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Large deviation principle for stochastic p-Laplacian reversible Selkov lattice systems

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Abstract

The Selkov system is a classical model for autocatalytic biochemical reactions such as glycolysis and exhibits complex spatio-temporal patterns including self-excitation, oscillations, and spatial chaos. In this paper, we consider stochastic p-Laplacian reversible Selkov lattice systems defined on the integer set $\mathbb{Z}$, which possess two pairs of oppositely signed nonlinear terms and whose nonlinear couplings can grow polynomially with any order $q \geq 1$, the inherent structure of the system precludes the possibility of any unidirectional dissipative influence arising from the interaction between the two coupled equations, thereby obstructing the emergence of a dominant energy-dissipation mechanism along a single directional pathway. In particular, our focus is on studying the asymptotic properties of the $(\psi^\gamma, \phi^\gamma)$ as the noise intensity $\gamma \to 0$ for the systems. Based on the well-posedness and the convergence of the solutions of the controlled system associated with the considered system, by the weak convergence method, we establish the large deviation principle in $C([0,T], \ell^2 \times \ell^2)$ for such infinite dimensional system. One of the advantages of the weak convergence method that does not rely on uniform exponential probability estimates of solutions. Moreover, a stopping time technique is used to prove $\lim_{\gamma\to 0} \left(\psi^\gamma_{z^\gamma} - \psi_{z^\gamma}, \phi^\gamma_{z^\gamma} - \phi_{z^\gamma}\right) = 0$ in probability in order to overcome the difficulty caused by the local monotonicity of the nonlinear and diffusion terms. Compared to the classical case of cubic nonlinearity, our results are not only more general in scope but also demonstrate a higher degree of novelty.

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1 Introduction

The Selkov system provides a simplified model for certain autocatalytic biochemical and chemical reactions. It has been shown [1, 27, 38, 46] that the system generates complex and

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irregular spatio-temporal patterns, including self-excitation, self-oscillations, self-replicating spikes, labyrinthine structures, and spatial chaos in one- and two-dimensional settings. The non-reversible Selkov system [55], similar to the Brusselator [37, 53], Gray–Scott [19, 20, 54], and Schnackenberg models [56], was proposed by Selkov [40, 41, 43] to represent the glycolysis process, whereby living cells obtain energy through the breakdown of sugars. These systems serve as prototypes for cubic-autocatalytic reactions, including the chlorite–iodide–malonic acid (CIMA) reaction and various enzyme and kinase-mediated processes in biochemistry and physiology [18, 30, 40, 41], such as the phosphofructokinase reaction involving macromolecules ATP, ADP, and AMP. The reversible cubic-autocatalytic Selkov system is derived from the following reaction:

$$\mathfrak{A} \underset{\hat{k}_{-1}}{\overset{\hat{k}_1}{\rightleftharpoons}} \mathfrak{S}, \quad \mathfrak{S} + 2\mathfrak{P} \underset{\hat{k}_{-2}}{\overset{\hat{k}_2}{\rightleftharpoons}} 3\mathfrak{P}, \quad \mathfrak{P} \underset{\hat{k}_{-3}}{\overset{\hat{k}_3}{\rightleftharpoons}} \mathfrak{B},$$

where $\hat{k}_i$ are rate constants of respective reactions. Although nonzero reverse reaction rates are involved, they are often neglected for simplicity. In this model, the concentrations of $\mathfrak{A}$ and $\mathfrak{B}$ are treated as constants, while those of $\mathfrak{P}$ and $\mathfrak{S}$ are considered unknown functions of time t and space x.

In this paper, we will investigate the large deviation principle (LDP) of the following stochastic p-Laplacian reversible Selkov lattice systems defined on the integer set $\mathbb{Z}$ and the coupling terms have a polynomial growth of any order $q \geq 1$:

$$\begin{cases} d\psi_i^\gamma(t) = \Big(-d_1(A\psi(t))_i - a_1\psi_i^\gamma(s) + b_1(\psi_i^\gamma)^{2q}(s)\phi_i^\gamma(s) - b_2(\psi_i^\gamma)^{2q+1}(s)\Big)\,ds \\ \qquad\qquad + f_{1,i}(s)ds + \sqrt{\gamma}\sum_{k=1}^{\infty}(h_{k,i}(t) + \delta_{k,i}\tilde{\sigma}_k(\psi_i^\gamma(t)))dW_k(t), \quad t > 0, \\ d\phi_i^\gamma(t) = \Big(-d_2(A\phi(t))_i - a_2\phi_i^\gamma(s) - b_1(\psi_i^\gamma)^{2q}(s)\phi_i^\gamma(s) + b_2(\psi_i^\gamma)^{2q+1}(s)\Big)\,ds \\ \qquad\qquad + f_{2,i}(s)ds + \sqrt{\gamma}\sum_{k=1}^{\infty}(h_{k,i}(t) + \delta_{k,i}\tilde{\sigma}_k(\phi_i^\gamma(t)))dW_k(t), \quad t > 0. \end{cases} \tag{1}$$

with initial data

$$\psi_i^\gamma(0) = \psi_{0,i}, \quad \phi_i^\gamma(0) = \phi_{0,i}, \tag{2}$$

where $i \in \mathbb{Z}$, $\psi = (\psi_i)_{i\in\mathbb{Z}}$, $\phi = (\phi_i)_{i\in\mathbb{Z}} \in \ell^2$, $0 < \gamma < 1$, $p \geq 1$. d_1, d_2, a_1, a_2, b_1, b_2 are positive constants, $f_1(t) = (f_{1i}(t))_{i\in\mathbb{Z}}$ and $f_2(t) = (f_{2i}(t))_{i\in\mathbb{Z}} \in \ell^2$ are time dependent random sequences. $\tilde{\sigma}_k : \mathbb{R} \to \mathbb{R}$ be a family of locally Lipschitz continuous functions, uniformly in k, such that for each $s \in \mathbb{R}$ and $k \in \mathbb{N}$, there exists a constant $\alpha > 0$ satisfying the inequality

$$|\tilde{\sigma}_k(s)| \leq \alpha(1 + |s|). \tag{3}$$

$\{W_k\}_{k\in\mathbb{N}}$ is a sequence of independent two-sided real-valued standard Wiener processes on a complete filtered probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, \mathbb{P})$. The discrete p-Laplace operator is defined by

$$(A\psi(t))_i = |\psi_i(t) - \psi_{i-1}(t)|^{p-2}(\psi_i(t) - \psi_{i-1}(t)) - |\psi_{i+1}(t) - \psi_i(t)|^{p-2}(\psi_{i+1}(t) - \psi_i(t)).$$

As one knows, a LDS has many applications in fluid dynamics, chemistry, and neural networks; see [3, 7, 24, 44]. The p-Laplace LDS (1) is the space-discretization of the corresponding p-Laplace partial differential equation (defined on the real line), while the dynamics of the (deterministic or stochastic) p-Laplace PDE was studied in [8, 14, 16, 17, 25, 28, 29, 32, 33, 34, 45, 49].

As one kind of lattice systems, the existence and stability of invariant/periodic measures of the reversible Selkov lattice systems driven by white noise have been discussed by Wang, see [51]. For stochastic lattice reversible Selkov equations driven by additive noises, the existence of random attractors was discussed by Li [31]. Uniform attractor for Selkov lattice systems was studied by Jia and Zhao. In the latest study, Wang et al in [50] studied upper semi-continuity of numerical attractors for deterministic and random lattice reversible Selkov systems. The deterministic reversible Selkov equations have been extensively studied with respect to the existence, regularity, and stability of global attractors (see [57, 59]). Numerous researchers have also explored the existence and robustness of random attractors in stochastic reversible Gray-Scott systems (see [21, 22, 23, 58]).

It is well known that the large deviation principle (LDP) provides an important tool for studying the asymptotic behavior of stochastic systems as the noise $\gamma \to 0$. There are two main methods to establish the LDP. One is the classical method based on time discretization and uniform exponential probability estimates; see, for example, [9, 15, 26, 39, 42]. The other is the weak convergence method developed in [4, 6, 13], which is based on variational representations of positive functionals of infinite-dimensional Brownian motions. It is worth noting that the weak convergence method does not require exponential-type probability estimates and avoids the complexity of time discretization.

The LDP for finite-dimensional stochastic differential equations has been thoroughly studied in [2, 35]. However, results for infinite-dimensional stochastic lattice systems remain very scarce. To the best of our knowledge, only a few works, including [11, 12, 47], have addressed this problem. In this paper, we apply the weak convergence method to establish the LDP for a stochastic reversible Selkov lattice system described by (1)–(2), see [13, 15]. Compared to the models considered in [12, 47], the system (1)–(2) is a p-Laplacian lattice system, where the coupled terms may grow polynomially with any order $q \geq 1$. The structure prevents the interaction between the two equations from exhibiting a unidirectional dissipative effect. As a result, more delicate analysis is required. Specifically, see the proof of Theorem 3.1.

It is also important to remark that the LDP for a class of stochastic partial differential equations satisfying a local monotonicity condition was established in [36]. In particular, the diffusion term in [36] is assumed to be Hölder continuous in time. In contrast, we do not impose such continuity assumptions here. Therefore, the results of [36] cannot be directly applied to our results.

The rest of this paper is organized as follows. In Section 2, we first discuss the assumptions on the nonlinear terms in system (1)-(2) and the well-posedness of this system, and then review some basic concepts and results of the weak convergence theory for the LDP. In Section 3,4,5 we prove the solution family $\{(\psi^\gamma, \phi^\gamma)\}$ of (1)-(2) satisfies the LDP by the weak convergence method.

2 Preliminaries

2.1 Well-posedness of stochastic p-Laplacian reversible Selkov lattice systems

In this section, some useful preliminaries and assumptions are given. Let $\ell^r := \{\psi = (\psi_i)_{i\in\mathbb{Z}} : \sum_{i\in\mathbb{Z}} |\psi_i|^r < +\infty\}$ for $r \geq 1$. The norm of ℓ^r is denoted by $\|\cdot\|_r$. The norm and inner product of ℓ^2 are written as $\|\cdot\|$ and $\langle\cdot,\cdot\rangle$, respectively.

