EJAM
www.ejamjournal.com

Electronic Journal of Applied Mathematics

Research article

Mersin University Publishing
DOI: ejam.202533113
2025, Vol. 3, No. 3, 64–82

Fishery model influenced by global warming with optimal harvesting policy

Josué Carlos Israël OULAI[1*], Hypolithe OKOU[1] and Albin Tetchi N'GUESSAN[1]

1. Research and Formation Unit of Mathematics and Computer Science, Félix Houphouët Boigny University of Abidjan Cocody, Abidjan, Ivory Coast

*Corresponding author

Abstract

In this article, we proposed a modified version of the Gordon-Schaefer model in which harvesting is non-linear and the mortality rate of the fish stock is a function of temperature. We showed that every solution of the system is globally bounded and that there is a single interior equilibrium point that is locally, asymptotically, and globally stable under certain conditions. We then determined the optimal levels of production and profit when the stock evolution is kept constant (MSY and MEY, respectively). Furthermore, using Pontryagin's maximum principle, we characterized the optimal harvesting policy that maximizes net present value. Finally, we performed numerical simulations to validate our theoretical results.

Keywords: equilibrium points, stability, profit, fishing effort, MSY, MEY, temperature
MSC (2020): 34B05, 34B08, 34D20, 34D23, 93D05, 49J15, 19K15
Article history: Received 10 Apr 2025; Accepted 15 Sep 2025; Online 25 Sep 2025

1 Introduction

The foundations of fisheries theory were developed by Gordon (1954) and Schaefer (1957). Their model (the first in bioeconomics) links biological and economic factors inspired by the Lotka-Volterra predator-prey model and is presented as follows: [2, 7]

$$\begin{cases} \dfrac{dB(S)}{dS} =a_1 B(S)(1-\dfrac{B(S)}{K_0}) - qE(S)B(S), \\ \dfrac{dE(S)}{dS} =\alpha[wqE(S)B(S) - c_0 E(S)], \end{cases} \tag{1}$$

where the prey is the fish stock denoted by B and the predator is the fishing effort E, i.e., all the material and human resources used to carry out fishing activities, time is denoted by S, and

Contact: Josué Carlos Israël OULAI josueoulai8@gmail.com; Hypolithe OKOU okouakpetihi@hotmail.com; Albin Tetchi N'GUESSAN albintetchi@gmail.com

$qE(S)B(S)$ is the harvest or production. As for the model parameters, we have a_1 the natural growth rate of stock B, K_0 the carrying capacity of the aquatic environment, q the coefficient of fishing intensity, w is the (constant) unit price at landing, c_0 is the (constant) unit cost of fishing effort, and α is the rigidity parameter measuring the reaction force of fishing effort to the perceived rent [7]. This model can be used to calculate the maximum sustainable yield (MSY) [5] and the maximum economic yield (MEY) [4] when stock remains constant. These thresholds enable fisheries management policies to reconcile the conservation of fish stocks with the economic profitability of the activity. Furthermore, it (the model) highlights that in the absence of fishing effort regulation, the fishery will tend to expand until the marginal profit is dissipated, which is the situation of open access (OA). Despite these important results, aquatic resource management faces significant challenges, including global warming [9], [13] and overexploitation [1]. In this article, we replace the capture qEB in the previous model with a non-linear Holling 2 function, $\dfrac{m_1 B(S)E(S)}{B(S)+K_1}$, to account for the saturation of the harvest due to the complexity of the seafloor. Here, m_1 is the maximum capture intensity coefficient, and K_1 is the maximum (optimal) capture value. The model also considers the impact of climate change on the fishery stock. Thus, we assume that stock's mortality can be natural or temperature-related. This allows us to introduce the term $-d_0 e^{-\frac{A}{b}(\frac{1}{T}-\frac{1}{T_{ref}})} B(S)$, where T is the temperature of the aquatic environment, T_{ref} is the maximum reference temperature of the exploited resource, d_0 is the mortality rate of the species at temperature T_{ref}, A is the activation energy, and $b = 8.314\ J.mol^{-1}.K^{-1}$ is the Boltzmann constant [3]. We obtain an extension of the Gordon-Schaefer model, which is defined as follows:

$$\begin{cases} \dfrac{dB(S)}{dS} = a_1 B(S)(1-\dfrac{B(S)}{K_0}) - \dfrac{m_1 B(S)E(S)}{B(S)+K_1} - d_0 e^{-\frac{A}{b}(\frac{1}{T}-\frac{1}{T_{ref}})} B(S) \\ \dfrac{dE(S)}{dS} = \alpha \left[w \dfrac{m_1 B(S)}{B(S)+K_1} E(S) - c_0 E(S) \right] \\ B(0) > 0, E(0) \geq 0; B, E \in C^1(]0,+\infty[;\mathbb{R}_+). \end{cases} \tag{2}$$

To facilitate the qualitative study of the system (2), we make a change of variables that reduces the number of parameters.
Let us set $t = a_1 S$, $u(t) = \frac{B(S)}{K_0}$, $v(t) = \frac{E(S)}{K_0 a_1}$, $K = \frac{K_1}{K_0}$, $d = \frac{d_0}{a_1}$, $a = \frac{c_0}{a_1}$ and $p = \frac{w}{c_0}$.
thus, we have

$$\begin{cases} \dfrac{du(t)}{dt} = ((1-u(t)) - \dfrac{m_1 v(t)}{u(t)+K} - de^{-\frac{A}{b}(\frac{1}{T}-\frac{1}{T_{ref}})})u(t) \\ \dfrac{dv(t)}{dt} = \alpha a(\dfrac{p m_1 u(t)}{u(t)+K} - 1)v(t) \\ u(0) > 0,\ v(0) \geq 0. \end{cases} \tag{3}$$

2 Boundedness of solutions

In this section, we give the bounding conditions for the fish stock u and the fishing effort v in order to define their limits. Consider the following notation:

- $\mathbb{R}^2_+ = \{(u,v) \in \mathbb{R}^2 : u \geq 0,\ v \geq 0\}$,
- $int(\mathbb{R}^2_+) = \{(u,v) \in \mathbb{R}^2 : u > 0,\ v > 0\}$,

Lemma 2.1. *The interior $int(\mathbb{R}^2_+)$ and the boundary $\partial(\mathbb{R}^2_+)$ of the positive quadrant are invariant for the system (3).*

Proof. We want to show that if the initial state $(u(0), v(0))$ is in $int(\mathbb{R}^2_+)$ or $\partial\mathbb{R}^2_+$, then the trajectories $(u(t), v(t))$ remain in these sets for all $t > 0$. Let us set

$$\begin{cases} h_1(t) = (1 - u(t)) - \dfrac{m_1 v(t)}{u(t) + K} - de^{-\frac{A}{b}(\frac{1}{T} - \frac{1}{T_{ref}})}, \\ h_2(t) = \alpha a \left[\dfrac{p m_1 u(t)}{u(t) + K} - 1 \right]. \end{cases}$$

Let $(u(0), v(0)) \in \mathbb{R}^2_+$, $\tau > 0$.
such as u and v are continuous on $\mathbb{R}_+$ and more particuliary on the compact set $[0;\tau] \subset \mathbb{R}_+$, then h_1 and h_2 are also continuous on $[0;\tau]$, hence h_1 and h_2 are uniformly continuous on $[0;\tau]$ and reach their bounds there. Therefore, $\int_0^\tau h_1(t)dt$ and $\int_0^\tau h_2(t)dt$ exist and are bounded. Starting from the system (3), the possible solutions are expressed as follows:
$u(0)exp(\eta_1 \tau) \leq u(\tau) = u(0)exp(\int_0^\tau h_1(t)dt)$ and $v(0)exp(\zeta_1 \tau) \leq v(\tau) = v(0)exp(\int_0^\tau h_2(t)dt)$ where $\eta_1 = \min_{t\in[0;\tau]}(h_1(t))$ and $\zeta_1 = \min_{t\in[0;\tau]}(h_2(t))$.

