,t) = 0, & x \in \partial\mathcal{D},\ t \in (0,T], \\ v(x,0) = \rho_1(x), & x \in \mathcal{D}, \\ v(x,t_0) = \rho_2(x), & x \in \mathcal{D},\ t_0 \in (0,T], \\ v(x,T) = \rho_3(x), & x \in \mathcal{D}, \end{cases} \tag{1.1}$$

where $0 < \gamma < 1$ and $0 < \alpha < 1$. $\mathcal{C}\left(t^{1-\alpha}\frac{\partial}{\partial t}\right)^{\gamma}$ is regularized Caputo-like counterpart of the hyper-Bessel operator, which is defined as follows [2]

$$\mathcal{C}\left(t^{1-\alpha}\frac{\partial}{\partial t}\right)^{\gamma} v(t) = \left(t^{1-\alpha}\frac{\partial}{\partial t}\right)^{\gamma} v(t) - \frac{v(0)t^{-\gamma\alpha}}{\alpha^{-\gamma}\Gamma(1-\gamma)},\ 0 < \alpha, \gamma < 1,$$

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Here the Riemann-Liouville fractional derivative $\left(t^{1-\alpha}\frac{\partial}{\partial t}\right)^{\gamma}$ is defined as follows [9]

$$\left(t^{1-\alpha}\frac{\partial}{\partial t}\right)^{\gamma} f(t) = \alpha^{\gamma} I_{\alpha}^{0,-\gamma} t^{-\alpha\gamma} f(t), \quad \alpha > 0,$$

where $I_{\alpha}^{0,-\gamma} t^{-\alpha\gamma} = \frac{\Gamma(1-\gamma)}{\Gamma(1-2\gamma)} t^{-\alpha\gamma}$. When $\alpha = 0$, $\left(t\frac{\partial}{\partial t}\right)^{\gamma}$ is the Hadamard fractional derivative, there are some studies on Hadamard-type fractional differential equations [1].

In problem (1.1), the initial value $\rho_1(x)$ and the source term $f(x)$ are both unknown and must be reconstructed. To this end, we employ the intermediate observation $v(x,t_0) = \rho_2(x)$ and the terminal data $v(x,T) = \rho_3(x)$ to simultaneously recover the initial state $\rho_1(x)$ and the source term $f(x)$. Since $\rho_2(x)$ and $\rho_3(x)$ are obtained from measurements, we assume that the exact and perturbed data satisfy

$$\|\rho_2 - \rho_2^{\delta}\|_{L^2(\mathcal{D})} + \|\rho_3 - \rho_3^{\delta}\|_{L^2(\mathcal{D})} \leq \delta, \tag{1.2}$$

where $\delta > 0$ denotes the noise level.

The analysis is closely related to hyper-Bessel operators. Introduced by Dimovski [7] as a generalization of the classical Bessel operator, these operators arise in various models of mathematical physics, including fractional relaxation [9] and heat diffusion [10]. Their fractional powers were characterized in [8], and a systematic theory was developed in [11, 12]. In particular, Garra et al. [9] derived an explicit representation of $(t^{\theta}\frac{d}{dt})^{\alpha}$ via the Erdélyi–Kober integral.

Inverse problems for fractional diffusion equations involving Caputo-like hyper-Bessel operators have been extensively studied. The recovery of initial data was considered in [14], while source identification problems were addressed in [17]. The fractional Landweber method was applied in [15] to treat the inverse initial value problem.

Further related contributions include the simultaneous identification of initial data and boundary flux [16], joint reconstruction of source and initial distribution via Fourier methods [13], and regularization-based recovery of spatial source terms [17]. The simultaneous identification of source and initial value for Caputo-type models was studied in [18] via problem decomposition.

The paper is organized as follows. Section 2 collects preliminary results. Section 3 constructs the solution to (1.1) and establishes its ill-posedness and conditional stability. Section 4 develops a modified fractional Landweber regularization and provides error estimates under a priori parameter choices.

2 Preliminaries

Definition 2.1 ([4, 5]). Let $\lambda_k, e_k(x)$ be the Dirichlet eigenvalues and eigenfunctions of the negative Laplacian operator $-\Delta$ on the domain $\mathcal{D}$, i.e.,

$$\begin{cases} -\Delta e_n(x) = \lambda_n e_n(x), & x \in \mathcal{D}, \ \forall n \in \mathbb{N}^+, \\ e_n(x) = 0, & x \in \partial\mathcal{D}, \ \forall n \in \mathbb{N}^+, \end{cases}$$

where $1 \leq \lambda_1 < \lambda_2 < \cdots < \lambda_n < \cdots$, $\lim_{n\to\infty} \lambda_n = +\infty$, and $\{e_n(x)\}_{n\in\mathbb{N}^+}$ is a complete and orthogonal basis in Hilbert space $L^2(\mathcal{D})$. For any $0 < s < 1$, define the Hilbert scale space

$$\mathcal{H}^s(\mathcal{D}) := \left\{ \psi \in L^2(\mathcal{D}) \middle| \sum_{n=1}^{\infty} \lambda_n^{2s} \langle \psi, e_n \rangle^2 < +\infty \right\}.$$

Then, if $\psi \in \mathcal{H}^s(\mathcal{D})$, the spectral fractional Laplacian $(-\Delta)^s$ is defined by

$$(-\Delta)^s \psi(x) = \sum_{n=1}^{\infty} \lambda_n^s \langle \psi, e_n \rangle e_n(x).$$

Definition 2.2. [6] The Mittag-Leffler function is defined as

$$E_{\alpha,\beta}(z) = \sum_{k=0}^{+\infty} \frac{z^k}{\Gamma(\alpha k + \beta)}, \quad z \in \mathbb{C},$$

where $\alpha > 0$ and $\beta \in \mathbb{R}$ are arbitrary constants.