Firstly, defining linear and bounded operators B, B^* from ℓ^2 to ℓ^2 as follows

$$(B\psi)_i = \psi_{i+1} - \psi_i, \ (B^*\psi)_i = \psi_i - \psi_{i-1}, \quad \text{for all } \psi = (\psi_i)_{i\in\mathbb{Z}} \in \ell^2.$$

Thus, we have $(B\psi, \phi) = (\psi, B^*\phi)$. Define the discrete p-Laplace operator

$$(A\psi)_i = \left(B^*\left(\left(|(B\psi)_i|^{p-2}(B\psi)_i\right)_{i\in\mathbb{Z}}\right)\right)_i.$$

By [10, 48], $A : \ell^2 \to \ell^2$ is locally Lipschitz continuous and monotonous. Therefore, we have

$$\begin{aligned} &\|(A\psi) - (A\phi)\| \le C\|\psi - \phi\|, \\ &(A(\psi) - A(\phi), \psi - \phi) \ge 0, \quad \text{for all } \psi, \phi \in \ell^2. \end{aligned} \tag{4}$$

We also define the operators $F : \ell^2 \times \ell^2 \to \ell^2$ and $G : \ell^2 \to \ell^2$ by $F(\psi, \phi) = (\psi_i^{2p}\phi_i)_{i\in\mathbb{Z}}$ and $G(\psi) = (\psi_i^{2p+1})_{i\in\mathbb{Z}}$ for $\psi, \phi \in \ell^2$.

$$\begin{aligned} &\|F(\psi_1, \phi_1) - F(\psi_2, \phi_2)\|^2 \le c_1(n)\big(\|\psi_1 - \psi_2\|^2 + \|\phi_1 - \phi_2\|^2\big), \\ &|\langle F(\psi_1, \phi_1) - F(\psi_2, \phi_2), \psi_1 - \psi_2\rangle| \le c_1(n)\big(\|\psi_1 - \psi_2\|^2 + \|\phi_1 - \phi_2\|^2\big), \\ &|\langle F(\psi_1, \phi_1) - F(\psi_2, \phi_2), \phi_1 - \phi_2\rangle| \le c_1(n)\big(\|\psi_1 - \psi_2\|^2 + \|\phi_1 - \phi_2\|^2\big), \end{aligned} \tag{5}$$

and

$$\begin{aligned} &\|G(\psi) - G(\phi)\|^2 \le c_2(n)\|\psi - \phi\|^2, \\ &|\langle G(\psi_1) - G(\psi_2), \psi_1 - \psi_2\rangle| \le c_2(n)\|\psi_1 - \psi_2\|^2, \\ &|\langle G(\psi_1) - G(\psi_2), \phi_1 - \phi_2\rangle| \le c_2(n)\big(\|\psi_1 - \psi_2\|^2 + \|\phi_1 - \phi_2\|^2\big), \end{aligned} \tag{6}$$

where $\psi, \phi, \psi_1, \psi_2, \phi_1, \phi_2 \in \ell^2$ and $\|\psi\| \le n$, $\|\psi_1\| \le n$, $\|\psi_2\| \le n$, $\|\phi\| \le n$, $\|\phi_1\| \le n$, $\|\phi_2\| \le n$; see [51] for the proof.

Let $\delta_k : \ell^2 \to \ell^2$ by $\sigma_k(\psi) = (\delta_{k,i}\tilde{\sigma}_k(\psi_i))_{i\in\mathbb{Z}}$ and $\sigma_k : \ell^2 \to \ell^2$ is also locally Lipschitz continuous. For any $k \in \mathbb{N}$, there exist $\alpha > 0$, $C_k(n) > 0$ such that

$$\sum_{k=1}^{\infty} \|\sigma_k(\psi_1) - \sigma_k(\psi_2)\|^2 \le C_k(n)\|\psi_1 - \psi_2\|^2, \tag{7}$$

and

$$\sum_{k=1}^{\infty} \|\sigma_k(\psi)\|^2 \le 2\alpha^2\|\delta\|^2(1 + \|\psi\|^2), \tag{8}$$

where $\|\delta\|^2 := \sum_{k\in\mathbb{N}}\sum_{i\in\mathbb{Z}} |\delta_{k,i}|^2 < \infty$. In addition,

$$\int_{\tau}^{\tau+T} \mathbb{E}\left(\|f_1(t)\|^2 + \|f_2(t)\|^2 + \sum_{k=1}^{\infty}\|h_k(t)\|^2\right) dt < \infty,\ \forall \tau \in \mathbb{R},\ T > 0. \tag{9}$$

Let $H = \{\psi = (\psi_j)_{j=1}^{\infty} : \sum_{j=1}^{\infty} |\psi_j|^2 < \infty\}$ be a Hilbert space. For each $k \in \mathbb{N}$, define $e_k = (e_{k,j})_{j=1}^{\infty}$,

$$e_{k,j} = \begin{cases} 1, & j = k, \\ 0, & j \ne k. \end{cases}$$

Consequently, $\{e_k : k \in \mathbb{N}\}$ forms an orthonormal basis of H. Let I denote the identity operator on H, and consider W to be a cylindrical Wiener process in H with covariance operator I, represented as

$$W(t) = \sum_{k\in\mathbb{N}} W_k(t)e_k, \quad t \in \mathbb{R}^+,$$

where the series converges in $L^2(\Omega; C([0,T]; U))$ for any $T > 0$. Here, U is a larger separable Hilbert space such that the embedding $H \hookrightarrow U$ is Hilbert-Schmidt. For any $\psi \in \ell^2$ and $t \geq 0$, define $\sigma(t,\psi) : H \to \ell^2$ by

$$\sigma(t,\psi)(z) = \sum_{k\in\mathbb{N}} (h_k(t) + \sigma_k(\psi))\, z_k, \quad \forall z = (z_k)_{k=1}^{\infty} \in H. \tag{10}$$

Based on equation (8) and (9), we conclude that $\sigma(t,\psi)$ is well-defined. What is more, this operator is Hilbert-Schmidt, and it satisfies the inequality

$$\|\sigma(t,\psi)\|_{L(H;\ell^2)} \leq \|\sigma(t,\psi)\|_{L_2(H;\ell^2)} = \left(\sum_{k\in\mathbb{N}} \|h_k(t) + \sigma_k(\psi)\|^2\right)^{\frac{1}{2}} < \infty, \tag{11}$$

where $L(H;\ell^2)$ denotes the space of bounded linear operators from H to ℓ^2, with norm $\|\cdot\|_{L(H;\ell^2)}$, and $L_2(H;\ell^2)$ refers to the space of Hilbert-Schmidt operators from H to ℓ^2, with norm $\|\cdot\|_{L_2(H;\ell^2)}$.

By (10), the system (1)-(2) can be rewritten as

$$\begin{cases} \mathrm{d}\psi^\gamma(t) = \Big(-d_1 A\psi^\gamma - a_1\psi^\gamma(s) + b_1(\psi^\gamma)^{2q}(t)\phi^\gamma(t) - b_2(\psi^\gamma)^{2q+1}(t) + f_1(t)\Big)\,\mathrm{d}t \\ \qquad\qquad + \sqrt{\gamma}\sigma(t,\psi^\gamma(t))\mathrm{d}W_k(t), \quad t > 0, \\ \mathrm{d}\phi^\gamma(t) = \Big(-d_2 A\phi^\gamma - a_2\phi^\gamma(t) - b_1(\psi^\gamma)^{2q}(t)\phi^\gamma(t) + b_2(\psi^\gamma)^{2q+1}(t) + f_2(t)\Big)\,\mathrm{d}t \\ \qquad\qquad + \sqrt{\gamma}\sigma(t,\phi^\gamma(t))\mathrm{d}W_k(t), \quad t > 0. \end{cases} \tag{12}$$

with initial conditions

$$(\psi^\gamma(0), \phi^\gamma(0)) = (\psi_0, \phi_0) \in \ell^2 \times \ell^2. \tag{13}$$

An ℓ^2-valued $\mathcal{F}_t$-adapted stochastic process $(\psi^\gamma(t), \phi^\gamma(t))$ with $t \in [0,\infty)$ is called a solution to problem (12)-(13) if $(\psi^\gamma, \phi^\gamma) \in L^2(\Omega; C([0,T]; \ell^2 \times \ell^2))$ for any $T > 0$, and for $\mathbb{P}$-almost surely $\omega \in \Omega$,

$$\begin{cases} \psi^\gamma(t) = \psi_0 + \int_0^t \Big(-d_1 A\psi^\gamma(s) - a_1\psi^\gamma(s) + b_1(\psi^\gamma)^{2q}(s)\phi^\gamma(s) - b_2(\psi^\gamma)^{2q+1}(s) + f_1(s)\Big)\, ds \\ \qquad\qquad + \sqrt{\gamma}\int_0^t \sigma(s,\psi^\gamma)\, dW_k(s), \quad \mathbb{P}\text{-a.s.}, \\ \phi^\gamma(t) = \phi_0 + \int_0^t \Big(-d_2 A\phi^\gamma(s) - a_2\phi^\gamma(s) - b_1(\psi^\gamma)^{2q}(s)\phi^\gamma(s) + b_2(\psi^\gamma)^{2q+1}(s) + f_2(s)\Big)\, ds \\ \qquad\qquad + \sqrt{\gamma}\int_0^t \sigma(s,\phi^\gamma)\, dW_k(s), \quad \mathbb{P}\text{-a.s.}. \end{cases} \tag{14}$$

where $t \geq 0$ and $(\psi_0, \phi_0) \in \ell^2 \times \ell^2$. Under assumptions (7)-(9), the existence and uniqueness of solutions to system (12)-(13) can be proved, see [51, 52].

2.2 Weak convergence method for large deviations

In this subsection, we recall some basic concepts and results of the weak convergence theory for the large deviation principle, which can be found in [4, 13, 47].

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a complete probability space, $\{\mathcal{F}_t\}_{t\geq 0}$ is a standard filtration. Suppose H is a separable Hilbert space, $\{W(t)\}_{t\geq 0}$ is a cylindrical Wiener process with identity covariance operator in H with respect to $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, P)$, which means that there exists a separable Hilbert space U such that the embedding $H \hookrightarrow U$ is Hilbert-Schmidt and $W(t)$ takes values in U.

Let $\mathcal{E}$ be a Polish space, $T > 0$. For every $\gamma > 0$, let $\mathcal{G}^\gamma : C([0,T],U) \to \mathcal{E}$ be a measurable map, and denote by

$$X^\gamma = \mathcal{G}^\gamma(W). \tag{15}$$

We study the large deviation principle of X^γ as $\gamma \to 0$. Given $N \in \mathbb{R}^+$, denote by

$$S_N = \left\{ z \in L^2(0,T;H) : \int_0^T \|z(t)\|_H^2 \, dt \leq N \right\}. \tag{16}$$

Then S_N is a Polish space endowed with the weak topology. Let $\mathcal{A}$ be the space of all H-valued stochastic processes z which are progressively measurable with respect to $\{\mathcal{F}_t\}_{t\in[0,T]}$ and $\int_0^T \|z(t)\|_H^2 \, dt < \infty$, P-almost surely. Denote by

$$\mathcal{A}_N = \{ z \in \mathcal{A} : z(\omega) \in S_N \text{ for almost all } \omega \in \Omega \}. \tag{17}$$

Definition 2.1. A function $I : \mathcal{E} \to [0,\infty)$ is called a rate function on $\mathcal{E}$ if it is lower semi-continuous in $\mathcal{E}$. A rate function I on $\mathcal{E}$ is said to be a good rate function on $\mathcal{E}$ if for every $0 \leq C < \infty$, the level set $\{x \in \mathcal{E} : I(x) \leq C\}$ is a compact subset of $\mathcal{E}$.

Definition 2.2. (Large Deviation Principle) The family $\{X^\gamma\}$ is said to satisfy the large deviation principle in $\mathcal{E}$ with a rate function $I : \mathcal{E} \to [0,\infty)$ if for every Borel subset B of $\mathcal{E}$,

$$-\inf_{x\in B^\circ} I(x) \leq \liminf_{\gamma\to 0} \gamma \ln \mathbb{P}(X^\gamma \in B) \leq \limsup_{\gamma\to 0} \gamma \ln \mathbb{P}(X^\gamma \in B) \leq -\inf_{x\in \overline{B}} I(x),$$

where B° and $\overline{B}$ are the interior and closure of B in $\mathcal{E}$, respectively.