- If $(u(0), v(0)) \in int(\mathbb{R}^2_+)$ $u(0) > 0$ and $v(0) > 0$, then $exp(\int_0^\tau h_1(t)dt)$ and $exp(\int_0^\tau h_2(t)dt)$ remain well defined and the solutions $u(\tau) > 0$ and $v(\tau) > 0$ for all $\tau > 0$. This guarantees that $(u(\tau), v(\tau)) \in int(\mathbb{R}^2_+)$.
- If $(u(0), v(0)) \in \partial(\mathbb{R}^2_+)$, i.e., $u(0) = 0$ or $v(0) = 0$ then $u(\tau) = 0$ and $v(\tau) = 0$ for all $\tau > 0$. This guarantees that $(u(\tau), v(\tau)) \in \partial(\mathbb{R}^2_+)$.

□

Theorem 2.2. *consider the set* $\Gamma = \left\{(u, v) \in \mathbb{R}^2_+ : 0 \leqslant u \leqslant 1;\ 0 \leqslant v \leqslant 1\right\}$. *All solution of the system* (3) *is globally bounded and remains in the domain* Γ.

Proof. Let us determine the upper limits of u and v. According to the first equation of the system (3), we have

$$\begin{cases} \dfrac{du(t)}{dt} \leqslant u(t)(1 - u(t)), \\ u(0) \geq 0. \end{cases}$$

According to the arguments of comparisons [8], we have $0 \leq u(t) \leq (1 + (\frac{1}{u(0)} - 1)e^{-t})^{-1}$, so we deduce that $\limsup_{t\to+\infty} u(t) \leqslant 1$. Therefore, $\forall t > 0$, we have $u(t) \leq 1$.
Starting from the second equation of the system (3), we have:

$$\begin{cases} \dfrac{dv(t)}{dt} = \alpha a(\dfrac{p m_1 u(t)}{u(t) + K} - 1)v(t), \\ v(0) \geq 0. \end{cases}$$

And for all $t > 0$, we have $u(t) \leq 1$, consequently

$$\frac{dv(t)}{dt} \leqslant \alpha a \left[\frac{p m_1}{K} - 1 \right] v(t).$$

Thus, according to the principle of comparison, we have

$$v(t) \leqslant \frac{v(0)}{v(0) + e^{-(\frac{p m_1}{K} - 1)t}} \leq 1,\ \forall t > 0.$$

□

3 Points of equilibrium

In this section and for the rest of our work, we define

$$\beta(T) = 1 - de^{-\frac{A}{b}\left(\frac{1}{T} - \frac{1}{T_{ref}}\right)}.$$

Consequently, $\beta(T^*) = 0$, with

$$T^* = \frac{A}{A + b\ln(d)\,T_{ref}}\,T_{ref} > 0 \quad \text{if} \quad d > e^{-\frac{A}{bT_{ref}}}.$$

The study of the sign of β on $(0, +\infty)$ showed that β is decreasing and that

$$1 - de^{-\frac{A}{bT_{ref}}} < \beta(T) < 1.$$

We have $\beta(T_{ref}) = 1 - d$. Thus, $T_{ref} < T^*$ is equivalent to $0 < d < 1$; in addition, $T_{ref} > T^*$ is equivalent to $d > 1$. For all $T \in (T_{ref}, T^*)$, we have

$$0 < \beta(T) < \beta(T_{ref}) < 1.$$

The reference temperature T_{ref} corresponds to the temperature at which the fish population develops normally. Thus, when discussing climate change, we consider that the temperature of the aquatic environment T is within the interval (T_{ref}, T^*) where $\beta(T) > 0$. However, beyond T^*, we have that $\beta(T) < 0$. In our work, the thermal zone defined by (T_{ref}, T^*) is equivalent to the thermal tolerance zone, as defined by Souchon et al. [11].

Theorem 3.1. *If*

$$pm_1 > 1 + \frac{K}{\beta(T)}, \tag{4}$$

is verified, then

1) *the system* (3) *has two trivial equilibrium points* $E_1 = (0,0)$ *and* $E_2 = (\beta(T), 0)$.
2) *The interior equilibrium of the model* (3) *is of the form* $E_3 = (u_e^*, v_e^*)$*, where* $u_e^* = \dfrac{K}{pm_1 - 1}$ *and* $v_e^* = (\beta(T) - u_e^*)\left(\frac{u_e^* + K}{m_1}\right)$.

Proof. To determine the equilibrium points (u_e, v_e), we solve the following equations $\dfrac{du}{dt} = 0$ and $\dfrac{dv}{dt} = 0$. We proceed as follows:

1) By solving $\dfrac{du}{dt} = 0$ for $v = 0$, and $\dfrac{dv}{dt} = 0$ for $u = 0$, we obtain the following equilibrium points $E_1 = (0,0)$ and $E_2 = (\beta(T), 0)$.

2) $u_e \neq 0$ and $v_e \neq 0$.
By solving the following system of equations

$$\begin{cases} \dfrac{du_e}{dt} = (1 - u_e) - \dfrac{m_1 v_e}{u_e + K} - de^{-\frac{A}{b}\left(\frac{1}{T} - \frac{1}{T_{ref}}\right)} = 0, \\ \dfrac{dv_e}{dt} = pm_1 \dfrac{u_e}{u_e + K} - 1 = 0, \end{cases}$$

we obtain

$$\begin{cases} v_e = (\beta(T) - u_e)\left(\dfrac{u_e + K}{m_1}\right), \\ u_e = \dfrac{K}{pm_1 - 1}. \end{cases}$$

such as $pm_1 > 1 + \dfrac{K}{\beta(T)}$ then $u_e^* = \dfrac{K}{pm_1 - 1}$ and $v_e^* = (\beta(T) - u_e^*)\left(\dfrac{u_e^* + K}{m_1}\right)$ are positive. Therefore, under the condition $pm_1 > 1 + \dfrac{K}{\beta(T)}$, the interior equilibrium of model (3) is of the form $E_3 = (u_e^*, v_e^*)$. □

Remark 3.2. The bionomic equilibrium E_3 refers to the state of equilibrium reached in a free-access fishery (with no restrictions on fishing effort), where total profit is zero. At this equilibrium, marginal profit (profit per unit of effort) and the growth rate of biomass are zero. Furthermore, zero profit implies zero rate of change in fishing effort, which is proportional to marginal profit.

4 Local stability of equilibrium points

In this section, we will define the conditions under which equilibrium points are stable or unstable.

Theorem 4.1. *If* (4) *and*

$$\beta(T) > K, \tag{5}$$

are verified, then

1) $E_1 = (0, 0)$ *is unstable.*
2) $E_2 = (\beta(T), 0)$ *is unstable.*
3) a) *If* $p < \dfrac{1}{m_1}\left(1 + \dfrac{2K}{\beta(T) - K}\right)$ *then* E_3 *is stable.*
   b) *If* $p > \dfrac{1}{m_1}\left(1 + \dfrac{2K}{\beta(T) - K}\right)$ *then* E_3 *is unstable.*

Proof. Let us determine the Jacobian matrix of the system (3) associated with each equilibrium point E_i, $i = 0, 1, 2$.