Lemma 2.3. *If $\lambda > 0$, then the following equation holds*

$$\int_c^\infty e^{-s \log \frac{t}{c}} \big(\log(t/c)\big)^{\beta k + \gamma - 1} E_{\beta,\gamma}^{(k)}\big(\pm \lambda \big(\log(t/c)\big)^\beta\big) \frac{dt}{t} = \frac{k! s^{\beta-\gamma}}{\big(s^\beta \mp \lambda\big)^{k+1}}, \mathrm{Re}(s) > |\lambda|^{\frac{1}{\beta}},$$

where $E_{\beta,\gamma}^{(m)}(y) := \dfrac{d^m}{dy^m} E_{\beta,\gamma}(y)$. The Lemma 2.3 means that the Laplace transformation of

$$\big(\log \frac{t}{c}\big)^{\beta k + \gamma - 1} E_{\beta,\gamma}^{(k)}\big(\pm \lambda \big(\log \frac{t}{c}\big)^\beta\big) \text{ is } \frac{k! s^{\beta-\gamma}}{(s^\beta \mp \lambda)^{k+1}}.$$

Lemma 2.4. *For $0 < \gamma < 1, z > 0$, we have $0 \le E_{\gamma,1}(-z) < 1$. Moreover, $E_{\gamma,1}(-z)$ is completely monotonic, that is,*

$$(-1)^n \frac{d^n}{dz^n} E_{\gamma,1}(-z) \ge 0, z \ge 0,$$

Lemma 2.5. *[6] Assume that $0 < \gamma_0 < \gamma_1 < 1$. Then, there exist constants $\mathcal{Z}_\pm > 0$, depending only on γ_0, γ_1 such that for all $\gamma \in [\gamma_0, \gamma_1]$, we obtain*

$$\frac{\mathcal{Z}_-}{\Gamma(1-\gamma)} \frac{1}{1-x} \le E_{\gamma,1}(x) \le \frac{\mathcal{Z}_+}{\Gamma(1-\gamma)} \frac{1}{1-x}, \forall x \le 0.$$

Lemma 2.6. *For $0 < \gamma < 1$, $z > 0$, we have $0 \le E_{\gamma,1}(-z) < 1$. In addition, $E_{\gamma,1}(-z)$ is completely monotonic because*

$$(-1)^n \frac{d^n}{dz^n} E_{\gamma,1}(-z) \ge 0, \ z \ge 0.$$

Lemma 2.7. *[2] For any λ_n that satisfies $0 < \lambda_1 \le \lambda_n$, there are positive numbers $\mathcal{Z}_1, \mathcal{Z}_2, \mathcal{Z}_3, \mathcal{Z}_4, \mathcal{Z}_5, \mathcal{Z}_6$ that depends on $\gamma, \alpha, \lambda_1, T$ or t_0 such that*

$$\begin{aligned}
&\frac{\mathcal{Z}_1}{\lambda_n} \le E_{\gamma,1}\big(-\frac{\lambda_n}{\alpha^\gamma} t_0^{\alpha\gamma}\big) \le \frac{\mathcal{Z}_2}{\lambda_n}, \ \frac{\mathcal{Z}_3}{\lambda_n} \le E_{\gamma,1}\big(-\frac{\lambda_n}{\alpha^\gamma} T^{\alpha\gamma}\big) \le \frac{\mathcal{Z}_4}{\lambda_n}, \\
&\frac{\mathcal{Z}_5}{\lambda_n} \le \frac{t_0^{\alpha\gamma}}{\alpha^\gamma} E_{\gamma,1+\gamma}\big(-\frac{\lambda_n}{\alpha^\gamma} t_0^{\alpha\gamma}\big) \le \frac{1}{\lambda_n}, \ \frac{\mathcal{Z}_6}{\lambda_n} \le \frac{T^{\alpha\gamma}}{\alpha^\gamma} E_{\gamma,1+\gamma}\big(-\frac{\lambda_n}{\alpha^\gamma} T^{\alpha\gamma}\big) \le \frac{1}{\lambda_n}, \\
&\frac{\mathcal{Z}_1 - \mathcal{Z}_4}{\lambda_n} \le E_{\gamma,1}\big(-\frac{\lambda_n}{\alpha^\gamma} t_0^{\alpha\gamma}\big) - E_{\gamma,1}\big(-\frac{\lambda_n}{\alpha^\gamma} T^{\alpha\gamma}\big) \le \frac{\mathcal{Z}_2}{\lambda_n},
\end{aligned}$$

where $\mathcal{Z}_1 > \mathcal{Z}_4$, $\mathcal{Z}_1 := \frac{\mathcal{Z}_-}{T(1-\alpha)} \frac{1}{\frac{1}{\lambda_1} + \frac{t_0^{\beta\alpha}}{\beta^\alpha}}$, $\mathcal{Z}_2 := \frac{\mathcal{Z}^-}{T(1-\alpha)} \frac{\beta^\alpha}{t_0^{\alpha\gamma}}$, $\mathcal{Z}_3 := \frac{\mathcal{Z}_-}{T(1-\gamma)} \frac{1}{\frac{1}{\lambda_1} + \frac{T^{\alpha\gamma}}{\alpha^\gamma}}$, $\mathcal{Z}_4 := \frac{\mathcal{Z}^-}{T(1-\gamma)} \frac{\alpha^\gamma}{T^{\alpha\gamma}}$, $\mathcal{Z}_5 := 1 - E_{\gamma,1}\big(-\frac{\lambda_1}{\alpha^\gamma} t_0^{\alpha\gamma}\big)$, $\mathcal{Z}_6 := 1 - E_{\gamma,1}\big(-\frac{\lambda_1}{\alpha^\gamma} T^{\alpha\gamma}\big)$.