In order to investigate the **LDP** of $\{X^\gamma\}$ in Polish space $\mathcal{E}$ by the weak convergence method, we need to recall the concept of Laplace principle of $\{X^\gamma\}$ as below.

Definition 2.3. (Laplace Principle) The family $\{X^\gamma\}$ is said to satisfy the Laplace principle in $\mathcal{E}$ with a rate function $I : \mathcal{E} \to [0,\infty)$ if and only if for every bounded and continuous function $J : \mathcal{E} \to \mathbb{R}$,

$$\lim_{\gamma\to 0} \ln \mathbb{E}\left[e^{-\frac{1}{\gamma} J(X^\gamma)} \right] = -\inf_{x\in\mathcal{E}} \{J(x) + I(x)\}.$$

The following result shows the equivalence of large deviation principle and Laplace principle in Polish space, which can be found in [13].

Proposition 2.4. *The family $\{X^\gamma\}$ satisfies the large deviation principle in a Polish space $\mathcal{E}$ with a rate function I, if and only if $\{X^\gamma\}$ satisfies the Laplace principle in $\mathcal{E}$ with the same rate function.*

In view of this equivalence, it is sufficient to verify the Laplace principle of X^γ. To this end, we need the following conditions for the family $\mathcal{G}^\gamma$: there exists a measurable map $\mathcal{G}^0 : C([0,T],U) \to \mathcal{E}$ such that

(i) For every $N < \infty$, the set $\{\mathcal{G}^0\left(\int_0^\cdot z(t)dt : z \in S_N\right)\}$ is a compact subset of $\mathcal{E}$.
(ii) If $N < \infty$ and $\{z^\gamma\} \subseteq \mathcal{A}_N$ such that $\{z^\gamma\}$ converges in distribution to z as S_N-valued random variables, then $\mathcal{G}^\gamma(W + \epsilon^{-\frac{1}{2}} \int_0 z^\gamma(t)dt)$ converges in distribution to $\mathcal{G}^0\left(\int_0 z(t)dt\right)$.

Define $I : \mathcal{E} \to [0, \infty)$ by, for every $x \in \mathcal{E}$,

$$I(x) = \inf \left\{ \frac{1}{2} \int_0^T \|z(t)\|_H^2 dt : z \in L^2(0,T;H) \text{ such that } G^\gamma \left(\int_0^T z(t)dt \right) = x \right\}, \tag{18}$$

with the convention that the infimum over an empty set is taken to be ∞. Under (i) and (ii), the following proposition can be obtained (see [5]), which is useful for establishing the LDP.

Proposition 2.5. *If $\mathcal{G}^\gamma$ satisfies (i)-(ii), then the family $\{X^\gamma\}$ as given by (15) satisfies the Laplace principle in $\mathcal{E}$ with rate function I as defined by (18).*

3 Large deviation of stochastic p-Laplacian reversible Selkov Lattice Systems

In this section, we aim to establish the large deviation principle for the stochastic p-Laplacian reversible Selkov lattice systems (12)–(13). The proof is based on the equivalence between the large deviation principle and the Laplace principle and the main result of this paper is stated as follows.

Theorem 3.1. *Suppose that (i)-(ii) hold, and $(\psi^\gamma, \phi^\gamma)$ is the solution of (12)-(13). Then the family $\{(\psi^\gamma, \phi^\gamma)\}$, as $\gamma \to 0$, satisfies the large deviation principle in $C([0,T], L^2 \times L^2)$, with the good rate function given by*

$$I(\varphi) = \inf \left\{ \frac{1}{2} \int_0^T \|z(s)\|_H^2 \, ds : z \in L^2(0,T;H), (\psi_z, \phi_z) = \varphi \right\}, \tag{19}$$

where $\varphi \in C([0,T], \ell^2 \times \ell^2)$, (ψ_z, ϕ_z) is the solution of (21)-(22) given below. As usual, the infimum of the empty set is taken to be ∞.

Remark 3.2. Since $(\psi^\gamma, \phi^\gamma)$ is the unique solution of (1)-(2), it follows that there exists a Borel measurable map $\mathcal{G}^\gamma : C([0,T], U) \to C([0,T], \ell^2 \times \ell^2)$ such that

$$(\psi^\gamma, \phi^\gamma) = \mathcal{G}^\gamma(W), \quad P\text{-almost surely.} \tag{20}$$

In order to obtain Theorem 3.1, we just need to show that $\mathcal{G}^\gamma$ (see (20)) and $\mathcal{G}^0$ (see (46)) satisfy condition (i) and (ii) with rate function I in (19) based on Propositions 2.4 and 2.5.

The rest of the paper is to prove Theorem 3.1.

3.1 The controlled p-Laplacian reversible Selkov lattice system

Firstly, we investigate a deterministic control system corresponding to (12), which is important for establishing the large deviation principle of $(\psi^\gamma, \phi^\gamma)$ as $\gamma \to 0$. For a given control $z \in L^2(0,T;H)$, consider the controlled system:

$$\begin{cases} \dfrac{\mathrm{d}\psi_z(t)}{\mathrm{d}t} = -d_1 A\psi_z(t) - a_1\psi_z(t) + b_1(\psi_z)^{2q}(t)\phi_z(t) - b_2(\psi_z)^{2q+1}(t) + f_1(t) \\ \qquad\qquad + \sigma(t, \psi_z(t))z(t), \\ \dfrac{\mathrm{d}\phi_z(t)}{\mathrm{d}t} = -d_2 A\phi_z(t) - a_2\phi_z(t) - b_1(\psi_z)^{2q}(t)\phi_z(t) + b_2(\psi_z)^{2q+1}(t) + f_2(t) \\ \qquad\qquad + \sigma(t, \phi_z(t))z(t), \end{cases} \tag{21}$$

with initial data

$$\psi_z(0), \phi_z(0)) = (\psi_0, \phi_0) \in \ell^2 \times \ell^2. \tag{22}$$

Definition 3.3. For any $T > 0$ and $(\psi_0, \phi_0) \in \ell^2 \times \ell^2$, $(\psi_z, \phi_z) \in C([0,T], \ell^2 \times \ell^2)$ is called a solution of (21)-(22), if for all $t \in [0,T]$,

$$\begin{cases} \psi_z(t) = \psi_0 + \int_0^t \left(-d_1 A\psi_z(s) - a_1\psi_z(s) + b_1(\psi_z)^{2q}(s)\phi_z(s) - b_2(\psi_z)^{2q+1}(s) + f_1(s)\right) ds \\ \qquad + \int_0^t \sigma(s, \psi_z(s)) z(s)\, \mathrm{d}s, \\ \phi_z(t) = \phi_0 + \int_0^t \left(-d_2 A\phi_z(s) - a_2\phi_z(s) - b_1(\psi_z)^{2q}(s)\phi_z(s) + b_2(\psi_z)^{2q+1}(s) + f_2(s)\right) ds \\ \qquad + \int_0^t \sigma(s, \phi_z(s)) z(s)\, \mathrm{d}s. \end{cases} \tag{23}$$

We first prove the existence and uniqueness of solutions to (21)-(22).

Lemma 3.4. *Suppose that (7)-(9) hold. Then for every $z \in L^2(0,T;H)$, system (21)-(22) has a unique solution $(\psi_z, \phi_z) \in C([0,T], \ell^2 \times \ell^2)$.*

Furthermore, for each $n_1 > 0$ and $n_2 > 0$, there exists $C = C(n_1, n_2, T) > 0$ such that for any $(\psi_{0,1}, \phi_{0,1}), (\psi_{0,2}, \phi_{0,2}) \in \ell^2 \times \ell^2$ with $\|\psi_{0,1}\| \vee \|\psi_{0,2}\| \vee \|\phi_{0,1}\| \vee \|\phi_{0,2}\| \le n_1$, and any $z_1, z_2 \in L^2(0,T;H)$ with $\|z_1\|_{L^2(0,T;H)} \vee \|z_2\|_{L^2(0,T;H)} \le n_2$, the solutions (ψ_{z_1}, ϕ_{z_1}) and (ψ_{z_2}, ϕ_{z_2}) of (21)–(22) with initial values $(\psi_{0,1}, \phi_{0,1})$ and $(\psi_{0,2}, \phi_{0,2})$, respectively, satisfy

$$\|(\psi_{z_1} - \psi_{z_2}, \phi_{z_1} - \phi_{z_2})\|^2_{C([0,T],\ell^2\times\ell^2)} \le C\left(\|\psi_{0,1} - \psi_{0,2}\|^2 + \|\phi_{0,1} - \phi_{0,2}\|^2 + \|z_1 - z_2\|^2_{L^2(0,T;H)}\right), \tag{24}$$

and

$$\|(\psi_{z_1}, \phi_{z_1})\|^2_{C([0,T],\ell^2\times\ell^2)} \le C. \tag{25}$$

Proof. Given $z \in L^2(0,T;H)$, we first prove the well-posedness of system (21)-(22). By dropping the subscript z, system (21)-(22) can be written as

$$\begin{cases} \frac{\mathrm{d}\psi(t)}{\mathrm{d}t} = -d_1 A\psi(t) - a_1\psi(t) + b_1(\psi)^{2q}(t)\phi(t) - b_2(\psi)^{2q+1}(t) + f_1(t) \\ \qquad + \sigma(t, \psi(t)) z(t) \\ \qquad =: G(t, \psi, \phi), \\ \frac{\mathrm{d}\phi(t)}{\mathrm{d}t} = -d_2 A\phi(t) - a_2\phi(t) - b_1(\psi)^{2q}(t)\phi(t) + b_2(\psi)^{2q+1}(t) + f_2(t) \\ \qquad + \sigma(t, \phi(t)) z(t) \\ \qquad =: \bar{G}(t, \psi, \phi) \end{cases} \tag{26}$$

with initial data

$$(\psi(0), \phi(0)) = (\psi_0, \phi_0) \in \ell^2 \times \ell^2. \tag{27}$$

Since $z \in L^2(0,T;H)$, by (4)-(6) and (11) we deduce that for every $R > 0$, there exists a constant $C_1 = C_1(n) > 0$ such that for all $t \in [0,T]$ and $\psi, \phi \in \ell^2$ with $\|\psi\| \le n$, $\|\phi\| \le n$,

$$\begin{aligned} \|G(t,\psi,\phi)\|^2 \le& 6\big(d_1^2\|A\psi\|^2 + a_1^2\|\psi\|^2 + b_1^2\|F(\psi,\phi)\|^2 + b_2^2\|G(\psi)\|^2 + \|f_1(t)\|^2 \\ &+ \|\sigma(t,\psi)\|^2_{L_2(H,\ell^2)}\|z(t)\|^2_H\big) \\ \le& 6\big(16d_1^2\|\psi\|^2 + a_1^2\|\psi\|^2 + b_1^2 c_1(n)(\|\psi\|^2 + \|\phi\|^2) + b_2^2 c_2(n)\|\psi\|^2 + \|f_1(t)\|\big) \\ &+ 6\big(2\sum_{k=1}^{\infty}\|h_k\|^2_{L^\infty(0,T;\ell^2)} + 4\alpha^2\|\delta\|^2(1 + \|\psi\|^2)\big)\|z(t)\|^2_H \end{aligned}$$

$$\leq C_1\left(1+\|f_1(t)\|^2+\|\psi\|^2+\|\phi\|^2\right)+C_1\left(1+\sum_{k=1}^{\infty}\|h_k\|^2_{L^\infty((0,T);\mathbb{R})}+\|\psi\|^2\right)\|z(t)\|_H^2. \tag{28}$$