Consider the functional F defined by

$$F : \mathbb{R}^2 \longrightarrow \mathbb{R}^2, \qquad (u(t), v(t)) \longmapsto \begin{pmatrix} f_1(u(t), v(t)) \\ f_2(u(t), v(t)) \end{pmatrix},$$

where

$$f_1(u(t), v(t)) = \left[\beta(T) - u(t)) - \frac{m_1 v(t)}{u(t) + K}\right] u(t), \qquad f_2(u(t), v(t)) = \alpha a \left[\frac{pm_1 u(t)}{u(t) + K} - 1\right] v(t).$$

The Jacobian matrix of F is

$$J_F(u(t), v(t)) = \begin{pmatrix} J_{11} & J_{12} \\ J_{21} & J_{22} \end{pmatrix},$$

where

$$J_{11} = \frac{\partial f_1(u(t), v(t))}{\partial u(t)} = \beta(T) - 2u(t) - \frac{m_1 K v(t)}{(u(t) + K)^2},$$

$$J_{12} = \frac{\partial f_1(u(t),v(t))}{\partial v(t)} = -\frac{m_1 u(t)}{u(t)+K},$$
$$J_{21} = \frac{\partial f_2(u(t),v(t))}{\partial u(t)} = \alpha a \frac{p m_1 K v(t)}{(u(t)+K)^2},$$
$$J_{22} = \frac{\partial f_2(u(t),v(t))}{\partial v(t)} = \alpha a \left[\frac{p m_1 u(t)}{u(t)+K} - 1\right].$$

1) For $E_1 = (0,0)$, the associated Jacobian matrix is

$$J_F(E_1) = \begin{pmatrix} \beta(T) & 0 \\ 0 & -\alpha a \end{pmatrix}.$$

So the eigenvalues of $J_F(E_1)$ are $\lambda_1 = \beta(T)$ and $\lambda_2 = -\alpha a$, whose respective associated eigenspaces are $E_{\lambda_1} =< (1\,0)^t >$ and $E_{\lambda_2} =< (0\,1)^t >$. Such as $T \in]T_{ref}; T^*[$ then $\beta(T) > 0$, so $\lambda_1 > 0$ and $\lambda_2 < 0$. Hence, E_1 is a saddle point, therefore unstable with an unstable variety E_{λ_1} and a stable variety E_{λ_2}.

2) For $E_2 = (\beta(T), 0)$, the associated Jacobian matrix is

$$J_F(E_2) = \begin{pmatrix} -\beta(T) & -\dfrac{m_1\beta(T)}{\beta(T)+K} \\ 0 & \alpha a \left(\dfrac{p m_1 \beta(T)}{\beta(T)+K} - 1\right) \end{pmatrix}.$$

Then its eigenvalues are $\lambda_1' = -\beta(T)$ and $\lambda_2' = \alpha a \left(\dfrac{p m_1 \beta}{\beta(T)+K} - 1\right)$, and its respective eigenspaces are $E_{\lambda_1'} =< (1\ 0)^t >$ and $E_{\lambda_2'} =< (\theta_1\ \ \theta_2)^t >$, where

$$\theta_1 = \frac{m_1\beta(T)}{\beta(T)+K}, \qquad \theta_2 = -\beta(T) - \lambda_2'.$$

Such as $\beta(T) > K$ and $p m_1 > 1 + \dfrac{K}{\beta(T)}$ then $\lambda_2' > 0$ and $\lambda_1' < 0$. Hence, E_2 is a saddle point therefore unstable, with $E_{\lambda_1'}$ a stable variety and $E_{\lambda_2'}$ an unstable variety.

3) For

$$E_3 = \left(u_e^*, (\beta(T) - u_e^*)\left(\frac{u_e^*+K}{m_1}\right)\right) \quad \text{where} \quad u_e^* = \frac{K}{p m_1 - 1},$$

the associated Jacobian matrix is

$$J_F(E_3) = \begin{pmatrix} J_{11} & J_{12} \\ J_{21} & J_{22} \end{pmatrix},$$

with

$$J_{11} = \beta(T) - 2u_e^* - \frac{m_1 K v_e^*}{(u_e^*+K)^2},$$
$$J_{12} = -\frac{m_1 u_e^*}{u_e^*+K},$$
$$J_{21} = \alpha a \frac{p m_1 K v_e^*}{(u_e^*+K)^2},$$
$$J_{22} = 0.$$

The characteristic polynomial of $J_F(E_3)$ is

$$P(X) = X^2 - tr(J_F(E_3))X + det(J_F(E_3)),$$

with $tr(J_F(E_3)) = J_{11}$ and $det(J_F(E_3)) = -J_{21}J_{12}$. Let λ_1^* and λ_2^* be solutions of

$$P(X) = X^2 - tr(J_F(E_3))X + det(J_F(E_3)) = 0,$$

such as the eigenspace associated with λ_1^* is $E_{\lambda_1^*} =< (J_{12}\ (\lambda_1^* - J_{11}))^t >$ and the eigenspace associated with λ_2^* is

$$E_{\lambda_2^*} =< (J_{21}\ \ (\lambda_2^* - J_{22}))^t > .$$

As $det(J_F(E_3)) = -J_{21}J_{12} > 0$ then λ_1^* and λ_2^* have the same signs. Furthermore, the sign of $tr(J_F(E_3))$ depends on that of J_{11}. Such as

$$J_{11} = \frac{u_e^*}{(u_e^* + K)}\left[-2u_e^* + (\beta(T) - K)\right] \quad \text{and} \quad \frac{u_e^*}{(u_e^* + K)} > 0,$$

then the sign of J_{11} depends on

$$C_1(u_e^*) = -2u_e^* + (\beta(T) - K).$$

Furthermore, since $u_e^* = \dfrac{K}{pm_1 - 1}$ and that $\beta(T) > K$, then we can deduce that for

a) $p < \dfrac{1}{m_1}\left(1 + \dfrac{2K}{\beta(T) - K}\right)$, we have $C_1(u_e^*) < 0$ therefore $tr(J_F(E_3)) < 0$, i.e., λ_1^* and λ_2^* are negative. Therefore E_3 is stable.

b) $p > \dfrac{1}{m_1}\left(1 + \dfrac{2K}{\beta(T) - K}\right)$, we have $C_1(u_e^*) > 0$ therefore $tr(J_F(E_3)) > 0$, i.e., λ_1^* and λ_2^* are positive. Therefore E_3 is unstable.

□

Remark 4.2. Under the conditions of the previous theorem, there is a Hopf bifurcation that occurs at the value $p = p_c$, where

$$p_c = \frac{1}{m_1}(1 + \frac{2K}{\beta(T) - K}).$$

5 Global stability

In this section, we determine the conditions ensuring the global stability of the interior equilibrium by using a carefully chosen Lyapunov functional.

Theorem 5.1. *If*

$$\frac{m_1 v_e^*}{(u + K)(u_e^* + K)} < 1, \tag{6}$$

the E_3 is globally asymptotically stable.

Proof. The proof of the theorem is based on the Lyapunov function. Let $V_1 : \mathbb{R}^2 \to \mathbb{R}$ and $V_2 : \mathbb{R}^2 \to \mathbb{R}$ such as:

$$V_1(u, v) = [u - u_e^* - u_e^* \ln\left(\frac{u}{u_e^*}\right)] = \int_{u_e^*}^{u} \left(1 - \frac{u_e^*}{\eta}\right) d\eta,$$

and

$$V_2(u,v) = [v - v_e^* - v_e^* \ln\left(\frac{v}{v_e^*}\right)] = \int_{v_e^*}^{v} \left(1 - \frac{v_e^*}{\eta}\right) d\eta,$$

with u and v depending on t.