3 Regularization and error estimate for unknown source (1.1)

Assume that Problem (1.1) has a solution u which has the form

$$\nu(x,t)=\sum_{n=1}^{\infty}\nu_n(t)e_n(x), \quad u_n(t)=\langle \nu(\cdot,t),e_n\rangle,$$

then we have

$$C\Big(t^{1-\alpha}\frac{\partial}{\partial t}\Big)^{\gamma}u(x,t)-Au(x,t)=\varphi(t)f(x)\cdot$$

Next, we know that

$$\nu(x,t)=\sum_{n=1}^{\infty}\Big(\langle\rho_1,e_n\rangle E_{\gamma,1}(-\frac{\lambda_n}{\alpha^{\gamma}}t^{\alpha\gamma})+\frac{\langle f,e_n\rangle}{\alpha^{\gamma}}t^{\alpha\gamma}E_{\alpha,1+\alpha}(-\frac{\lambda_n}{\beta^{\alpha}}t^{\beta\alpha})\Big)e_n(x).$$

Letting $t=t_0$, we receive

$$\rho_2(x)=\sum_{n=1}^{\infty}\Big(\langle\rho_1,e_n\rangle E_{\gamma,1}\big(-\frac{\lambda_n}{\alpha^{\gamma}}t_0^{\alpha\gamma}\big)+\frac{\langle f,e_n\rangle}{\alpha^{\gamma}}T^{\alpha\gamma}E_{\gamma,\gamma+1}\big(-\frac{\lambda_n}{\alpha^{\gamma}}t_0^{\alpha\gamma}\big)\Big)e_n(x), \tag{3.3}$$

Next, with $t=T$, one has

$$\rho_3(x)=\sum_{n=1}^{\infty}\Big(\langle\rho_1,e_n\rangle E_{\gamma,1}\Big(-\frac{\lambda_n}{\alpha^{\gamma}}T^{\alpha\gamma}\Big)+\frac{\langle f,e_n\rangle}{\alpha^{\gamma}}T^{\alpha\gamma}E_{\gamma,1+\gamma}\Big(-\frac{\lambda_n}{\alpha^{\gamma}}T^{\alpha\gamma}\Big)\Big)e_n(x). \tag{3.4}$$

Through (3.3) and (3.4), we can calculate that

$$f(x)=\sum_{n=1}^{\infty}\lambda_n\frac{E_{\gamma,1}(-\frac{\lambda_n}{\alpha^{\gamma}}t_0^{\alpha\gamma})\langle\rho_3,e_n\rangle-E_{\gamma,1}(-\frac{\lambda_n}{\alpha^{\gamma}}T^{\alpha\gamma})\langle\rho_2,e_n\rangle}{E_{\gamma,1}(-\frac{\lambda_n}{\alpha^{\gamma}}t_0^{\alpha\gamma})-E_{\gamma,1}(-\frac{\lambda_n}{\alpha^{\gamma}}T^{\alpha\gamma})}e_n(x), \tag{3.5}$$

and

$$\rho_1(x)=\sum_{n=1}^{\infty}\frac{\lambda_n}{\alpha^{\gamma}}\frac{t_0^{\alpha\gamma}E_{\gamma,1+\gamma}(-\frac{\lambda_n}{\alpha^{\gamma}}t_0^{\alpha\gamma})\langle\rho_3,e_n\rangle-T^{\alpha\gamma}E_{\gamma,1+\gamma}(-\frac{\lambda_n}{\alpha^{\gamma}}T^{\alpha\gamma})\langle\rho_2,e_n\rangle}{E_{\gamma,1}(-\frac{\lambda_n}{\alpha^{\gamma}}T^{\alpha\gamma})-E_{\gamma,1}(-\frac{\lambda_n}{\alpha^{\gamma}}t_0^{\alpha\gamma})}e_n(x). \tag{3.6}$$

Form (3.5) and (3.6), we know that $f^{\delta},\rho_1^{\delta}$ with the measured data as follows:

$$f^{\delta}(x)=\sum_{n=1}^{\infty}\lambda_n\frac{E_{\gamma,1}(-\frac{\lambda_n}{\alpha^{\gamma}}t_0^{\alpha\gamma})\langle\rho_3^{\delta},e_n\rangle-E_{\gamma,1}(-\frac{\lambda_n}{\alpha^{\gamma}}T^{\alpha\gamma})\langle\rho_2^{\delta},e_n\rangle}{E_{\gamma,1}(-\frac{\lambda_n}{\alpha^{\gamma}}t_0^{\alpha\gamma})-E_{\gamma,1}(-\frac{\lambda_n}{\alpha^{\gamma}}T^{\alpha\gamma})}e_n(x), \tag{3.7}$$

and

$$\rho_1^{\delta}(x)=\sum_{n=1}^{\infty}\frac{\lambda_n}{\alpha^{\gamma}}\frac{t_0^{\alpha\gamma}E_{\gamma,1+\gamma}(-\frac{\lambda_n}{\alpha^{\gamma}}t_0^{\alpha\gamma})\langle\rho_3^{\delta},e_n\rangle-T^{\alpha\gamma}E_{\gamma,1+\gamma}(-\frac{\lambda_n}{\alpha^{\gamma}}T^{\alpha\gamma})\langle\rho_2^{\delta},e_n\rangle}{E_{\gamma,1}(-\frac{\lambda_n}{\alpha^{\gamma}}T^{\alpha\gamma})-E_{\gamma,1}(-\frac{\lambda_n}{\alpha^{\gamma}}t_0^{\alpha\gamma})}e_n(x). \tag{3.8}$$

By combining the above (3.5) and (3.6), We define them as linear operator $\mathcal{L}:(f,\rho_1)\mapsto(\rho_2,\rho_3)$. From [18], the operator $A:(L_2(\mathcal{D}))^2\mapsto(L_2(\mathcal{D}))^2$ is compact, we obtain that this problem is ill-posed. Next, we define the priori bound condition of $\rho_1(x)$ and $f(x)$ as follows

$$\max\Big\{\|f\|_{\mathcal{H}^s(\mathcal{D})},\|\rho_1\|_{\mathcal{H}^s(\mathcal{D})}\Big\}\le\mathcal{C}, \tag{3.9}$$

where $\mathcal{C}$ and p are both positive constants.