Similarly, by (11), we have

$$\begin{aligned}\|\bar{G}(t,\psi,\phi)\|^2 \leq & 6\big(d_2^2\|A\phi\|^2+a_2^2\|\phi\|^2+b_1^2\|F(\psi,\phi)\|^2+b_2^2\|G(\psi)\|^2+\|f_2(t)\|^2 \\ & +\|\sigma(t,\phi)\|^2_{L_2(H,\ell^2)}\|z(t)\|_H^2\big) \\ \leq & C_1\left(1+\|f_2(t)\|^2+\|\psi\|^2+\|\phi\|^2\right)+C_1\left(1+\sum_{k=1}^{\infty}\|h_k\|^2_{L^\infty((0,T);\mathbb{R})}+\|\phi\|^2\right)\|z(t)\|_H^2.\end{aligned}$$

By (7) and (5)-(6), we obtain that there exists $C_2=C_2(n)>0$ such that for all $t\in[0,T]$ and $\psi_1,\psi_2,\phi_1,\phi_2\in\ell^2$ with $\|\psi_1\|\leq n$, $\|\psi_2\|\leq n$, $\|\phi_1\|\leq n$, $\|\|\phi_2\|\leq n$,

$$\begin{aligned}&\|G(t,\psi_1,\phi_1)-G(t,\psi_2,\phi_2)\|^2 \\ =&\|-Ad_1(\psi_1-\psi_2)+b_1(F(\psi_1,\phi_1)-F(\psi_2,\phi_2))-b_2(G(\psi_1)-G(\psi_2)) \\ &-a_1(\psi_1-\psi_2)+(\sigma(t,\psi_1)-\sigma(t,\psi_2))z(t)\|^2 \\ \leq&5\Big(\|A\psi_1-A\psi_2\|^2+b_1\|F(\psi_1,\phi_1)-F(\psi_2,\phi_2)\|^2+b_2\|G(\psi_1)-G(\psi_2)\|^2+a_1\|\psi_1-\psi_2\|^2 \\ &+\|\sigma(t,\psi_1)-\sigma(t,\psi_2)\|^2_{L_2(H,\ell^2)}\|z(t)\|_H^2\Big) \\ \leq&C_2(1+\|z(t)\|_H^2)\left(\|\psi_1-\psi_2\|^2+\|\phi_1-\phi_2\|^2\right).\end{aligned}$$

Similarly,

$$\begin{aligned}&\|\bar{G}(t,\psi_1,\phi_1)-\bar{G}(t,\psi_2,\phi_2)\|^2 \\ =&\|-Ad_2(\phi_1-\phi_2)-b_1(F(\psi_1,\phi_1)-F(\psi_2,\phi_2))+b_2(G(\psi_1)-G(\psi_2)) \\ &-a_2(\phi_1-\phi_2)+(\sigma(t,\phi_1)-\sigma(t,\phi_2))z(t)\|^2 \\ \leq&C_2(1+\|z(t)\|_H^2)\left(\|\psi_1-\psi_2\|^2+\|\phi_1-\phi_2\|^2\right).\end{aligned} \tag{29}$$

From (28)-(29), it follows that for each $(\psi_0,\phi_0)\in\ell^2\times\ell^2$, system (26)-(27) has a unique local maximal solution $(\psi,\phi)\in C([0,T_0],\ell^2\times\ell^2)$, where $0<T_0\leq T$. Next we will prove that this solution is actually defined on the entire interval $[0,T]$ by the uniform estimates of solutions.

By (26) we have for all $t\in(0,T_0)$,

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\|\psi(t)\|^2 \\ =&-d_1\|A\psi(t)\|^2+\langle f_1(t),\psi(t)\rangle-a_1\|\psi(t)\|^2+b_1\langle F(\psi(t),\phi(t)),\psi(t)\rangle-b_2\langle G(\psi(t)),\psi(t)\rangle \\ &+\langle\sigma(t,\psi(t))z(t),\psi(t)\rangle,\end{aligned}$$

and

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\|\phi(t)\|^2 \\ =&-d_2\|A\phi(t)\|^2+\langle f_2(t),\phi(t)\rangle-a_2\|\phi(t)\|^2-b_1\langle F(\psi(t),\phi(t)),\phi(t)\rangle+b_2\langle G(\psi(t)),\phi(t)\rangle \\ &+\langle\sigma(t,\phi(t))z(t),\phi(t)\rangle,\end{aligned}$$

which imply that

$$\frac{d}{dt}\left(\|\psi(t)\|^2+\|\phi(t)\|^2\right)$$

$$\leq -2C(\|\psi(t)\|^2+\|\phi(t)\|^2)+2\langle f_1(t),\psi(t)\rangle+2\langle f_2(t),\phi(t)\rangle+2\langle\sigma(t,\psi(t))z(t),\psi(t)\rangle$$
$$+2\langle\sigma(t,\phi(t))z(t),\phi(t)\rangle. \tag{30}$$

For the third and the fourth term on the right-hand side of (30), we have

$$2\langle f_1(t),\psi(t)\rangle \leq \|f_1(t)\|+\|\psi(t)\|^2, \tag{31}$$

and

$$2\langle f_2(t),\phi(t)\rangle \leq \|f_2(t)\|+\|\phi(t)\|^2. \tag{32}$$

For the last two terms on the right-hand side of (30), by (8) and (11), we have

$$\begin{aligned}&2\,|\langle\sigma(t,\psi(t))z(t),\psi(t)\rangle|\\ \leq&\|\sigma(t,\psi(t))\|^2_{L_2(H,\ell^2)}\|z(t)\|^2_H+\|\psi(t)\|^2\\ \leq&2\|z(t)\|^2_H\sum_{k=1}^{\infty}\|h_k(t)\|^2+4\alpha^2\|\delta\|^2(1+\|\psi(t)\|^2)\|z(t)\|^2_H+\|\psi(t)\|^2,\end{aligned} \tag{33}$$

similarly,

$$\begin{aligned}&2\,|\langle\sigma(t,\phi(t))z(t),\phi(t)\rangle|\\ \leq&2\|z(t)\|^2_H\sum_{k=1}^{\infty}\|h_k(t)\|^2+4\alpha^2\|\delta\|^2(1+\|\phi(t)\|^2)\|z(t)\|^2_H+\|\phi(t)\|^2.\end{aligned} \tag{34}$$

By (30)-(34), we get for all $t\in(0,T_0)$,

$$\begin{aligned}&\frac{d}{dt}\left(\|\psi(t)\|^2+\|\phi(t)\|^2\right)\\ \leq&2(C+C\|\delta\|^2\|z(t)\|^2_H)\left(\|\psi(t)\|^2+\|\phi(t)\|^2\right)+\|f_1(t)\|^2+\|f_2(t)\|^2\\ &+2\left(4\alpha^2\|\delta\|^2+2\sum_{k=1}^{\infty}\|h_k(t)\|^2\right)\|z(t)\|^2_H\\ =:&\,a(t)(\|\psi(t)\|^2+\|\phi(t)\|^2)+b(t).\end{aligned} \tag{35}$$

Then by Gronwall's lemma, we obtain that for all $t\in[0,T_0]$,

$$\|\psi(t)\|^2+\|\phi(t)\|^2\leq e^{\int_0^t a(s)\,ds}\left(\|\psi_0\|^2+\|\phi_0\|^2\right)+\int_0^t e^{\int_s^t a(r)\,dr}b(s)\,ds. \tag{36}$$

In addition, notice that

$$\int_0^T a(t)dt\leq 2CT+2C\|\delta\|^2\|z(t)\|^2_{L_2(0,T;H)}=:a_0<\infty,$$

and

$$\begin{aligned}&\int_0^T b(t)dt\\ \leq&\int_0^T\left(\|f_1(t)\|^2+\|f_2(t)\|^2+2\sum_{k=1}^{\infty}\|h_k\|^2_{L_\infty(0,T;\ell^2)}+8\alpha^2\|\delta\|^2\right)\|z(t)\|^2_{L^2(0,T;H)}dt\\ =&b_0<\infty.\end{aligned}$$

It then follows from (36) that

$$\|\psi(t)\|^2 + \|\phi(t)\|^2 \leq e^{a_0}\left(\|\psi_0\|^2 + \|\phi_0\|^2\right) + e^{a_0} b_0 < \infty. \tag{37}$$

By (9) and (37), we infer that for each $n_1 > 0$ and $n_2 > 0$, there exists $C_3 = C_3(n_1, n_2, T) > 0$ such that for any $(\psi_0, \phi_0) \in \ell^2 \times \ell^2$ with $\|\psi_0\| \leq n_1$ and $\|\phi_0\| \leq n_1$, and for any $z \in L^2(0, T; H)$ with $\|z\|_{L^2(0,T;H)} \leq n_2$, the solution (ψ, ϕ) satisfies

$$\|\psi(t)\|^2 + \|\phi(t)\|^2 \leq C_3, \quad \forall t \in [0, T_0), \tag{38}$$

which implies that $T_0 = T$ and hence the solution (ψ, ϕ) of (26)-(27) is defined on the entire interval $[0, T]$. Next, we prove (24) and (25). Let z_1, z_2 be given in $L^2(0, T; H)$, and denote by

$$(\psi_1, \phi_1) = (\psi_{z_1}, \phi_{z_1}) \quad \text{and} \quad (\psi_2, \phi_2) = (\psi_{z_2}, \phi_{z_2}). \tag{39}$$

Suppose that $\|\psi_{0,1}\| \vee \|\psi_{0,2}\| \vee \|\phi_{0,1}\| \vee \|\phi_{0,2}\| \leq n_1$, $\|z_1\|_{L^2(0,T;H)}\| \vee \|z_2\|_{L^2(0,T;H)} \leq n_2$. Then by (38), we know that

$$\|\psi_1(t)\|^2 + \|\phi_1(t)\|^2 \leq C_3, \quad \forall t \in [0, T],$$

which implies (25).