Let $V(u,v) = V_1(u,v) + V_2(u,v)$ and proceed to calculate $\frac{dV(u,v)}{dt}$. We have

$$\frac{dV}{dt} = \frac{dV_1}{dt} + \frac{dV_2}{dt} = \frac{dV_1}{du}\frac{du}{dt} + \frac{dV_2}{dv}\frac{dv}{dt}.$$

According to the system (3), we arrive at

$$\begin{cases} \frac{\dot{u}}{u} = \left(\beta(T) - u - \frac{m_1 v}{u+K}\right), \\ \frac{\dot{v}}{v} = \alpha a \left(\frac{p m_1 u}{u+K} - 1\right). \end{cases}$$

Furthermore, taking into account the interior equilibrium point $E_3 = (u_e^*, v_e^*)$, we obtain:

$$\beta(T) = u_e^* + \frac{m_1 v_e^*}{u_e^* + K} \qquad \text{and} \qquad 1 = \frac{p m_1 u_e^*}{u_e^* + K}.$$

Thus, we have

$$\begin{cases} \frac{\dot{u}}{u} = \left(u_e^* + \frac{m_1 v_e^*}{u_e^* + K} - u - \frac{m_1 v}{u+K}\right), \\ \frac{\dot{v}}{v} = \alpha a \left(\frac{p m_1 u}{u+K} - \frac{p m_1 u_e^*}{u_e^* + K}\right). \end{cases}$$

Then we have

$$\frac{dV}{dt} = (u - u_e^*)\left[-(u - u_e^*) + m_1 \frac{v_e^*(u - u_e^*) - u_e^*(v - v_e^*)}{(u+K)(u_e^*+K)}\right] + (v - v_e^*)\left[\frac{p m_1 K (u - u_e^*)}{(u+K)(u_e^*+K)}\right],$$

implies that

$$\frac{dV}{dt} = \left[-1 + \frac{m_1 v_e^*}{(u+K)(u_e^*+K)}\right](u - u_e^*)^2 + \left[\frac{p m_1 K - m_1 u_e^*}{(u+K)(u_e^*+K)}\right](v - v_e^*)(u - u_e^*).$$

Therefore, we have

$$\frac{dV}{dt} = (u - u_e^* \quad v - v_e^*) \begin{pmatrix} A_{11} & \frac{A_{21}}{2} \\ \frac{A_{21}}{2} & A_{22} \end{pmatrix} \begin{pmatrix} u - u_e^* \\ v - v_e^* \end{pmatrix},$$

with

$$A_{11} = -1 + \frac{m_1 v_e^*}{(u+K)(u_e^*+K)}, \qquad A_{22} = 0, \qquad A_{12} = A_{21} = \frac{p m_1 K - m_1 u_e^*}{(u+K)(u_e^*+K)}.$$

According to condition (6), we have $A_{11} < 0$, and such as $A_{22} = 0$, then $det(S) = -(\frac{A_{21}}{2})^2 \leq 0$. Therefore, we can deduce that the following matrix

$$\begin{pmatrix} A_{11} & \frac{A_{21}}{2} \\ \frac{A_{21}}{2} & A_{22} \end{pmatrix}$$

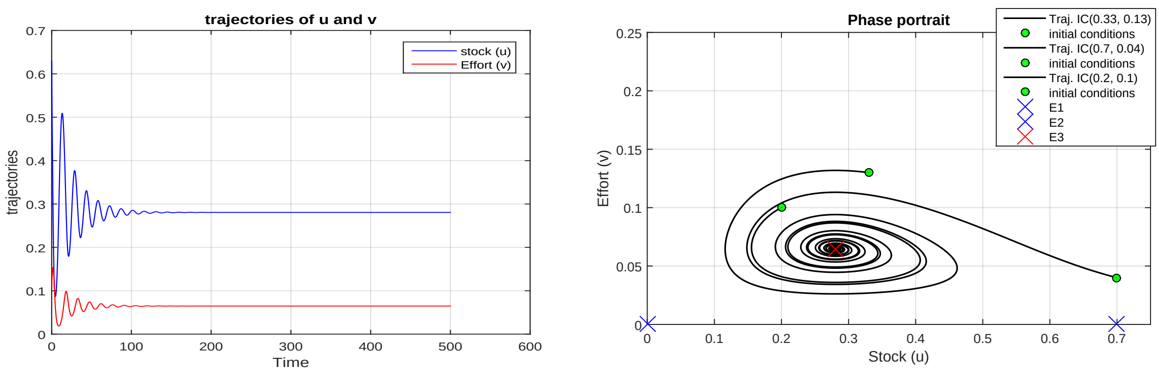
is a negative definite symmetric matrix. Hence E_3 is globally asymptotically stable. □

5.1 Simulations

Table 1: Data

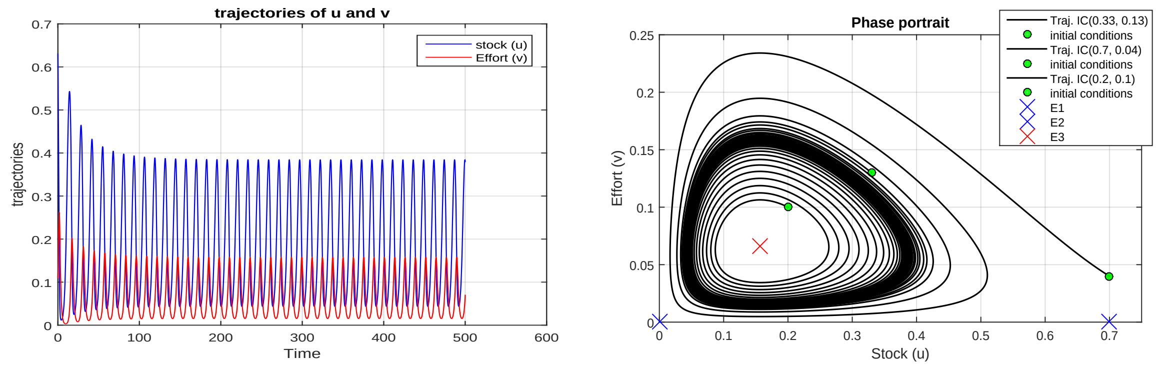
d	b	K	T	A	a	α	T_{ref}	m_1
0.3	8.314	0.3	301.9250	90	2.5	0.35	298.25	3.7625

In this subsection, we numerically analyze the impact of the unit price/unit cost ratio of fishing effort $(p = w/c_0)$ on model stability by analyzing the Hopf bifurcation. Thus, we construct the trajectories of stock u and fishing effort v on the one hand, and on the other hand, a phase portrait defining the curve of fishing effort evolution relative to stock before and after a critical threshold of the bifurcation parameter. All these curves are constructed by simulating using the data in Table 1. and Scilab codes for the fourth-order Runge-Kutta method (RK4) for a period $t_f = 500$ and a discretization step $h = 1/2L$, where $L = 11.8418$ is an estimate of the Lipschitz constant of the model under study. Table 1 describes a situation where the temperature of the marine environment is 301.9250 K (or 28.7750°C), and the maximum reference temperature favorable to the development of fishery resources is 298.25 K (or 25.10°C), an activation energy A equal to 90 kJ/mol, the maximum catch value is 30% of the carrying capacity of the environment ($K = 0.3$, or $K_1 = 0.3 * K_0$). Furthermore, at 25.10°C, the mortality rate of the resource is 30% of its growth rate ($d = 0.3$, or $d_0 = 0.3 * a_1$), the maximum value of the catch intensity is equal to 3.7625 tons/effort/year, and the unit cost per effort is 2.5 times the natural growth of the stock ($a = 2.5$ or $c_0 = 2.51 * a_1$). This gives us the figures below: (See Figure 1 and Figure 2.)