Theorem 3.1. (see [2]) *If f and ρ_1 satisfy a priori bound condition 1.2, then we have*

$$\|f\|_{L^2(\mathcal{D})} \leq \left(\frac{\sqrt{2}\mathcal{Z}_2}{\mathcal{Z}_1 - \mathcal{Z}_4}\right)^{\frac{s}{s+2}} \left(\|\rho_2\|^2_{L^2(\mathcal{D})} + \|\rho_3\|^2_{L^2(\mathcal{D})}\right)^{\frac{s}{2(s+2)}} \mathcal{C}^{\frac{2}{s+2}}, \quad s > 0,$$

and

$$\|\rho_1\|_{L^2(\mathcal{D})} \leq \left(\frac{\sqrt{2}}{\mathcal{Z}_1 - \mathcal{Z}_4}\right)^{\frac{s}{s+2}} \left(\|\rho_2\|^2_{L^2(\mathcal{D})} + \|\rho_3\|^2_{L^2(\mathcal{D})}\right)^{\frac{s}{2(s+2)}} \mathcal{C}^{\frac{2}{s+2}}, \quad s > 0.$$

3.1 Modified fractional Landweber iterative method

We know that if the fractional Landweber iterative regularization method is used to solve problem (1.1), then the regularization solution can be expressed by [3]

$$\Xi^{0,\delta}(x) = 0,$$
$$\Xi^{m,\delta}(x) = \left(\mathcal{I} - s(\mathscr{K}^*\mathscr{K})^{\frac{r+1}{2}}\right)\Xi^{m-1,\delta}(x) + s\ (\mathscr{K}^*\mathscr{K})^{\frac{r-1}{2}}\ \mathscr{K}^*\Psi_i^\delta(x), \quad m = 1, 2, 3, \cdots, \ i = 1, 2.$$

where $0 < r \leq 1$, $\mathcal{I}$ is a unit operator, m denotes the iterative step number, and the reciprocal of m is the regularization parameter. The coefficient s is called the relaxation factor and satisfies $0 < s < \frac{1}{\|\mathscr{K}\|^{r+1}}$, Ψ_1^δ and Ψ_2^δ denote the noisy data,

$$\Psi_1^\delta(x) = \sum_{n=1}^{\infty} \frac{\lambda_n}{\alpha^\gamma}\left(t_0^{\alpha\gamma} E_{\gamma,1+\gamma}\left(-\frac{\lambda_n}{\alpha^\gamma} t_0^{\alpha\gamma}\right)\langle \rho_3^\delta, e_n\rangle - T^{\alpha\gamma} E_{\gamma,1+\gamma}\left(-\frac{\lambda_n}{\alpha^\gamma} T^{\alpha\gamma}\right)\langle \rho_2^\delta, e_n\rangle\right) e_n(x), \quad (3.10)$$

and

$$\Psi_2^\delta(x) = \sum_{n=1}^{\infty} \left(E_{\gamma,1}\left(-\frac{\lambda_n}{\alpha^\gamma} t_0^{\alpha\gamma}\right)\langle \rho_3^\delta, e_n\rangle - E_{\gamma,1}\left(-\frac{\lambda_n}{\alpha^\gamma} T^{\alpha\gamma}\right)\langle \rho_2^\delta, e_n\rangle\right) e_n(x). \quad (3.11)$$

Define the operator $\Upsilon_m : L^2(\mathcal{D}) \to L^2(\mathcal{D})$ as

$$\Upsilon_m = s\sum_{n=0}^{m-1} \left(I - s(\mathscr{K}^*\mathscr{K})^{\frac{r+1}{2}}\right)^n (\mathscr{K}^*\mathscr{K})^{\frac{r-1}{2}} \mathscr{K}^*, \quad 0 < r \leq 1, \ m = 1, 2, 3, \cdots,$$

we have the regularization solution as follows:

$$f^{m,\delta}(x) = \Upsilon_m \Psi_1^\delta(x) = s\sum_{n=0}^{m-1} \left(I - s(\mathscr{K}^*\mathscr{K})^{\frac{r+1}{2}}\right)^n (\mathscr{K}^*\mathscr{K})^{\frac{r-1}{2}} \mathscr{K}^*\Psi_1^\delta(x).$$

Defined the following iterative regularization solution

$$\Xi^{m,\delta}(x) = \sum_{n=1}^{\infty} \frac{1 - \left(1 - \bar{s}|\lambda_n^{-1}|^{r+1}\right)^m}{\mathcal{Q}_n(\alpha, \gamma, t_0, T)} \langle \Psi_1^\delta, e_n\rangle e_n(x),$$

where $0 < \bar{s} < \frac{1}{|\lambda_1^{-1}|^{r+1}}$, in which $\mathcal{Q}_n(\alpha, \gamma, t_0, T) = E_{\gamma,1}(-\frac{\lambda_n}{\alpha^\gamma} t_0^{\alpha\gamma}) - E_{\gamma,1}(-\frac{\lambda_n}{\alpha^\gamma} T^{\alpha\gamma})$. Now, by using the Fourier truncation technique, we make a modification on the regularization solution (3.5) to obtain our regularization solution

$$f_{\mathcal{M}}^{m,\delta}(x) = \mathcal{R}_m \Psi_2^\delta(x) = \sum_{n=1}^{\mathcal{M}} \frac{1 - \left(1 - \bar{s}|\lambda_n^{-1}|^{r+1}\right)^m}{\mathcal{Q}_n(\alpha, \gamma, t_0, T)} \langle \Psi_2^\delta, e_n\rangle e_n(x),$$

and in a similar way, we can also obtain the following.

$$\rho_{1,\mathcal{M}}^{m,\delta}(x) = \mathcal{R}_m \Psi_1^\delta(x) = \sum_{n=1}^{\mathcal{M}} \frac{1-\left(1-\bar{s}|\lambda_n^{-1}|^{r+1}\right)^m}{\mathcal{Q}_n(\alpha,\gamma,t_0,T)} \langle \Psi_1^\delta, e_n \rangle e_n(x),$$

where $\mathcal{M}$ represents the number of truncated items. According to the assumption on $\{\lambda_n\}_{n=1}^{+\infty}$ in Definition 2.1, we know that $\mathcal{M}$ and $\lambda_{\mathcal{M}}$ are one-to-one correspondences. $\lambda_{\mathcal{M}}$ will be given in Subsection 3.2.