By (26) we have

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\|\psi_1(t) - \psi_2(t)\|^2 \\
=& -d_1\langle A\psi_1(t) - A\psi_2(t), \psi_1(t) - \psi_2(t)\rangle + b_1\langle F(\psi_1(t), \phi_1(t)) - F(\psi_2(t), \phi_2(t)), \psi_1(t) - \psi_2(t)\rangle \\
& - b_2\langle G(\psi_1) - G(\psi_2), \psi_1 - \psi_2\rangle - a_1\|\psi_1 - \psi_2\|^2 \\
& + \langle \sigma(t, \psi_1(t))z_1(t) - \sigma(t, \psi_2(t))z_2(t),\ \psi_1(t) - \psi_2(t)\rangle,
\end{aligned} \tag{40}$$

and

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\|\phi_1(t) - \phi_2(t)\|^2 \\
=& -d_2\langle A\phi_1(t) - A\phi_2(t), \phi_1(t) - \phi_2(t)\rangle - b_1\langle F(\psi_1(t), \phi_1(t)) - F(\psi_2(t), \phi_2(t)), \phi_1(t) - \phi_2(t)\rangle \\
& + b_2\langle G(\psi_1) - G(\psi_2), \phi_1 - \phi_2\rangle - a_2\|\phi_1 - \phi_2\|^2 \\
& + \langle \sigma(t, \phi_1(t))z_1(t) - \sigma(t, \phi_2(t))z_2(t),\ \phi_1(t) - \phi_2(t)\rangle.
\end{aligned} \tag{41}$$

Then by (4)-(6), (40)-(41) we have

$$\begin{aligned}
&\frac{d}{dt}\left(\|\psi_1(t) - \psi_2(t)\|^2 + \|\phi_1(t) - \phi_2(t)\|^2\right) \\
\leq& 2C\left(\|\psi_1(t) - \psi_2(t)\|^2 + \|\phi_1(t) - \phi_2(t)\|^2\right) \\
& + 2\left\langle \sigma(t, \psi_1(t))z_1(t) - \sigma(t, \psi_2(t))z_2(t),\ \psi_1(t) - \psi_2(t)\right\rangle \\
& + 2\left\langle \sigma(t, \phi_1(t))z_1(t) - \sigma(t, \phi_2(t))z_2(t),\ \phi_1(t) - \phi_2(t)\right\rangle.
\end{aligned} \tag{42}$$

For the third term on the right-hand side of (42), by (11) and (25), we know there exist $C_4 = C_4(n_1, n_2, T) > 0$ and $C_5 = C_5(n_1, n_2, T) > 0$ such that for all $t \in [0, T]$,

$$\begin{aligned}
&2\left\langle \sigma(t, \psi_1(t))z_1(t) - \sigma(t, \psi_2(t))z_2(t),\ \psi_1(t) - \psi_2(t)\right\rangle \\
&\leq 2\|\sigma(t, \psi_1(t))z_1(t) - \sigma(t, \psi_2(t))z_2(t)\| \cdot \|\psi_1(t) - \psi_2(t)\| \\
&\leq 2\|\left(\sigma(t, \psi_1(t)) - \sigma(t, \psi_2(t))\right)z_1(t)\| \cdot \|\psi_1(t) - \psi_2(t)\| \\
&\quad + \|\sigma(t, \psi_2(t))(z_1(t) - z_2(t))\| \cdot \|\psi_1(t) - \psi_2(t)\|
\end{aligned}$$

$$
\begin{aligned}
&\le 2\|\sigma(t,\psi_1(t))-\sigma(t,\psi_2(t))\|_{L_2(H,\ell^2)}\|z_1(t)\|_H\|\psi_1(t)-\psi_2(t)\| \\
&\quad + 2\|\sigma(t,\psi_2(t))\|_{L_2(H,\ell^2)}\|z_1(t)-z_2(t)\|_H\|\psi_1(t)-\psi_2(t)\| \\
&\le 2\left(\left(\sum_{k=1}^{\infty}\|h_k\|^2_{L^\infty(0,T;\ell^2)}\right)^{1/2} + \sqrt{2}\alpha\|\delta\|(1+\|\psi_2(t)\|^2)^{1/2}\right) \\
&\quad \times \|z_1(t)-z_2(t)\|_H\|\psi_1(t)-\psi_2(t)\| + C_4\|z_1(t)\|_H\|\psi_1(t)-\psi_2(t)\|^2 \\
&\le C_4\|z_1(t)\|_H\|\psi_1(t)-\psi_2(t)\|^2 + 2C_5\|z_1(t)-z_2(t)\|_H\|\psi_1(t)-\psi_2(t)\| \\
&\le (1+C_4\|z_1(t)\|_H)\,\|\psi_1(t)-\psi_2(t)\|^2 + C_5^2\|z_1(t)-z_2(t)\|_H^2.
\end{aligned}
$$

Similarly, there exist $C_4 = C_4(n_1,n_2,T) > 0$ and $C_5 = C_5(n_1,n_2,T) > 0$ such that for all $t \in [0,T]$,

$$
\begin{aligned}
&2\langle \sigma(t,\phi_1(t))z_1(t) - \sigma(t,\phi_2(t))z_2(t), \phi_1(t)-\phi_2(t)\rangle \\
\le&(1+C_4\|z_1(t)\|_H)\|\phi_1(t)-\phi_2(t)\|^2 + C_5^2\|z_1(t)-z_2(t)\|_H^2.
\end{aligned} \tag{43}
$$

It follows from (42)-(43) that for all $t\in[0,T]$,

$$
\begin{aligned}
&\frac{d}{dt}\left(\|\psi_1(t)-\psi_2(t)\|^2 + \|\phi_1(t)-\phi_2(t)\|^2\right) \\
\le&(C_6+C_7\|z_1(t)\|_H)\left(\|\psi_1(t)-\psi_2(t)\|^2 + \|\phi_1(t)-\phi_2(t)\|^2\right) \\
&+2C_5^2\|z_1(t)-z_2(t)\|_H^2
\end{aligned} \tag{44}
$$

By Gronwall's lemma, we obtain that

$$
\begin{aligned}
&\|\psi_1(t)-\psi_2(t)\|^2 + \|\phi_1(t)-\phi_2(t)\|^2 \\
\le& e^{\int_0^t (C_6+C_7\|z_1(s)\|_H)ds}\left(\|\psi_{0,1}-\psi_{0,2}\|^2 + \|\phi_{0,1}-\phi_{0,2}\|^2\right) \\
&+2C_5^2\int_0^t e^{\int_s^t (C_6+C_7\|z_1(r)\|_H)dr}\|z_1(s)-z_2(s)\|_H^2 ds \\
\le& e^{C_6T+C_7\sqrt{T}\|z_1\|_{L^2(0,T;H)}}\left(\|\psi_{0,1}-\psi_{0,2}\|^2 + \|\phi_{0,1}-\phi_{0,2}\|^2\right) \\
&+2C_5^2 e^{C_6T+C_7\sqrt{T}\|z_1\|_{L^2(0,T;H)}}\|z_1-z_2\|^2_{L^2(0,T;H)}.
\end{aligned} \tag{45}
$$

Then (24) follows from (45) immediately. □

Based on Lemma 3.4, we can define $\mathcal{G}^0 : C([0,T],U) \to C([0,T],\ell^2\times\ell^2)$ by, for every $\xi \in C([0,T],U)$,

$$
\mathcal{G}^0(\xi) = \begin{cases} (\psi_z,\phi_z), & \text{if } \xi = \int_0^t z(t)dt \text{ for some } z\in L^2(0,T;H), \\ (0,0), & \text{otherwise,} \end{cases} \tag{46}
$$

where (ψ_z,ϕ_z) is the solution of (21)-(22) corresponding to the control term z.

4 Proof of the condition (i)

In this subsection, we will show that the condition (i) is fulfilled. Firstly we prove the following lemma about continuity in weak-strong topology.

Lemma 4.1. *Suppose that (7)-(9) hold. For fixed $(\xi,\bar{\xi}) \in L^\infty(0,T;\ell^2\times\ell^2)$, define an operator $\mathcal{T} : L^2(0,T;H) \to C([0,T],\ell^2\times\ell^2)$ by*

$$
\mathcal{T}(z)(t) = \left(\int_0^t \sigma(s,\xi(s))z(s)ds, \int_0^t \sigma(s,\bar{\xi}(s))z(s)ds\right), \quad \textit{for any } z\in L^2(0,T;H). \tag{47}
$$

Then the operator $\mathcal{T}$ is continuous from the weak topology of $L^2(0,T;H)$ to the strong topology of $C([0,T],\ell^2\times\ell^2)$.

Proof. At first we verify that the operator $\mathcal{T}: L^2(0,T;H)\to C([0,T],\ell^2\times\ell^2)$ is well defined. In fact, for every $z\in L^2(0,T;H)$, by (9) we have

$$\begin{aligned}
&\int_0^T \left(\|\sigma(s,\xi(s))z(s)\|^2+\|\sigma(s,\bar{\xi}(s))z(s)\|^2\right)ds\\
\le&\int_0^T \left(\|\sigma(s,\xi(s))\|^2_{L_2(H,\ell^2)}\|z(s)\|^2_H+\|\sigma(s,\bar{\xi}(s))\|^2_{L_2(H,\ell^2)}\|z(s)\|^2_H\right)ds\\
\le&2\int_0^T\left(\sum_{k=1}^{\infty}\|h_k\|^2_{L_\infty(0,T;\ell^2)}+2\alpha^2\|\delta\|^2(1+\|\xi(s)\|^2)\right)\|z(s)\|^2_H ds\\
&+2\int_0^T\left(\sum_{k=1}^{\infty}\|h_k\|^2_{L_\infty(0,T;\ell^2)}+2\alpha^2\|\delta\|^2(1+\|\bar{\xi}(s)\|^2)\right)\|z(s)\|^2_H ds\\
\le&2\left(\sum_{k=1}^{\infty}\|h_k\|^2_{L_\infty(0,T;\ell^2)}+\sum_{k=1}^{\infty}\|h_k\|^2_{L_\infty(0,T;\ell^2)}\right)\int_0^T\|z(s)\|^2_H ds\\
&+4\alpha^2\|\delta\|^2\left(2+\|\xi\|^2_{L^\infty(0,T;\ell^2)}+\|\bar{\xi}\|^2_{L^\infty(0,T;\ell^2)}\right)\int_0^T\|z(s)\|^2_H ds\\
=&C_1\int_0^T\|z(s)\|^2_H ds<\infty,
\end{aligned}\tag{48}$$

where C_1 is a positive constant, which implies that $\mathcal{T}(z)$ as given by (47) belongs to $C([0,T],\ell^2\times\ell^2)$ for all $z\in L^2(0,T;H)$. It is clear that $\mathcal{T}: L^2(0,T;H)\to C([0,T],\ell^2\times\ell^2)$ is linear. On the other hand, by (47) we have for every $z\in L^2(0,T;H)$,

$$\begin{aligned}
\|\mathcal{T}(z)(t)\|^2_{\ell^2\times\ell^2}=&\int_0^t\left(\|\sigma(s,\xi(s))z(s)\|^2+\|\sigma(s,\bar{\xi}(s))z(s)\|^2\right)ds\\
\le&t\int_0^t\left(\|\sigma(s,\xi(s))z(s)\|^2+\|\sigma(s,\bar{\xi}(s))z(s)\|^2\right)ds,
\end{aligned}\tag{49}$$

which along with (48) shows that

$$\begin{aligned}
\|\mathcal{T}(z)\|^2_{C([0,T],\ell^2\times\ell^2)}\le&T\int_0^T\left(\|\sigma(s,\xi(s))z(s)\|^2+\|\sigma(s,\bar{\xi}(s))z(s)\|^2\right)ds\\
\le&C_1T\|z\|^2_{L^2(0,T;H)}.
\end{aligned}\tag{50}$$

Thus $\mathcal{T}: L^2(0,T;H)\to C([0,T],\ell^2\times\ell^2)$ is bounded and hence continuous. Since $\mathcal{T}: L^2(0,T;H)\to C([0,T],\ell^2\times\ell^2)$ is linear and continuous in the strong topology, we can deduce that if $z_n\to z$ weakly in $L^2(0,T;H)$, then $\mathcal{T}(z_n)\to\mathcal{T}(z)$ in $L^2(0,T;H)$, then $\mathcal{T}(z_n)\to\mathcal{T}(z)$ weakly in $C([0,T],\ell^2\times\ell^2)$. Next, by using Ascoli-Arzelà theorem, we will prove actually that $\mathcal{T}(z_n)$ converges strongly in $C([0,T],\ell^2\times\ell^2)$. Indeed, from (48) and the boundedness of $\{z_n\}_{n=1}^{\infty}$ in $L^2(0,T;H)$, it follows that the sequence $\{\mathcal{T}(z_n)\}_{n=1}^{\infty}$ is equicontinuous on $[0,T]$. It remains to show that for every $t\in[0,T]$, the set $\{\mathcal{T}(z_n)(t): n\in\mathbb{Z}_+\}$ is totally bounded in $\ell^2\times\ell^2$. To do that, we need to prove the set $\{\mathcal{T}(z_n)(t): n\in\mathbb{Z}_+\}$ is totally bounded in $\ell^2\times\ell^2$.