(a) Trajectories **(b)** Phase portrait

Figure 1: Trajectories and phase portrait of the predator-prey system for $p < p_c = 0.6646$, illustrating the stability of the interior equilibrium $E_3 = (0.281, 0.065)$



(a) Trajectories **(b)** Phase portrait

Figure 2: Trajectories and phase portrait of the predator-prey system for $p > p_c = 0.6646$, illustrating the unstability of the interior equilibrium $E_3 = (0.157, 0.066)$

5.1.1 Interpretation

According to Figures (1) and (2), we observe that for $p < 0.6646$, in other words, the unit price at landing is less than the product of 0.6646 and the unit cost of fishing effort ($w < 0.6646 * c_0$), then the fish stock and fishing effort converge towards their steady states, i.e., $E_3 = (0.281, 0.065)$ where stock B is 28.1% of the carrying capacity of the environment ($B = 0.281 * K_0$) and fishing effort E is 6.5% of $a_1 * K_0$. Now, if we keep the other parameters fixed and gradually increase the value of p, we obtain a critical value of p (w/c_0) equal to 0.6646, where a Hopf bifurcation occurs. Furthermore, for a unit landing price greater than 66.46% of the unit cost of effort ($p > 0.6646$), the positive equilibrium $(0.157 * K_0, 0.066 * a_1 * K_0)$ or $E_3 = (0.157, 0.066)$ is unstable and we have a limit cycle.

6 Static Optimization

In this subsection, we will determine the maximum sustainable yield (MSY) and maximum economic yield (MEY) when $\frac{du}{dt} = 0$. In MSY and MEY, we will determine the level of stock and fishing effort in order to deduce the optimal yield and optimal profit. These MSY and MEY thresholds allow fisheries management policies to set the total allowable catch (TAC) [12] over a given period, in order to avoid overexploitation of the specie and ensure the sustainability of fishing activity.

Theorem 6.1. *If condition (4) is satisfied, then*

1) *the maximum sustainable stock production u is*

$$H_{MSY} = \frac{\beta^2(T)}{4},$$

where stock $u_{MSY} = \frac{\beta(T)}{2}$ *and fishing effort* $v_{MSY} = (\beta(T) - u_{MSY})(\frac{u_{MSY} + K}{m_1})$.

2) *The maximum economic profit at stock equilibrium u is*

$$\pi_{MEY} = a[\frac{pm_1 u_{MEY}}{u_{MEY} + K} - 1]v_{MEY},$$

where the value of stock is $u_{MEY} = \frac{\beta(T)(pm_1 - 1) + K}{2(pm_1 - 1)}$*, and the fishing effort is* $v_{MEY} = (\beta(T) - u_{MEY})(\frac{u_{MEY} + K}{m_1})$.

Proof.

1) To calculate the maximum sustainable yield (MSY) of stock u, we need to figure out the max harvest possible while keeping the stock balanced, meaning $\frac{du}{dt} = 0$.

Given that $\frac{du}{dt} = 0$ implies that

$$\left[\beta(T) - u - \frac{m_1 v}{u+K}\right] u = 0,$$

implies that

$$\frac{m_1 v u}{u+K} = [\beta(T) - u]\, u.$$

Let

$$H(u) = \frac{m_1 v u}{u+K} = [\beta(T) - u]\, u$$

be the production. Then the problem is as follows

$$Find\ u_{MSY} \in [0,1]\ as\ is : \begin{cases} Max_{\{0 \le u \le 1\}} H(u), \\ s.c\ \dfrac{du}{dt} = 0. \end{cases}$$

The derivative of H is $\frac{dH(u)}{du} = \beta(T) - 2u(t)$ and it is zero for $u = \frac{\beta(T)}{2}$. Furthermore, $\frac{d^2 H(u)}{du^2} = -2 < 0$, so we can deduce that for $u_{MSY} = \frac{\beta(T)}{2}$ and

$$v_{MSY} = (\beta(T) - u_{MSY}) \left(\frac{u_{MSY} + K}{m_1}\right),$$

the maximum production is $H_{MSY} = \frac{\beta^2(T)}{4}$.

2) Determine the maximum economic yield at stock equilibrium u (MEY), i.e., maximize net profit while maintaining stock at equilibrium.

Since $\frac{du}{dt} = 0$ implies that

$$\left[\beta(T) - u - \frac{m_1 v}{u+K}\right] u = 0,$$

implies that

$$\frac{m_1 v u}{u+K} = [\beta(T) - u]\, u,$$

implies that

$$v = (\beta(T) - u) \left(\frac{u+K}{m_1}\right).$$

Let us set the net profit

$$\Pi(u,v) = a \left[\frac{p m_1 u}{u+K} - 1\right] v.$$

And by replacing v with its expression in Π, we obtain

$$\pi(u) = apu(\beta(T) - u) - a(\beta(T) - u) \left(\frac{u+K}{m_1}\right).$$

Then the problem is as follows:

$$Find\ u_{MEY} \in [0,1]\ as\ is: \begin{cases} Max_{\{0\leq u\leq 1\}}\pi(u), \\ s.c\ \dfrac{du}{dt} = 0. \end{cases}$$

The derivative of π is

$$\frac{d\pi(u)}{du} = \frac{a}{m_1}\Big[2(1-pm_1)u + \big(\beta(T)(pm_1-1)+K\big)\Big]$$

and it is zero for

$$u = \frac{\beta(T)(pm_1-1)+K}{2(pm_1-1)}.$$

Furthermore, $\dfrac{d^2\pi(u)}{du^2} = 2a(1-pm_1) < 0$ because $pm_1 > 1$, so we can deduce that for

$$u_{MEY} = \frac{\beta(T)(pm_1-1)+K}{2(pm_1-1)} \quad \text{and} \quad v_{MEY} = (\beta(T)-u_{MEY})\left(\frac{u_{MEY}+K}{m_1}\right),$$

the maximum profit is

$$\pi_{MEY} = a\left[\frac{pm_1u_{MEY}}{u_{MEY}+K} - 1\right]v_{MEY}.$$

□

6.1 Numerical result

Table 2: Data.

d	b	K	T	A	a	α	p	T_{ref}	m_1
0.3	8.314	0.3	301.9250	90	2.5	0.35	0.550	298.25	3.7625

The table 2 verifies the stability of the interior equilibrium and the instability of trivial equilibrium points. But here, it is used because it verifies condition (4) of theorem 6.1, and allows us to numerically calculate profit, production, fish stock, and fishing effort in MSY, MEY, and open access (OA). Indeed, in MSY we have production H_{MSY} equal to 0.1225, for a fish stock level u_{MSY} equal to 0.3499, i.e., $B_{MSY} = 0.3499 * K_0$ and a fishing effort v_{MSY} equal to 0.0604, i.e., $E_{MSY} = 0.0604 * a_1 * K_0$. Then, in MEY we have the profit π_{MEY} equal to 0.0312, for a fish stock level u_{MEY} equal to 0.4902, i.e. $B_{MEY} = 0.0312 * K_0$ and a fishing effort v_{MEY} equal to 0.0440, i.e. $E_{MEY} = 0.0440 * a_1 * K_0$. Finally, in a situation known as open access (OA), which results in the interior equilibrium $(0.2805 * K_0, 0.0647 * a_1 * K_0)$, the fisherman's profit is zero, i.e. π_{OA}= 0, for a stock level u_{OA} is 0.2805, i.e. $B_{OA} = 0.2805 * K_0$, and a fishing effort v_{OA} equal to 0.0647, i.e. $E_{OA} = 0.0647 * a_1 * K_0$. Based on the data in Table 2, we can also deduce that $B_{OA} < B_{MSY} < B_{MEY}$ and $E_{OA} > E_{MSY} > E_{MEY}$.