3.2 An a- priori parameter choice rule

Lemma 3.2. *[3] Let $0 < r \le 1$. Then, we have the inequality*

$$\frac{1-\left(1-\bar{s}|\lambda_n^{-1}|^{r+1}\right)^m}{|\lambda_n^{-1}|} \le m^{\frac{1}{r+1}} \bar{s}^{\frac{1}{r+1}}.$$

Lemma 3.3. *Let $\Psi_1^\delta(x)$ be given by (3.10), and $\Psi_1(x)$ defined similarly but without δ, then we have the estimate*

$$\left\|\Psi_1^\delta - \Psi_1\right\|_{L^2(\mathcal{D})} \le 2\alpha^{-\gamma}\delta.$$

Proof. From (3.10), and Lemma 3.2, it can be derived that

$$\begin{aligned}
\left\|\Psi_1^\delta - \Psi_1\right\|_{L^2(\mathcal{D})} &= \left\| \sum_{n=1}^{\infty} \frac{\lambda_n}{\alpha^\gamma} \Big(t_0^{\alpha\gamma} E_{\gamma,1+\gamma}\big(-\frac{\lambda_n}{\alpha^\gamma} t_0^{\alpha\gamma}\big) \langle \rho_3^\delta - \rho_3, e_n \rangle \Big) e_n(x) \right\|_{L^2(\mathcal{D})} \\
&\quad + \left\| \sum_{n=1}^{\infty} \frac{\lambda_n}{\alpha^\gamma} \Big(T^{\alpha\gamma} E_{\gamma,1+\gamma}(-\frac{\lambda_n}{\alpha^\gamma} T^{\alpha\gamma}) \Big) \langle \rho_2^\delta - \rho_2, e_n \rangle e_n(x) \right\|_{L^2(\mathcal{D})} \\
&= \left\| \frac{1}{\alpha^\gamma} \sum_{n=1}^{\infty} \Big((\rho_3^\delta - \rho_3) + (\rho_2^\delta - \rho_2) \Big) e_n(x) \right\|_{L^2(\mathcal{D})} \le 2\alpha^{-\gamma}\delta.
\end{aligned}$$

□

Lemma 3.4. *Let $\Psi_2^\delta(x)$ be given by (3.11), and $\Psi_2(x)$ defined similarly but without δ, then we have the estimate*

$$\left\|\Psi_2^\delta - \Psi_2\right\|_{L^2(\mathcal{D})} \le 2\lambda_1^{-1}(\mathcal{Z}_2 + \mathcal{Z}_4)\delta.$$

Proof. From (3.10), and Lemma 3.2, it can be derived that

$$\begin{aligned}
\left\|\Psi_2^\delta - \Psi_2\right\|_{L^2(\mathcal{D})} &= \left\| \sum_{n=1}^{\infty} \Big(E_{\gamma,1}\big(-\frac{\lambda_n}{\alpha^\gamma} t_0^{\alpha\gamma}\big) \langle \rho_3^\delta - \rho_3, e_n \rangle \Big) e_n(x) \right\|_{L^2(\mathcal{D})} \\
&\quad + \left\| \sum_{n=1}^{\infty} \Big(E_{\gamma,}(-\frac{\lambda_n}{\alpha^\gamma} T^{\alpha\gamma}) \Big) \langle \rho_2^\delta - \rho_2, e_n \rangle e_n(x) \right\|_{L^2(\mathcal{D})} \\
&\le 2\lambda_1^{-1}(\mathcal{Z}_2 + \mathcal{Z}_4)\delta.
\end{aligned}$$

□

Lemma 3.5. ([3]) *For $0 < r \le 1$, $m > 1$, we can obtain that*

$$\begin{aligned}
\left(1-\bar{s}|\lambda_n^{-1}|^{r+1}\right)^{m+1} \lambda_1^{-(p+1)} &\le \bar{s}^{-\frac{p+1}{r+1}} (m+1)^{-\frac{p+1}{r+1}} \left(\frac{p+1}{r+1}\right)^{\frac{p+1}{r+1}}, \\
\left(1-\bar{s}|\lambda_n^{-1}|^{r+1}\right)^{m-1} \lambda_n^{-(p+2)} &\le \bar{s}^{-\frac{p+2}{r+1}} m^{-\frac{p+2}{r+1}} (p+2)^{\frac{p+2}{r+1}}.
\end{aligned}$$

Theorem 3.6. *Let $\rho_1(x), \rho_1^\delta(x) \in L^2(\mathcal{D})$ be given in (3.6) and (3.8), assume that the a priori condition (3.9) is satisfied, and the noise data satisfy (1.2). For $\delta, E, p > 0$, $0 < r \le 1$, choosing the iteration step number*

$$m = \Big[\big(\frac{\mathcal{C}}{\delta}\big)^{\frac{r+1}{p+2}} \Big],$$

we have the convergence estimate

$$\|\rho_{1,\mathcal{M}}^{m,\delta} - \rho_1\|_{L^2(\mathcal{D})} \text{ is of order } \delta^{\frac{p+1}{p+2}},$$