For every $i\in\mathbb{Z}$, let $e_i=(\delta_{i,j})_{j\in\mathbb{Z}}$, where

$$\delta_{i,j}=\begin{cases}1, & i=j,\\ 0, & i\neq j.\end{cases}$$

Then $\{(e_i,0),(0,e_j)\}_{i,j\in\mathbb{Z}}$ is an orthonormal basis of $\ell^2\times\ell^2$. Given $m\in\mathbb{Z}_+$, let $P_m:\ell^2\to\text{span}\{e_i:|i|\le m\}$ be the projection operator and $Q_m=I-P_m$. Define $\tilde{P}_m:\ell^2\times\ell^2\to\text{span}\{(e_i,0),(0,e_j):|i|\le m,|j|\le m\}$ by $\tilde{P}_m(\psi,\phi)=(P_m\psi,P_m\phi)$. Let $\tilde{Q}_m=I-\tilde{P}_m$. Then $\tilde{Q}_m(\psi,\phi)=(Q_m\psi,Q_m\phi)$. Given $t\in[0,T]$, since $\sigma(t,\xi(t))$, $\sigma(t,\bar{\xi}(t)):H\to\ell^2$ are Hilbert-Schmidt, we have

$$\lim_{m\to\infty}\|Q_m(\sigma(t,\xi(t))\|^2_{L_2(H,\ell^2)}=0,$$
$$\lim_{m\to\infty}\|Q_m(\sigma(t,\bar{\xi}(t))\|^2_{L_2(H,\ell^2)}=0.$$

Then we get that

$$\lim_{m\to\infty}\|\tilde{Q}_m(\sigma(s,\xi(s)),\sigma(s,\bar{\xi}(s)))\|^2_{L_2(H,\ell^2\times\ell^2)}ds=0,$$

which, together with the dominated convergence theorem, implies that for every $t\in[0,T]$,

$$\lim_{m\to\infty}\int_0^t\|\tilde{Q}_m(\sigma(s,\xi(s)),\sigma(s,\bar{\xi}(s)))\|^2_{L_2(H,\ell^2\times\ell^2)}ds=0. \tag{51}$$

It follows from (47) that

$$\begin{aligned}\|\mathcal{T}(z_n)(t)\|_{\ell^2\times\ell^2}&\le\int_0^t\left(\|\sigma(s,\xi(s))z_n(s)\|^2+\|\sigma(s,\bar{\xi}(s))z_n(s)\|^2\right)^{\frac12}ds\\&\le\int_0^t\Big(\|\sigma(s,\xi(s))\|_{L_2(H,\ell^2)}\|z_n(s)\|_H+\|\sigma(s,\bar{\xi}(s))\|_{L_2(H,\ell^2)}\|z_n(s)\|_H\Big)ds\\&\le\left(\int_0^t\|\sigma(s,\xi(s))\|^2_{L_2(H,\ell^2)}\right)^{\frac12}\left(\int_0^t\|z_n(s)\|^2_Hds\right)^{\frac12}\\&\quad+\left(\int_0^t\|\sigma(s,\bar{\xi}(s))\|^2_{L_2(H,\ell^2)}\right)^{\frac12}\left(\int_0^t\|z_n(s)\|^2_Hds\right)^{\frac12}.\end{aligned} \tag{52}$$

Similarly, for every $m\in\mathbb{Z}_+$, we have

$$\begin{aligned}&\|\tilde{Q}_mT(z_n)(t)\|_{\ell^2\times\ell^2}\\&\le\int_0^t\|\tilde{Q}_m(\sigma(s,\xi(s))z_n(s)),\sigma(s,\bar{\xi}(s))z_n(s)\|ds\\&\le\left(\int_0^t\|Q_m(\sigma(s,\xi(s)))\|^2_{L_2(H,l^2)}ds\right)^{\frac12}\left(\int_0^t\|z_n(s)\|^2_Hds\right)^{\frac12}\\&\quad+\left(\int_0^t\|Q_m(\sigma(s,\bar{\xi}(s)))\|^2_{L_2(H,\ell^2)}ds\right)^{\frac12}\left(\int_0^t\|z_n(s)\|^2_Hds\right)^{\frac12}.\end{aligned} \tag{53}$$

Since $\{z_n\}_{n=1}^\infty$ is bounded in $L^2(0,T;H)$, it follows from (11), and (52)-(53) that there exists $C_2>0$ independent of $n,m\in\mathbb{Z}_+$ such that for any $n\in\mathbb{Z}_+$,

$$\|\mathcal{T}(z_n)(t)\|_{\ell^2\times\ell^2}\le C_2, \tag{54}$$

and for any $n,m\in\mathbb{Z}_+$,

$$\begin{aligned}&\|\tilde{Q}_mT(z_n)(t)\|_{l^2\times l^2}\\\le&C_2\left[\left(\int_0^t\|Q_m(\sigma(s,\xi(s)))\|^2_{L_2(H,\ell^2)}ds\right)^{\frac12}+\left(\int_0^t\|Q_m(\sigma(s,\bar{\xi}(s)))\|^2_{L_2(H,\ell^2)}ds\right)^{\frac12}\right].\end{aligned} \tag{55}$$

By (51) and (55), we find that for every $\eta > 0$, there exists $m_0 \in \mathbb{Z}_+$ such that for all $n \in \mathbb{Z}_+$ and $m \geq m_0$,

$$\|\tilde{Q}_m \mathcal{T}(z_n)(t)\|_{\ell^2\times\ell^2} \leq \frac{1}{4}\eta. \tag{56}$$

By (54) we know that $\{\tilde{P}_{m_0}(\mathcal{T}(z_n(t)))\}_{n=1}^{\infty}$ is bounded in a $(4m_0+2)$-dimensional space, and hence precompact. Therefore, $\{\tilde{P}_{m_0}(\mathcal{T}(z_n(t)))\}_{n=1}^{\infty}$ has a finite open cover of radius $\frac{1}{4}\eta$, which along with (56), shows that the sequence $\{\mathcal{T}(z_n)(t)\}_{n=1}^{\infty}$ has a finite open cover of radius η. Therefore, $\{\mathcal{T}(z_n)(t)\}_{n=1}^{\infty}$ is totally bounded and hence precompact in $\ell^2 \times \ell^2$.

By Ascoli-Arzelà theorem, there exists a subsequence $\{z_{n_k}\}_{k=1}^{\infty}$ of $\{z_n\}_{n=1}^{\infty}$ such that $\mathcal{T}(z_{n_k}) \to \mathcal{T}(z)$ strongly in $C([0,T],\ell^2\times\ell^2)$. Moreover, by a contradiction argument, we find that the entire sequence $T(z_n) \to T(z)$ strongly in $C([0,T],\ell^2\times\ell^2)$. This completes the proof. □

Based on Lemma 4.1, we will further prove the strong continuity of solution of control system (21)-(22) with respect to control in weak topology.

Lemma 4.2. *Suppose that (7)-(9) hold. Let $z, z_n \in L^2(0,T;H)$ for all $n \in \mathbb{Z}_+$, and (ψ_z,ϕ_z), (ψ_{z_n},ϕ_{z_n}) be the solutions of (21)-(22) corresponding to the controls z and z_n, respectively. If $z_n \to z$ weakly in $L^2(0,T;H)$, then $(\psi_{z_n},\phi_{z_n}) \to (\psi_z,\phi_z)$ strongly in $C([0,T],\ell^2\times\ell^2)$.*

Proof. Suppose $z_n \to z$ weakly in $L^2(0,T;H)$. Then $\{z_n\}_{n=1}^{\infty}$ is bounded in $L^2(0,T;H)$. Similar to (48), we find that there exists $C_1 = C_1(T) > 0$ such that

$$\sup_{0\leq t\leq T}\left\{(\|\psi_{z_n}(t)\|^2+\|\phi_{z_n}(t)\|^2)\vee(\|\psi_z(t)\|^2+\|\phi_z(t)\|^2)\right\} \leq C_1, \quad \text{for all} \quad n\in\mathbb{Z}_+. \tag{57}$$

By (21)-(22) we have

$$\begin{aligned}&\frac{d}{dt}\left(\psi_{z_n}(t)-\psi_z(t)\right)\\ =&-d_1\left(A\psi_{z_n}(t)-A\psi_z(t)\right)+b_1\left(F(\psi_{z_n}(t),\phi_{z_n}(t))-F(\psi_z(t),\phi_z(t))\right)-a_1(\psi_{z_n}(t)-\psi_z(t))\\ &-b_2\left(G(\psi_{z_n})-G(\psi_z)\right)+\left(\sigma(t,\psi_{z_n}(t))z_n(t)-\sigma(t,\psi_z(t))z(t)\right),\end{aligned} \tag{58}$$

and

$$\begin{aligned}&\frac{d}{dt}\left(\phi_{z_n}(t)-\phi_z(t)\right)\\ =&-d_2\left(A\phi_{z_n}(t)-A\phi_z(t)\right)-b_1\left(F(\psi_{z_n}(t),\phi_{z_n}(t))-F(\psi_z(t),\phi_z(t))\right)-a_2(\phi_{z_n}(t)-\phi_z(t))\\ &+b_2\left(G(\phi_{z_n})-G(\phi_z)\right)+\left(\sigma(t,\phi_{z_n}(t))z_n(t)-\sigma(t,\phi_z(t))z(t)\right).\end{aligned} \tag{59}$$

By (5)-(6), (7)-(9) and (57) we get for all $n \in \mathbb{Z}_+$

$$\left\|\frac{d}{dt}\left(\psi_{z_n}(t)-\psi_z(t)\right)\right\| \leq C_2\left(1+\|z_n(t)\|_H+\|z(t)\|_H\right), \tag{60}$$

and

$$\left\|\frac{d}{dt}\left(\phi_{z_n}(t)-\phi_z(t)\right)\right\| \leq C_2\left(1+\|z_n(t)\|_H+\|z(t)\,\|\right. \tag{61}$$

where $C_2 = C_2(T) > 0$ is a constant independent of n. Similar to (40)-(41), by (58)-(59) and discrete p-Laplace operator A, $F(\psi,\phi)$, $G(\psi)$ satisfy the locally Lipschitz condition we have

$$\frac{d}{dt}\left(\|\psi_{z_n}(t)-\psi_z(t)\|^2+\|\phi_{z_n}(t)-\phi_z(t)\|^2\right)$$

$$
\begin{aligned}
&\leq 2C\left(\|\psi_{z_n}(t)-\psi_z(t)\|^2+\|\phi_{z_n}(t)-\phi_z(t)\|^2\right)\\
&\quad+\underbrace{2\left\langle\sigma(t,\psi_{z_n}(t))z_n(t)-\sigma(t,\psi_z(t))z(t),\ \psi_{z_n}(t)-\psi_z(t)\right\rangle}_{I_1}\\
&\quad+\underbrace{2\left\langle\sigma(t,\phi_{z_n}(t))z_n(t)-\sigma(t,\phi_z(t))z(t),\ \phi_{z_n}(t)-\phi_z(t)\right\rangle}_{I_2}. \qquad (62)
\end{aligned}
$$