7 Optimal harvesting policy

The primary challenge in fisheries management is to find a balance between the quantity of fish caught, revenue, and costs associated with the activity so that the fishery is profitable but sustainable. To do this, we consider t_f to be the period over which the activity is carried out and U to be the set of admissible harvesting strategies defined by $\{m_1 : [0; t_f] \to [m_{1min}; m_{1max}],\ Lesbegue\ mesurable\}$. And our control problem is given by:

$$\begin{cases} \max\limits_{m_1 \in U} J(m_1) = \max\limits_{m_1 \in U} \int\limits_0^{t_f} g(t,u,v,m_1)\, dt \\ (s.c) \begin{cases} \dfrac{du(t)}{dt} = \left[1 - u(t) - \dfrac{m_1(t)v(t)}{u(t)+K} - de^{-\frac{A}{b}\left(\frac{1}{T} - \frac{1}{T_{ref}}\right)}\right] u(t), \\ \dfrac{dv(t)}{dt} = \alpha a \left[\dfrac{pm_1(t)u(t)}{u(t)+K} - 1\right] v(t), \\ u(0) > 0, \quad v(0) \geqslant 0, \quad u(t_f) > u_{MSY}, \quad v(t_f) < 1, \end{cases} \end{cases} \tag{7}$$

where

- $g(t,u,v,m_1) = a\left[\dfrac{pm_1(t)u(t)}{u(t)+K} - 1\right] v(t)e^{-\delta t}$,
- $\delta > 0$: constant representing the discount rate [10],
- J: the present value (in continuous time) of total net revenue.

Let us assume

$$F(t,u,v,m_1) = \begin{pmatrix} f_1(t,u,v,m_1) \\ f_2(t,u,v,m_1) \end{pmatrix} = \begin{pmatrix} \left(1 - u(t) - \dfrac{m_1(t)v(t)}{u(t)+K} - de^{-\frac{A}{b}\left(\frac{1}{T} - \frac{1}{T_{ref}}\right)}\right) u(t) \\ \alpha a \left(\dfrac{pm_1(t)u(t)}{u(t)+K} - 1\right) v(t) \end{pmatrix}.$$

As for the existence of optimal control, we rely on the following proposition, which is an example of a result extracted from [6].

7.1 Existence of optimal control

Proposition 7.1. *Suppose*

$$\begin{cases} \min\limits_{u \in U} J(u) = \int\limits_{t_0}^{t_f} f(t,x,u)\, dt \\ (s.c) \begin{cases} \dfrac{dx(t)}{dt} = g(t,x,u), \\ x(t_0) = x_0, \quad x(t_f)\ free. \end{cases} \end{cases}$$

where U is the set of controls for the above problem consisting of Lebesgue integrable functions on $[t_0, t_f]$ with values in $\mathbb{R}$. We assume that $f(t,x,u)$ is convex in u and that there exist constants $C_1, C_2, C_3 > 0$, C_4 and $\mu > 1$ such that for all $t \in [t_0, t_f]$ and all $x, x_1, u \in \mathbb{R}$, we have:

1) $g(t,x,u) = \alpha(t,x) + \beta(t,x)u$,
2) $|g(t,x,u)| \leq C_1(1 + |x| + |u|)$,
3) $|g(t,x_1,u) - g(t,x,u)| \leq C_2\,|x_1 - x|(1 + |u|)$,
4) $f(t,x,u) \geq C_3|u|^\mu - C_4$.

Then there exists an optimal control u^* that minimizes $J(u)$.

Remark 7.2. For a maximization case, we must show that $f(t,x,u)$ is concave, and we retain 1), 2), 3) and replace 4) with $f(t,x,u) \leq A_1 - A_2|u|^\mu$, *with* $A_1 \in \mathbb{R}$ *and* $A_2 \in \mathbb{R}$.

For proposition 7.3 and its proof, we use proposition 7.1 and the remark 7.2.

Proposition 7.3. *Problem* (7) *admits an optimal control* $m_{1\delta}$ *that maximizes the objective function* J.

Proof. 1) Let us show that g is concave with respect to m_1.
$g(t,u,v,m_1) = \pi(t,u,v,m_1)e^{-\delta t} = a\left[\dfrac{pm_1(t)u(t)}{u(t)+K} - 1\right]v(t)e^{-\delta t}$, implies that g is affine with respect to m_1, so g is concave.
2) Let's show that $F(t,u,v,m_1) = R(t,u,v) + Y(t,u,v)m_1$,
where

$$R(t,u,v) = \begin{pmatrix} (\beta(T)-u(t))u(t) \\ -\alpha a v(t) \end{pmatrix} \quad \text{and} \quad Y(t,u,v) = \begin{pmatrix} -\dfrac{v(t)u(t)}{u(t)+K} \\ \dfrac{\alpha a p u(t) v(t)}{u(t)+K} \end{pmatrix}.$$

We have

$$F(t,u,v,m_1) = \begin{pmatrix} (\beta(T) - u(t) - \dfrac{m_1(t)v(t)}{u(t)+K})u(t) \\ \alpha a(\dfrac{pm_1(t)u(t)}{u(t)+K} - 1)v(t) \end{pmatrix}$$

implies that

$$F(t,u,v,m_1) = \begin{pmatrix} (\beta(T)-u(t))u(t) \\ -\alpha a v(t) \end{pmatrix} + m_1(t)\begin{pmatrix} -\dfrac{v(t)u(t)}{u(t)+K} \\ \dfrac{\alpha a p u(t) v(t)}{u(t)+K} \end{pmatrix}.$$

3) Let us show that

$$|f_1(t,u,v,m_1)| \leq C_1(1 + |u(t)| + |v(t)| + |m_1(t)|)$$

and

$$|f_2(t,u,v,m_1)| \leq C_2(1 + |u(t)| + |v(t)| + |m_1(t)|)$$

where $C_1 = \max\{1, \frac{1}{K}\}$ and $C_2 = \alpha max\{\frac{ap}{K}; a\}$. Firstly, we have

$$|f_1(t,u,v,m_1)| = |(\beta(T) - u(t) - \frac{m_1(t)v(t)}{u(t)+K})u(t)|,$$

and as $u(t) \leq 1$, then

$$|f_1(t,u,v,m_1)| \leq |1 + u(t) + \frac{m_1(t)v(t)}{u(t)+K}|.$$

Furthermore, $v(t) \leq 1$, and we have

$$|f_1(t,u,v,m_1)| \leq 1 + |u(t)| + \frac{1}{K}(|v(t)| + |m_1(t)|) \leq C_1(1 + |u(t)| + |v(t)| + |m_1(t)|).$$

Secondly, we have

$$|f_2(t,u,v,m_1)| \leq |\alpha a(\frac{pm_1(t)u(t)}{u(t)+K} + 1)|,$$

and as $v(t) \leq 1$ then

$$|f_2(t,u,v,m_1)| \leq C_2(1 + |m_1(t)| + |v(t)| + |u(t)|).$$

4) Let us show that f_1 and f_2 are Lipschitz with relation to u and v. Suppose $(u, v, u_1, v_1) \in \Gamma^4$.
Firstly, we have

$$|f_1(t, u, v, m_1) - f_1(t, u_1, v_1, m_1)| = \left| -u(t) - \frac{m_1(t)v(t)}{u(t)+K} - \left(-u_1(t) - \frac{m_1(t)v_1(t)}{u_1(t)+K} \right) \right|,$$

which implies that

$$|f_1(t, u, v, m_1) - f_1(t, u_1, v_1, m_1)| \leq |u_1(t) - u(t)| + \frac{m_1(t)}{K}|v_1(t) - v(t)|.$$

Consequently,

$$|f_1(t, u, v, m_1) - f_1(t, u_1, v_1, m_1)| \leq C_3\big(|u_1(t) - u(t)| + |v_1(t) - v(t)|\big)(1 + m_1(t)),$$

with $C_3 = \max\left\{1; \frac{1}{K}\right\}$.
Secondly,

$$|f_2(t, u, v, m_1) - f_2(t, u_1, v_1, m_1)| = \alpha a \left| \frac{pm_1(t)u(t)v(t)}{K+u(t)} - \frac{pm_1(t)u_1(t)v_1(t)}{K+u_1(t)} \right|,$$

which implies that

$$|f_2(t, u, v, m_1) - f_2(t, u_1, v_1, m_1)| \leq \frac{\alpha pam_1(t)}{K}|u(t)v(t) - u_1(t)v_1(t)|,$$

and consequently,

$$|f_2(t, u, v, m_1) - f_2(t, u_1, v_1, m_1)| \leq C_4\big(|u(t) - u_1(t)| + |v(t) - v_1(t)|\big)(1 + m_1(t)),$$

with $C_4 = \max\left\{1; \frac{\alpha ap}{K}\right\}$.