Proof. By the triangle inequality, we have

$$\|\rho_{1,\mathcal{M}}^{m,\delta} - \rho\|_{L^2(\mathcal{D})} \le \|\rho_{1,\mathcal{M}}^{m,\delta} - \rho_{1,\mathcal{M}}^{m}\|_{L^2(\mathcal{D})} + \|\rho_{1,\mathcal{M}}^{m} - \rho\|_{L^2(\mathcal{D})}.$$

Note that

$$\begin{aligned}
&\Big\|\rho_{1,\mathcal{M}}^{m,\delta} - \rho_{1,\mathcal{M}}^{m}\Big\|^2_{L^2(\mathcal{D})} \\
&\quad = \Big\| \sum_{n=1}^{\infty} \frac{1-(1-\bar{s}|\lambda_n^{-1}|^{r+1})^m}{\mathcal{Q}_n(\alpha,\gamma,t_0,T)} \langle \Psi_1^\delta, e_n\rangle e_n(x) - \sum_{n=1}^{\infty} \frac{1-(1-\bar{a}|\lambda_k^{-1}|^{r+1})^m}{K_{k,s}(\alpha,T,\varphi)} \langle \Psi_1, e_n\rangle e_n(x) \Big\|^2_{L^2(\mathcal{D})} \\
&\quad \le \sum_{n=1}^{\infty} \Big(1-\Big(1-\bar{s}|\lambda_n^{-1}|^{r+1}\Big)^m\Big)^2 \Big(\frac{\langle \Psi_1^\delta, e_n\rangle}{\mathcal{Q}_n(\alpha,\gamma,t_0,T)} - \frac{\langle \Psi_1, e_n\rangle}{\mathcal{Q}_n(\alpha,\gamma,t_0,T)} \Big)^2 \\
&\quad \le \sum_{n=1}^{\infty} \Big(1-\Big(1-\bar{s}|\lambda_n^{-1}|^{r+1}\Big)^m\Big)^2 \Big(\frac{\langle \Psi_1^\delta - \Psi_1, e_n\rangle}{\mathcal{Q}_n(\alpha,\gamma,t_0,T)} \Big)^2 \\
&\quad = \sum_{n=1}^{\infty} \Big(1-\Big(1-\bar{s}|\lambda_n^{-1}|^{r+1}\Big)^m\Big)^2 (\lambda_n^{-1})^{-2} (\lambda_n^{-1})^2 \left(\frac{\langle \Psi_1^\delta - \Psi_1, e_n\rangle}{\mathcal{Q}_n(\alpha,\gamma,t_0,T)} \right)^2 \\
&\quad \le m^{\frac{2}{r+1}} s^{\frac{2}{r+1}} 4\alpha^{-2\gamma}\delta^2.
\end{aligned}$$

This implies that

$$\|\rho_{1,\mathcal{M}}^{m,\delta} - \rho_{1,\mathcal{M}}^{m}\|_{L^2(\mathcal{D})} \le m^{\frac{1}{r+1}} s^{\frac{1}{r+1}} 2\alpha^{-\gamma}\delta;$$

$$\begin{aligned}
&\|\rho_{1,\mathcal{M}}^{m} - \rho_1\|^2_{L^2(\mathcal{D})} \\
&\quad = \left\| \sum_{n=1}^{\infty} \frac{1}{\mathcal{Q}_n(\alpha,\gamma,t_0,T)} \langle \Psi_1, e_n\rangle e_n(x) - \sum_{n=1}^{\infty} \frac{1-\big(1-\bar{s}\,\big|\lambda_n^{-1}\big|^{r+1}\big)^m}{\mathcal{Q}_n(\alpha,\gamma,t_0,T)} \langle \Psi_1, e_n\rangle e_n(x) \right\|^2_{L^2(\mathcal{D})} \\
&\quad = \Big\| \sum_{n=1}^{\mathcal{M}} \big(1-\bar{s}\,\big|\lambda_n^{-1}\big|^{r+1}\big)^m \langle f, e_n\rangle e_n(x) + \sum_{n=\mathcal{M}+1}^{+\infty} \frac{1}{\mathcal{Q}_n(\alpha,\gamma,t_0,T)} \langle \Psi_1, e_n\rangle e_n(x) \Big\|^2_{L^2(\mathcal{D})} \\
&\quad \le 2\Big\| \sum_{n=1}^{\mathcal{M}} (1-\bar{s}|\lambda_n^{-1}|^{r+1})^m \lambda_n^{-(p+1)} \lambda_n^{p+1} \langle \rho_1, e_n\rangle e_n(x) \Big\|^2_{L^2(\mathcal{D})} \\
&\qquad + 2\Big\| \sum_{n=\mathcal{M}+1}^{+\infty} \frac{\lambda_n^{-(p+1)}\lambda_n^{p+1}}{\mathcal{Q}_n(\alpha,\gamma,t_0,T)} \langle \Psi_1, e_n\rangle e_n(x) \Big\|^2_{L^2(\mathcal{D})} \\
&\quad \le 2\Big| \sum_{n=1}^{\mathcal{M}} (1-\bar{s}|\lambda_n^{-1}|^{r+1})^m \lambda_n^{-(p+1)} \lambda_n^{(p+1)} \langle \rho_1, e_n\rangle \frac{\mathcal{Q}_n(\alpha,\gamma,t_0,T)}{\mathcal{Q}_n(\alpha,\gamma,t_0,T)} \Big|^2
\end{aligned}$$

$$+2\Big|\sum_{n=\mathcal{M}+1}^{+\infty}\frac{\lambda_n^{-(p+1)}\lambda_n^{(p+1)}}{\mathcal{Q}_n(\alpha,\gamma,t_0,T)}\langle\Psi_1,e_n\rangle\frac{\mathcal{Q}_n(\alpha,\gamma,t_0,T)}{\mathcal{Q}_n(\alpha,\gamma,t_0,T)}\Big|^2$$