For the second term on the right-hand side of (62), we have

$$
\begin{aligned}
I_1 &=2\langle\sigma(t,\psi_{z_n}(t))z_n(t)-\sigma(t,\psi_z(t))z(t),\psi_{z_n}(t)-\psi_z(t)\rangle\\
&=2\langle(\sigma(t,\psi_{z_n}(t))-\sigma(t,\psi_z(t)))z_n(t),\ \psi_{z_n}(t)-\psi_z(t)\rangle\\
&\quad+2\langle\sigma(t,\psi_z(t))(z_n(t)-z(t)),\ \psi_{z_n}(t)-\psi_z(t)\rangle\\
&=:I_{11}+I_{12}. \qquad (63)
\end{aligned}
$$

By (7) and (57) we know that

$$
\begin{aligned}
I_{11} &\leq 2\|\sigma(t,\psi_{z_n}(t))-\sigma(t,\psi_z(t))\|_{L_2(H,\ell^2)}\|z_n(t)\|_H\|\psi_{z_n}(t)-\psi_z(t)\|\\
&\leq C_3\|\delta\|^2\|z_n(t)\|_H\|\psi_{z_n}(t)-\psi_z(t)\|^2, \qquad (64)
\end{aligned}
$$

where $C_3=C_3(T)>0$ is a constant independent of $n\in\mathbb{Z}_+$. For each $n\in\mathbb{Z}_+$ and $t\in[0,T]$, set

$$
\widehat{\varphi}_n(t):=\int_0^t\sigma(s,\psi_z(s))(z_n(s)-z(s))ds. \qquad (65)
$$

Since $z_n\to z$ weakly in $L^2(0,T;H)$, by Lemma 4.1 we have

$$
\widehat{\varphi}_n\to 0 \text{ in } C([0,T],\ell^2) \text{ as } n\to\infty. \qquad (66)
$$

By (60) we have

$$
\begin{aligned}
I_{12} &=2\left\langle\frac{d}{dt}\widehat{\varphi}_n(t),\psi_{z_n}(t)-\psi_z(t)\right\rangle\\
&=2\frac{d}{dt}\left(\widehat{\varphi}_n(t),\psi_{z_n}(t)-\psi_z(t)\right)-2\left\langle\widehat{\varphi}_n(t),\frac{d}{dt}(\psi_{z_n}(t)-\psi_z(t))\right\rangle\\
&\leq 2\frac{d}{dt}\left(\widehat{\varphi}_n(t),\psi_{z_n}(t)-\psi_z(t)\right)+2C_2\left(1+\|z_n(t)\|_H+\|z(t)\|_H\right)\|\widehat{\varphi}_n(t)\|. \qquad (67)
\end{aligned}
$$

It follows from (63)-(64) and (67) that

$$
\begin{aligned}
I_1 &\leq 2C_3\|\delta\|\|z_n(t)\|_H\|\psi_{z_n}(t)-\psi_z(t)\|^2+2\frac{d}{dt}\left(\widehat{\varphi}_n(t),\psi_{z_n}(t)-\psi_z(t)\right)\\
&\quad+2C_2\left(1+\|z_n(t)\|_H+\|z(t)\|_H\right)\|\widehat{\varphi}_n(t)\|. \qquad (68)
\end{aligned}
$$

For each $n\in\mathbb{Z}_+$ and $t\in[0,T]$, set

$$
\widetilde{\varphi}_n(t):=\int_0^t\sigma(s,\phi_z(s))(z_n(s)-z(s))ds. \qquad (69)
$$

Similar to (66), we obtain

$$
\widetilde{\varphi}_n\to 0 \text{ in } C([0,T],l^2),\ \text{as } n\to\infty. \qquad (70)
$$

And similar to (68), we have

$$I_2 \le 2C_3\|\delta\|\|z_n(t)\|_H\|\phi_{z_n}(t)-\phi_z(t)\|^2 + 2\frac{d}{dt}\langle\widetilde{\varphi}_n(t),\phi_{z_n}(t)-\phi_z(t)\rangle \\ + 2C_2\left(1+\|z_n(t)\|_H+\|z(t)\|_H\right)\|\widetilde{\varphi}_n(t)\|. \tag{71}$$

where C_2, C_3 are constants as above.

It follows from (62), (68) and (71) that

$$\begin{aligned}&\frac{d}{dt}\left(\|\psi_{z_n}(t)-\psi_z(t)\|^2+\|\phi_{z_n}(t)-\phi_z(t)\|^2\right)\\ &\le\left(C+2C_3\|\delta\|\|z_n(t)\|_H\|\psi_{z_n}(t)-\psi_z(t)\|^2\right)+\left(C+2C_3\|\delta\|\|z_n(t)\|_H\|\phi_{z_n}(t)-\phi_z(t)\|^2\right)\\ &\quad+2\frac{d}{dt}\left(\varphi_n(t),\psi_{z_n}(t)-\psi_z(t)\right)+2\frac{d}{dt}\left(\psi_n(t),\phi_{z_n}(t)-\phi_z(t)\right)\\ &\quad+\left(2C_2\|\widehat{\varphi}_n(t)\|+2C_2\|\widetilde{\varphi}_n(t)\|\right)\left(1+\|z_n(t)\|_H+\|z(t)\|_H\right).\end{aligned} \tag{72}$$

Since $\psi_{z_n}(0)=\psi_0$, $\phi_{z_n}(0)=\phi_0$, integrating (72) on $(0,t)$, we have

$$\begin{aligned}&\|\psi_{z_n}(t)-\psi_z(t)\|^2+\|\phi_{z_n}(t)-\phi_z(t)\|^2\\ &\le\int_0^t\left(C+2C_3\|\delta\|\|z_n(s)\|_H\|\psi_{z_n}(s)-\psi_z(s)\|^2+C+2C_3\|\delta\|\|z_n(s)\|_H\|\phi_{z_n}(s)-\phi_z(s)\|^2\right)ds\\ &\quad+2\left(\varphi_n(t),\psi_{z_n}(t)-\psi_z(t)\right)+2\left(\psi_n(t),\phi_{z_n}(t)-\phi_z(t)\right)\\ &\quad+\int_0^t\left(2C_2\|\widehat{\varphi}_n(s)\|+2C_2\|\widetilde{\varphi}_n(s)\|\right)\left(1+\|z_n(s)\|_H+\|z(s)\|_H\right)ds.\end{aligned} \tag{73}$$

For the third and fourth terms on the right-hand side of (73), we have

$$2\langle\widehat{\varphi}_n(t),\psi_{z_n}(t)-\psi_z(t)\rangle\le 2\|\widehat{\varphi}_n(t)\|^2_{C([0,T],\ell^2)}+\frac{1}{2}\|\psi_{z_n}(t)-\psi_z(t)\|^2, \tag{74}$$

and

$$2\langle\widetilde{\varphi}_n(t),\phi_{z_n}(t)-\phi_z(t)\rangle\le 2\|\widetilde{\varphi}_n(t)\|^2_{C([0,T],\ell^2)}+\frac{1}{2}\|\phi_{z_n}(t)-\phi_z(t)\|^2. \tag{75}$$

For the last term on the right-hand side of (73), we obtain

$$\begin{aligned}&\int_0^t\left(2C_2\|\widehat{\varphi}_n(s)\|+2C_2\|\widetilde{\varphi}_n(s)\|\right)\left(1+\|z_n(s)\|_H+\|z(s)\|_H\right)ds\\ &\le\left(2C_2T^{\frac12}\|\widehat{\varphi}_n\|_{C([0,T],\ell^2)}+2C_2T^{\frac12}\|\widetilde{\varphi}_n\|_{C([0,T],\ell^2)}\right)\left(T^{\frac12}+\|z_n\|_{L^2(0,T;H)}+\|z\|_{L^2(0,T;H)}\right).\end{aligned} \tag{76}$$

By (73)-(76) we have for all $t\in[0,T]$,

$$\begin{aligned}&\|\psi_{z_n}(t)-\psi_z(t)\|^2+\|\phi_{z_n}(t)-\phi_z(t)\|^2\\ &\le\int_0^t\left(C\|\delta\|\|z_n(s)\|_H\left(\|\psi_{z_n}(s)-\psi_z(s)\|^2+\|\phi_{z_n}(s)-\phi_z(s)\|^2\right)\right)ds\\ &\quad+\left(2C_2T^{\frac12}\|\widehat{\varphi}_n\|_{C([0,T],\ell^2)}+2C_2T^{\frac12}\|\widetilde{\varphi}_n\|_{C([0,T],\ell^2)}\right)\left(T^{\frac12}+\|z_n\|_{L^2(0,T;H)}+\|z\|_{L^2(0,T;H)}\right)\\ &\quad+2\left(\|\widehat{\varphi}_n\|^2_{C([0,T],\ell^2)}+\|\widetilde{\varphi}_n\|^2_{C([0,T],\ell^2)}\right)\\ &=:\int_0^t a(s)\left(\|\psi_{z_n}(s)-\psi_z(s)\|^2+\|\phi_{z_n}(s)-\phi_z(s)\|^2\right)ds+b_n.\end{aligned} \tag{77}$$

By (77) and Gronwall's lemma we obtain, for all $t \in [0, T]$,

$$\|\psi_{z_n}(t) - \psi_z(t)\|^2 + \|\phi_{z_n}(t) - \phi_z(t)\|^2 \le b_n e^{\int_0^t a(s)ds}. \tag{3.60}$$

Note that the integral of

$$\int_0^t a(s)\, ds \le CT^{\frac{1}{2}} \|\delta\| \|z_n\|_{L^2(0,T;H)}. \tag{78}$$

Since $\{z_n\}_{n=1}^{\infty}$ is bounded in $L^2(0, T; H)$, by (77)-(78) we know there exists constant $C_4 = C_4(T) > 0$ independent of $n \in \mathbb{Z}_+$ such that for any $t \in [0, T]$,

$$\begin{aligned} &\|\psi_{z_n}(t) - \psi_z(t)\|^2 + \|\phi_{z_n}(t) - \phi_z(t)\|^2 \\ \le &C_4 \left(\|\widehat{\varphi}_n\|^2_{C([0,T],\ell^2)} + \|\widetilde{\varphi}_n\|^2_{C([0,T],\ell^2)} + \|\widehat{\varphi}_n\|_{C([0,T],\ell^2)} + \|\widetilde{\varphi}_n\|_{C([0,T],\ell^2)} \right). \end{aligned} \tag{79}$$

It follows from (66), (70) and (79) that

$$\sup_{0 \le s \le T} \left(\|\psi_{z_n}(s) - \psi_z(s)\|^2 + \|\phi_{z_n}(s) - \phi_z(s)\|^2 \right) \to 0 \quad \text{as} \quad n \to \infty,$$

which concludes the proof. □

The following lemma confirms that condition (i) is satisfied.