5) $g(t, u, v, m_1) = a((\frac{pm_1(t)u(t)}{u(t)+K} - 1)e^{-\delta t})v(t)$ and $\forall (t, u, v, m_1), g(t, u, v, m_1) \leq C_5 - C_6|m_1(t)|^2$, with $C_5 = pam_{1max}^2$, $C_6 = \frac{ap}{1+K}$ *and* $\mu = 2$. So there exists an optimal control $m_{1\delta}$ that maximizes the objective function J. □

7.2 Characterization of optimal control

After proving the existence of optimal control for the controlled system (3), we proceed to its characterization. To do so, we use Pontryagin's maximum principle to define the necessary conditions for optimality.

Proposition 7.4. *Let $m_{1\delta}$ be an optimal control associated with an optimal state $E_{opt} = (u_{opt}, v_{opt})$. Then there exist two adjoint functions γ_1 and γ_2 defined from $[t_0, t_f]$ to $\mathbb{R}$ such that:*

1) a)

$$\frac{d\gamma_1}{dt} = -\Big[ae^{-\delta t}\Big(\frac{pm_1Kv_{opt}}{(u_{opt}+K)^2}\Big) + \gamma_1\Big(1 - 2u_{opt} - \frac{m_1Kv_{opt}}{(u_{opt}+K)^2} - de^{-\frac{A}{b}\left(\frac{1}{T} - \frac{1}{T_{ref}}\right)}\Big) + \gamma_2\frac{\alpha apm_1Kv_{opt}}{(u_{opt}+K)^2}\Big], \qquad \gamma_1(t_f) = 0.$$

b)

$$\frac{d\gamma_2}{dt} = -\Big[ae^{-\delta t}\Big(\frac{pm_1u_{opt}}{u_{opt}+K}-1\Big) - \gamma_1\frac{m_1u_{opt}}{u_{opt}+K} + \gamma_2 a\alpha\Big(\frac{pm_1u_{opt}}{u_{opt}+K}-1\Big)\Big], \qquad \gamma_2(t_f)=0.$$

2) According to Pontryagin's maximum principle, the characterization of optimal control is

$$\begin{cases} m_{1max},\ if\ \phi(t)>0,\\ m_{1min}\le m_{1\delta}\le m_{1max},\ if\ \phi(t)=0,\\ m_{1min},\ if\ \phi(t)<0, \end{cases}$$

where ϕ is the commutative function defined by

$$\phi(t) = ape^{-\delta t} - \gamma_1(t) + \alpha ap\gamma_2(t).$$

Proof. Let $(u,v)\in\Gamma$, $m_1\in U$ and (γ_1,γ_2) be the adjoint variables where $\gamma_i(t_f)=0, \forall i\in\{0;1\}$. The Hamiltonian associated with problem (7) is defined by:

$$\mathcal{H} = g(t,u,v,m_1) + \gamma_1\Big[(1-u(t) - \frac{m_1(t)v(t)}{u(t)+K} - de^{-\frac{A}{b}\left(\frac{1}{T}-\frac{1}{T_{ref}}\right)}]u(t) + \gamma_2\alpha a\Big[\frac{pm_1(t)u(t)}{u(t)+K}-1\Big]v(t).$$

First, we have

$$\dot{\gamma}_1 = \frac{d\gamma_1}{dt} = -\frac{\partial\mathcal{H}}{\partial u}|_{E_{opt}}$$

implies that

$$\dot{\gamma}_1 = -\Big[ae^{-\delta t}\Big(\frac{pm_1Kv_{opt}}{(u_{opt}+K)^2}\Big) + \gamma_1\Big(1-2u_{opt} - \frac{m_1Kv_{opt}}{(u_{opt}+K)^2} - de^{-\frac{A}{b}\left(\frac{1}{T}-\frac{1}{T_{ref}}\right)}\Big) + \gamma_2\frac{\alpha apm_1Kv_{opt}}{(u_{opt}+K)^2}\Big], \qquad \gamma_1(t_f)=0. \tag{8}$$

Next, we have

$$\frac{d\gamma_2}{dt} = -\frac{\partial\mathcal{H}}{\partial v}|_{E_{opt}}$$

implies

$$\frac{d\gamma_2}{dt} = -\Big[ae^{-\delta t}\Big(\frac{pm_1u_{opt}}{u_{opt}+K}-1\Big) - \gamma_1\frac{m_1u_{opt}}{u_{opt}+K} + \gamma_2 a\alpha\Big(\frac{pm_1u_{opt}}{u_{opt}+K}-1\Big)\Big], \qquad \gamma_2(t_f)=0.$$

Finally, the optimality condition is

$$\frac{\partial\mathcal{H}}{\partial m_1} = (ape^{-\delta t} - \gamma_1 + \gamma_2\alpha ap)\frac{uv}{u+K}.$$

In E_{opt}, we have $\frac{\partial\mathcal{H}}{\partial m_1}|_{E_{opt}} = 0$, and we obtain the switching function ϕ defined by