$$=2\Big|\sum_{n=1}^{\mathcal{M}}(1-\bar{s}\left|\lambda_n^{-1}\right|^{r+1})^m(1-\bar{s}|\lambda_n^{-1}|^{r+1})^{-1}\lambda_n^{-(p+1)}\lambda_n^{(p+1)}\langle\rho_1,e_n\rangle\frac{\mathcal{Q}_n(\alpha,\gamma,t_0,T)}{\mathcal{Q}_n(\alpha,\gamma,t_0,T)}\Big|^2$$

$$+2\Big|\sum_{n=\mathcal{M}+1}^{+\infty}\frac{\lambda_n^{-(p+1)}\lambda_n^{(p+1)}}{\mathcal{Q}_n(\alpha,\gamma,t_0,T)}\langle\Psi_1,e_n\rangle\frac{\mathcal{Q}_n(\alpha,\gamma,t_0,T)}{\mathcal{Q}_n(\alpha,\gamma,t_0,T)}\Big|^2.$$

Next, we have

$$\|\rho_{1,\mathcal{M}}^m-\rho_1\|_{L^2(\mathcal{D})}^2=2\sum_{n=1}^{\mathcal{M}}\left|(1-\bar{s}|\lambda_n^{-1}|^{r+1})^m(1-\bar{s}\left|\lambda_n^{-1}\right|^{r+1})^{-1}\lambda_n^{-(p+1)}\lambda_n^{(p+1)}\frac{\langle\Psi_1,e_n\rangle}{\mathcal{Q}_n(\alpha,\gamma,t_0,T)}\right|^2$$

$$+2\sum_{n=\mathcal{M}+1}^{+\infty}\left|\frac{\lambda_n^{-(p+1)}\lambda_n^{(p+1)}}{\mathcal{Q}_n(\alpha,\gamma,t_0,T)}\langle\Psi_1,e_n\rangle\right|^2$$

$$\le 2(1-\bar{s}\left|\lambda_1^{-1}\right|^{r+1})\sum_{n=1}^{\mathcal{M}}\left|\left(1-\bar{s}\left|\lambda_n^{-1}\right|^{r+1}\right)^m\lambda_n^{-(p+1)}\lambda_n^{p+1}\frac{\langle\Psi_1,e_n\rangle}{\mathcal{Q}_n(\alpha,\gamma,t_0,T)}\right|^2$$

$$+2\sum_{n=\mathcal{M}+1}^{+\infty}\left|\frac{\lambda_n^{-(p+1)}\lambda_n^{(p+1)}}{\mathcal{Q}_n(\alpha,\gamma,t_0,T)}\langle\Psi_1,e_n\rangle\right|^2$$

$$\le 2(1-\bar{s}|\lambda_1^{-1}|^{r+1})(\bar{s})^{p+1}(m+1)^{\frac{p+1}{r+1}}(\frac{p+1}{r+1})^{p+1}\sum_{n=1}^{\mathcal{M}}\lambda_n^{2(p+1)}|\langle\rho_1,e_n\rangle|^2$$

$$+2\sum_{n=\mathcal{M}+1}^{+\infty}\lambda_n^{-2(p+1)}\lambda_n^{2(p+1)}|\langle\rho_1,e_n\rangle|^2$$

$$\le 2(1-\bar{s}|\lambda_1^{-1}|^{r+1})\,(\bar{s})^{p+1}\,(m+1)^{\frac{p+1}{r+1}}\left(\frac{p+1}{r+1}\right)^{p+1}\mathcal{C}+2\lambda_{\mathcal{M}}^{-2(p+1)}\sum_{k=N+1}^{+\infty}\lambda_n^{2(p+1)}|\langle\rho_1,e_n\rangle|^2$$

$$\le 2\left(1-\bar{s}\left|\lambda_1^{-1}\right|^{r+1}\right)^{-2}(\bar{s}^{-\frac{p+1}{r+1}}(m+1)^{-\frac{p+1}{r+1}}\left(\frac{p+1}{r+1}\right)^{\frac{p+1}{r+1}}\mathcal{C})^2+2\lambda_{\mathcal{M}}^{-2(p+1)}\mathcal{C}^2$$

$$\le 2(1-\bar{s}\left|\lambda_1^{-1}\right|^{r+1})^{-2}\left(\bar{s}^{-\frac{p+1}{r+1}}(m+1)^{-\frac{p+1}{r+1}}(\frac{p+1}{r+1})^{\frac{p+1}{r+1}}\mathcal{C}\right)^2+2\mathcal{C}^2((\frac{\mathcal{C}}{\delta})^{\frac{1}{(p+2)}})^{-2(p+1)}$$

$$\le 2(1-\bar{s}|\lambda_1^{-1}|^{r+1})^{-2}\left(\bar{s}^{-\frac{p+1}{r+1}}\delta^{\frac{p+1}{r+1}}\mathcal{C}^{\frac{1}{p+2}}(\frac{p+1}{r+1})^{\frac{p+1}{r+1}}\right)^2+2(\mathcal{C}^{\frac{1}{p+2}}\delta^{\frac{p+1}{p+2}})^2.$$

Thus,

$$\|\rho_{1,\mathcal{M}}^m-\rho_1\|_{L^2(\mathcal{D})}\le\left(2(1-\bar{s}\left|\lambda_1^{-1}\right|^{r+1})^{-2}\bar{s}^{-\frac{2(p+1)}{r+1}}(\frac{p+1}{r+1})^{\frac{2(p+1)}{r+1}}+2\right)^{\frac{1}{2}}\mathcal{C}^{\frac{1}{p+2}}\delta^{\frac{p+1}{p+2}}.$$