$$\phi(t) = ape^{-\delta t} - \gamma_1(t) + \alpha ap\gamma_2(t).$$

□

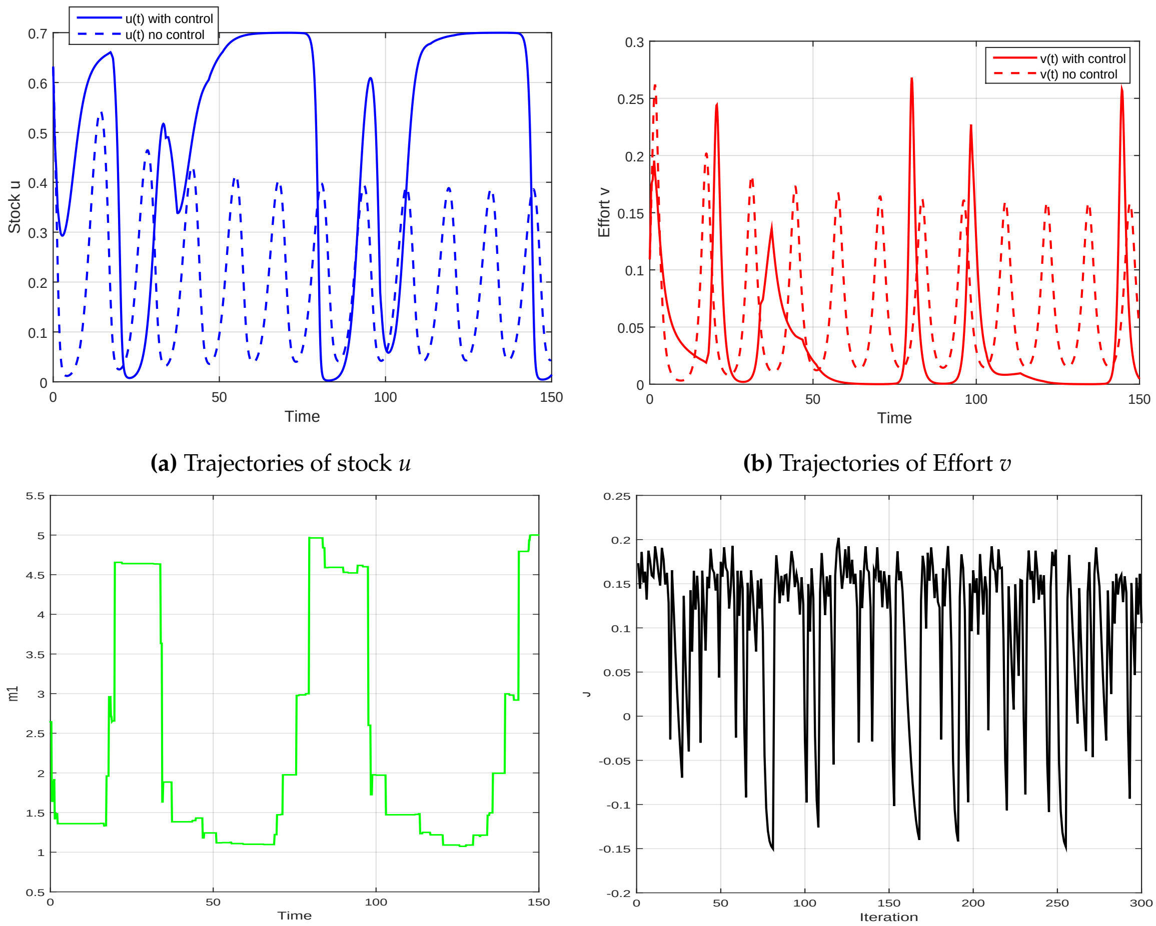
7.3 Simulations

Table 3: Data

d	b	K	T	Tr	p	A	δ	a	α	m_1
0.30	8.314	0.30	301.9250	298.25	0.7750	90	0.05	2.5	0.35	3.7625

In order to numerically justify the existence of an optimal control linked to the variation in maximum catch intensity $m_1 \in [1;5]$, maximizing the total profit from fishing activity, we consider the data in the table 3 ensure the existence of a single unstable interior equilibrium E_3. To estimate the objective function J, we use the rectangle method, and the optimal control will be sought using the forward-backward sweep method with a relaxation $\mu = 0.5$ and a convergence tolerance (threshold) of 10^{-10}. To construct the controlled and uncontrolled trajectories of stock and fishing effort, we use the RK4 method with time t = 150 and number of steps N = 1000, $u_0 = 0.63$ and $v_0 = 0.11$. We obtain the following curves:

- The blue curve shows the evolution of fish stock u over time t.
- The red curve shows the evolution of fishing effort v over time t.
- The green curve shows the evolution of control m_1 over time t.
- The black curve shows the evolution of criterion J over the number of iterations.



(a) Trajectories of stock u

(b) Trajectories of Effort v

(c) Trajectory of m_1

(d) Trajectory of J

Figure 3: The evolution of u, v, m_1 and J

7.4 Interpretation

For a maximum catch intensity $m_1 = 3.7625$, we have the interior equilibrium E_3 with a fishing effort of $v = 0.0647$, i.e., $E = 0.0647 * a_1 * K_0$ and the fish stock level is $u = 0.2805$, or $B = 0.2805 * K_0$, where the marginal profit from fishing is zero. After checking, we obtain the optimal value for maximum catch intensity given by $m_{1\delta} = 2.3539$. For this control $m_{1\delta}$, we obtain a level of effort of 0.0947, or $E_{opt} = 0.0947 * a_1 * K_0$, and a fish stock level of 0.3640, or $B_{opt} = 0.3640 * K_0$. In addition, the fisherman's total discounted profit is 0.10623 over a horizon of 150.

8 Conclusion

In this article, we proposed a modified Gordon-Schaefer model considering the following aspects: (i) the mortality of fish stock is a function of temperature, (ii) production is non-linear. And to emphasize global warming, we focused on a thermal tolerance zone where the temperature varies between 25.10°C and 36°C (298.25 K to 309.15 K). We were able to show that the system is uniformly bounded, i.e., that the behavior of the species is biologically acceptable. Next, we discussed the existence and stability of equilibrium points. It was shown that if p is less than 0.6646, i.e., if the unit price at landing is less than the product of 0.6646 and the unit cost of fishing effort, then all equilibrium points will converge to $(0.2805 * K_0, 0.0647 * a_1 * K_0)$. However, if the unit price at landing is greater than the product of 0.6646 and the unit cost of fishing effort, then a limited cycle occurs around $(0.2805 * K_0, 0.0647 * a_1 * K_0)$. Finally, we studied static and dynamic optimization. At first, in terms of static optimization, we determined the maximum production at equilibrium and the maximum profit at equilibrium and deduced that the biomass level in OA is lower than that in MSY and MEY, and that the effort level in OA is higher than that in MSY and MEY. Secondly, we focused on dynamic optimization and demonstrated the existence of optimal control. Then, using Pontryagin's maximum principle, we found the value of optimal control that maximizes our criterion J analytically, which we were able to obtain numerically when the interior equilibrium E_3 is unstable.

9 Declarations

Funding

This research received no external funding.

Competing Interests

The authors declare that they have no competing interests.

Ethical Approval

Not applicable.

Authors's Contributions

All authors contributed equally. All the authors read and approved the final manuscript.

Availability Data and Materials

Not applicable.

Acknowledgements

The authors would like to thank the editors and reviewers for their time, effort, and valuable commentswhich led to improvements in our manuscript.

References

[1] H. Berkowitz and A. C. Clarke, *Le problème de la surpêche et sa gestion*, Le Libellio d'AEGIS **10** (2014), 37–42.

[2] R. Bravo de la Parra, J. C. Poggiale, and P. Auger, *The effect of connecting sites in the environment of a harvested population*, Mathematical Modelling of Natural Phenomena **18** (2023), 4–19.

[3] U. Daugaard, O. L. Petchey, and F. Pennekamp, *Warming can destabilize predator-prey interactions by shifting the functional response from type iii to type ii*, Journal of Animal Ecology **88** (2019), 1575–1586.

[4] C. Dichmont, S. Pascoe, T. Kompas, A. E. Punt, and R. Deng, *On implementing maximum economic yield in commercial fisheries*, Proceedings of the National Academy of Sciences **107** (2010), 16–21.

[5] M. C. Finley and N. Oreskes, *Maximum sustained yield: a policy disguised as science*, ICES Journal of Marine Science **70** (2013), 245–250.

[6] W. H. Fleming and R. W. Rishel, *Deterministic and stochastic optimal control*, Springer-Verlag, 1975.

[7] T. K. Kar, *Conservation of a fishery through optimal taxation: a dynamic reaction model*, Communications in Nonlinear Science and Numerical Simulation **10** (2005), 121–131.

[8] A. F. Nindjin, A. T. N'Guessan, H. Okou, and K. T. Tia, *Cyclic growth and global stability of economic dynamics of kaldor type in two dimensions*, International Journal of Differential Equations (2017), 1–12.

[9] A. J. Pershing, M. A. Alexander, C. M. Hernandez, L. A. Kerr, A. L. Bris, and K. E. Mills, *Slow adaptation in the face of rapid warming leads to collapse of the gulf of maine cod fishery*, Science **350** (2015), 809–812.

[10] D. L. Ragozin and G. Brown, *Harvest policies and nonmarket valuation in a predator-prey system*, J. Envirn. Econ. Manag. **12** (1985), 155–168.

[11] Y. Souchon and L. Tisson, *Synthesis of thermal tolerances of the common freshwater fish species in large western europe rivers*, Knowledge and Management of Aquatic Ecosystems (2012), no. 405.

[12] I. V. Soukotta, A. N. Bambang, L. Sya'rani, and S. W. Saputra, *Estimation of MSY and MEY of skipjack tuna (Katsuwonus pelamis) fisheries of banda sea, moluccas aquaculture*, Aquarium, Conservation & Legislation-International Journal of the Bioflux Society (AACL Bioflux) **10** (2017), 44–435.

[13] K. I. Stergiou, S. Somarakis, G. Triantafyllou, K. P. Tsiaras, M. Giannoulaki, and G. Petihakis, *Trends in productivity and biomass yields in the mediterranean sea large marine ecosystem during climate change*, Environ. Dev. **17** (2016), 57–74.