Finally, we can obtain the a priori convergence estimate as follows:

$$\|\rho_{1,\mathcal{M}}^{m,\delta}-\rho_1\|_{L^2(\mathcal{D})}\le\bar{s}^{\frac{1}{r+1}}(2\alpha^{-\gamma})^{\frac{1}{2}}\mathcal{C}^{\frac{1}{p+2}}\delta^{\frac{p+1}{p+2}}+\left(\frac{2\bar{s}^{-\frac{2(p+1)}{r+1}}}{(1-\bar{s}|\lambda_1^{-1}|^{r+1})^2}\left(\frac{p+1}{r+1}\right)^{\frac{2(p+1)}{r+1}}+2\right)^{\frac{1}{2}}\mathcal{C}^{\frac{1}{p+2}}\delta^{\frac{p+1}{p+2}}.$$

□

Theorem 3.7. *Let $f(x), f^\delta(x)\in L^2(\mathcal{D})$ be given in (3.5) and (3.7), assume that the a priori condition (3.9) is satisfied, and the noise data satisfy (1.2). For $\delta,\mathcal{C},p>0$, $0<r\le 1$, choosing the iteration step*

number

$$m = \Big[\big(\frac{\mathcal{C}}{\delta}\big)^{\frac{r+1}{p+2}}\Big],$$

we have

$$\begin{aligned}\|f^{m,\delta}_{\mathcal{M}} - f\|_{L^2(\mathcal{D})} &\leq \bar{s}^{\frac{1}{r+1}} \big(2\lambda^{-1}(\mathcal{Z}_2 + \mathcal{Z}_4)\big)^{\frac{1}{2}} \mathcal{C}^{\frac{1}{p+2}} \delta^{\frac{p+1}{p+2}} \\ &+ \Bigg(\frac{2\bar{s}^{-\frac{2(p+1)}{r+1}}}{\big(1-\bar{s}|\lambda_1^{-1}|^{r+1}\big)^2}\Big(\frac{p+1}{r+1}\Big)^{\frac{2(p+1)}{r+1}} + 2\Bigg)^{\frac{1}{2}} \mathcal{C}^{\frac{1}{p+2}} \delta^{\frac{p+1}{p+2}}.\end{aligned}$$

Proof. By the triangle inequality, we have

$$\|f^{m,\delta}_{\mathcal{M}} - f\|_{L^2(\mathcal{D})} \leq \|f^{m,\delta}_{\mathcal{M}} - f^m_{\mathcal{M}}\|_{L^2(\mathcal{D})} + \|f^m_{\mathcal{M}} - f\|_{L^2(\mathcal{D})}.$$

Note that,

$$\begin{aligned}&\|f^{m,\delta}_{\mathcal{M}} - f^m_{\mathcal{M}}\|^2_{L^2(\mathcal{D})} \\ &= \Big\|\sum_{n=1}^{\infty} \frac{1-(1-\bar{s}|\lambda_n^{-1}|^{r+1})^m}{\mathcal{Q}_n(\alpha,\gamma,t_0,T)}\langle \Psi_2^\delta, e_n\rangle e_n(x) - \sum_{n=1}^{\infty} \frac{1-(1-\bar{a}|\lambda_k^{-1}|^{r+1})^m}{K_{k,s}(\alpha,T,\varphi)}\langle \Psi_2, e_n\rangle e_n(x)\Big\|^2_{L^2(\mathcal{D})} \\ &\leq \sum_{n=1}^{\infty}\Big(1-\Big(1-\bar{s}|\lambda_n^{-1}|^{r+1}\Big)^m\Big)^2\Big(\frac{\langle \Psi_2^\delta, e_n\rangle}{\mathcal{Q}_n(\alpha,\gamma,t_0,T)} - \frac{\langle \Psi_2, e_n\rangle}{\mathcal{Q}_n(\alpha,\gamma,t_0,T)}\Big)^2 \\ &\leq \sum_{n=1}^{\infty}\Big(1-\Big(1-\bar{s}|\lambda_n^{-1}|^{r+1}\Big)^m\Big)^2\Big(\frac{\langle \Psi_2^\delta - \Psi_2, e_n\rangle}{\mathcal{Q}_n(\alpha,\gamma,t_0,T)}\Big)^2 \\ &= \sum_{n=1}^{\infty}\Big(1-\Big(1-\bar{s}|\lambda_n^{-1}|^{r+1}\Big)^m\Big)^2(\lambda_n^{-1})^{-2}(\lambda_n^{-1})^2\Big(\frac{\langle \Psi_2^\delta - \Psi_2, e_n\rangle}{\mathcal{Q}_n(\alpha,\gamma,t_0,T)}\Big)^2 \\ &\leq m^{\frac{2}{r+1}} s^{\frac{2}{r+1}} 4\lambda_1^{-2}(\mathcal{Z}_2+\mathcal{Z}_4)^2\delta^2.\end{aligned}$$

This implies that

$$\|f^{m,\delta}_{\mathcal{M}} - f^m_{\mathcal{M}}\|_{L^2(\mathcal{D})} \leq m^{\frac{1}{r+1}} s^{\frac{1}{r+1}} 2\lambda_1^{-1}(\mathcal{Z}_2+\mathcal{Z}_4)\delta.$$

By proving similarly, we also obtained

$$\begin{aligned}\|f^{m,\delta}_{\mathcal{M}} - f\|_{L^2(\mathcal{D})} &\leq \bar{s}^{\frac{1}{r+1}} \big(2\lambda^{-1}(\mathcal{Z}_2 + \mathcal{Z}_4)\big)^{\frac{1}{2}} \mathcal{C}^{\frac{1}{p+2}} \delta^{\frac{p+1}{p+2}} \\ &+ \Bigg(\frac{2\bar{s}^{-\frac{2(p+1)}{r+1}}}{\big(1-\bar{s}|\lambda_1^{-1}|^{r+1}\big)^2}\Big(\frac{p+1}{r+1}\Big)^{\frac{2(p+1)}{r+1}} + 2\Bigg)^{\frac{1}{2}} \mathcal{C}^{\frac{1}{p+2}} \delta^{\frac{p+1}{p+2}}.\end{aligned}$$

The proof is complete. □

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