Lemma 4.3. *Suppose that* (7)-(9) *hold. Then for every $N < \infty$, the set*

$$K_N = \left\{ \mathcal{G}^0 \left(\int_0^{\cdot} z(t) dt \right) : z \in S_N \right\}, \tag{80}$$

is a compact subset of $C([0, T], \ell^2 \times \ell^2)$, where S_N is the set as defined by (16).

Proof. By (46) and (80) we find that

$$K_N = \{(\psi_z, \phi_z) : z \in S_N\} = \left\{ (\psi_z, \phi_z) : z \in L^2(0, T; H), \int_0^T \|z(t)\|_H^2 dt \le N \right\}, \tag{81}$$

where (ψ_z, ϕ_z) is the solution of (21)-(22). Let $\{(\psi_{z_n}, \phi_{z_n})\} \subset K_N$. Then $z_n \in L^2(0, T; H)$ and $\int_0^T \|z_n(t)\|_H^2 dt \le N$, which means there exists $z \in S_N$ and a subsequence $\{z_{n_k}\}_{k=1}^{\infty}$ such that $z_{n_k} \to z$ weakly in $L^2(0, T; H)$. Then by Lemma 4.2 we find that $(\psi_{z_{n_k}}, \phi_{z_{n_k}}) \to (\psi_z, \phi_z)$ strongly in $C([0, T], \ell^2 \times \ell^2)$, as desired. □

In the subsection, we will verify the condition (ii). For this purpose we first obtain the property of the measurable map $\mathcal{G}^\gamma$ in the following lemma.

Lemma 4.4. *Suppose that* (7)-(9) *hold, and $z \in \mathcal{A}_N$ for some $N < \infty$, where $\mathcal{A}_N$ is the set as defined by* (17). *If $(\psi_z^\gamma, \phi_z^\gamma) = \mathcal{G}^\gamma \left(W + \sqrt{\gamma} \int_0 z(t)\, dt \right)$, then $(\psi_z^\gamma, \phi_z^\gamma)$ is the unique solution of the system.*

$$\begin{cases} d\psi_z^\gamma(t) = \left(d_1 A \psi_z^\gamma(t) - a_1 \psi_z^\gamma(t) + b_1 F(\psi_z^\gamma, \phi_z^\gamma) - b_2 G(\psi_z^\gamma(t)) \right) dt \\ \qquad\qquad + f_1(t) dt + \sqrt{\gamma}\, \sigma(t, \psi_z^\gamma(t)) dW(t) + \sigma(t, \psi_z^\gamma(t)) z(t)\, dt, \quad t > 0, \\ d\phi_z^\gamma(t) = \left(d_2 A \phi_z^\gamma(t) - a_2 \phi_z^\gamma(t) - b_1 F(\psi_z^\gamma, \phi_z^\gamma) + b_2 G(\psi_z^\gamma(t)) \right) dt \\ \qquad\qquad + f_2(t) dt + \sqrt{\gamma}\, \sigma(t, \phi_z^\gamma(t)) dW_k(t) + \sigma(t, \phi_z^\gamma(t)) z(t)\, dt, \quad t > 0, \end{cases} \tag{82}$$

with initial data

$$(\psi_z^\gamma(0), \phi_z^\gamma(0)) = (\psi_0, \phi_0) \in \ell^2 \times \ell^2, \tag{83}$$

Furthermore, for each $n > 0$ there exists $C = C(n, T, N) > 0$ such that for any $(\psi_0, \phi_0) \in \ell^2 \times \ell^2$ with $\|\psi_0\| \le n$ and $\|\phi_0\| \le n$ and for any $z \in \mathcal{A}_N$, the solution $(\psi_z^\gamma, \phi_z^\gamma)$ satisfies for all $\gamma \in (0,1)$

$$E\left[\|(\psi_z^\gamma, \phi_z^\gamma)\|^2_{C([0,T],\ell^2\times\ell^2)}\right] \le C. \tag{84}$$

Proof. By Girsanov's theorem, we deduce that (u_z^γ, v_z^γ) is the unique solution of (82)-(83). Next we show (84). From (82)-(83) and It'o formula, it follows that for all $t \in [0, T]$, P-almost surely,

$$\begin{aligned}
\|\psi_z^\gamma(t)\|^2 =& \|\psi_0\|^2 - 2d_1 \int_0^t \langle A\psi_z^\gamma(s), \psi_z^\gamma(s)\rangle ds + 2b_1 \int_0^t \langle F(\psi_z^\gamma(s), \phi_z^\gamma(s)), \psi_z^\gamma(s)\rangle ds \\
&- 2b_2 \int_0^t \langle G(\psi_z^\gamma(s)), \psi_z^\gamma(s)\rangle ds - 2a_1 \int_0^t \|\psi_z^\gamma(s)\|^2 ds + 2\int_0^t \langle f_1(s), \psi_z^\gamma(s)\rangle ds \\
&+ 2\int_0^t \langle \sigma(s, \psi_z^\gamma(s))z(s), \psi_z^\gamma(s)\rangle ds + \gamma \int_0^t \|\sigma(s, \psi_z^\gamma(s))\|^2_{L_2(H,\ell^2)} ds \\
&+ 2\sqrt{\gamma} \int_0^t \langle \psi_z^\gamma(s), \sigma(s, \psi_z^\gamma(s))\rangle dW(s),
\end{aligned} \tag{85}$$

and

$$\begin{aligned}
\|\phi_z^\gamma(t)\|^2 =& \|\phi_0\|^2 - 2d_2 \int_0^t \langle A\phi_z^\gamma(s), \phi_z^\gamma(s)\rangle ds - 2b_1 \int_0^t \langle F(\psi_z^\gamma(s), \phi_z^\gamma(s)), \phi_z^\gamma(s)\rangle ds \\
&+ 2b_2 \int_0^t \langle G(\psi_z^\gamma(s)), \phi_z^\gamma(s)\rangle ds - 2a_2 \int_0^t \|\phi_z^\gamma(s)\|^2 ds + 2\int_0^t \langle f_2(s), \phi_z^\gamma(s)\rangle ds \\
&+ 2\int_0^t \langle \sigma(s, \phi_z^\gamma(s))z(s), \phi_z^\gamma(s)\rangle ds + \gamma \int_0^t \|\sigma(s, \phi_z^\gamma(s))\|^2_{L_2(H,\ell^2)} ds \\
&+ 2\sqrt{\gamma} \int_0^t \langle \phi_z^\gamma(s), \sigma(s, \phi_z^\gamma(s))\rangle dW(s).
\end{aligned} \tag{86}$$

It follows from (85) and (86) that

$$\begin{aligned}
& b_2\|\psi_z^\gamma(t)\|^2 + b_1\|\phi_z^\gamma(t)\|^2 \\
\le& b_2\|\psi_0\|^2 + b_1\|\phi_0\|^2 + a_1^{-1}b_2 \int_\tau^t \|f_1(s)\|^2 ds + a_2^{-1}b_1 \int_\tau^t \|f_2(s)\|^2 ds \\
&+ \gamma \int_0^t \|\sigma(s, \phi_z^\gamma(s))\|^2_{L_2(H,\ell^2)} ds + \gamma \int_0^t \|\sigma(s, \psi_z^\gamma(s))\|^2_{L_2(H,\ell^2)} ds \\
&+ 2\int_0^t \langle \sigma(s, \phi_z^\gamma(s))z(s), \phi_z^\gamma(s)\rangle ds + 2\int_0^t \langle \sigma(s, \psi_z^\gamma(s))z(s), \psi_z^\gamma(s)\rangle ds \\
&+ 2\sqrt{\gamma} \int_0^t \langle \psi_z^\gamma(s), \sigma(s, \psi_z^\gamma(s))\rangle dW(s) + 2\sqrt{\gamma} \int_0^t \langle \phi_z^\gamma(s), \sigma(s, \phi_z^\gamma(s))\rangle dW(s),
\end{aligned} \tag{87}$$

note that

$$b_2 b_1\langle F(\psi, \phi), \psi\rangle - b_2^2\langle G(\psi), \psi\rangle - b_1^2\langle F(\psi, \phi), \phi\rangle + b_2 b_1\langle G(\psi), \phi\rangle \le 0.$$

Since $z \in \mathcal{A}_N$, by (11) we obtain

$$2\int_0^t |\langle \sigma(s, \psi_z^\gamma(s))z(s), \psi_z^\gamma(s)\rangle|\, ds$$

$$
\begin{aligned}
&\leq 2\int_0^t \|\sigma(s,\psi_z^\gamma(s))\|_{L_2(H,\ell^2)}\|z(s)\|_H\|\psi_z^\gamma(s)\|\,ds \\
&\leq 2\left(\int_0^t \|\sigma(s,\psi_z^\gamma(s))\|_{L_2(H,\ell^2)}\|\psi_z^\gamma(s)\|^2\,ds\right)^{1/2}\left(\int_0^t \|z(s)\|_H^2\,ds\right)^{1/2} \\
&\leq 2N^{\frac{1}{2}}\sup_{0\leq s\leq t}\|\psi_z^\gamma(s)\|\left(\int_0^t \|\sigma(s,\psi_z^\gamma(s))\|_{L_2(H,\ell^2)}\,ds\right)^{1/2} \\
&\leq \frac{1}{4}\sup_{0\leq s\leq t}\|\psi_z^\gamma(s)\|^2+4N\int_0^t \|\sigma(s,\psi_z^\gamma(s))\|_{L_2(H,\ell^2)}\,ds \\
&\leq \frac{1}{4}\sup_{0\leq s\leq t}\|\psi_z^\gamma(s)\|^2+8NT\sum_{k=1}^{\infty}\|h_k\|^2_{L^\infty(0,T;\ell^2)}+16N\alpha^2\|\delta\|^2T \\
&\quad +16N\alpha^2\|\delta\|^2\int_0^t \|\psi_z^\gamma(s)\|^2\,ds, \qquad (88)
\end{aligned}
$$

and

$$
\begin{aligned}
&2\int_0^t |\langle\sigma(s,\phi_z^\gamma(s))z(s),\phi_z^\gamma(s)\rangle|\,ds \\
&\leq \frac{1}{4}\sup_{0\leq s\leq t}\|\phi_z^\gamma(s)\|^2+8NT\sum_{k=1}^{\infty}\|h_k\|^2_{L^\infty(0,T;\ell^2)}+16N\alpha^2\|\delta\|^2T \\
&\quad +16N\alpha^2\|\delta\|^2\int_0^t \|\phi_z^\gamma(s)\|^2\,ds. \qquad (89)
\end{aligned}
$$

Similarly, for all $\epsilon\in(0,1)$ we have

$$
\gamma\int_0^t \|\sigma(s,\psi_z^\gamma(s))\|_{L_2(H,\ell^2)}\,ds \leq 2T\sum_{k=1}^{\infty}\|h_k\|^2_{L^\infty(0,T;\ell^2)}+4\alpha^2\|\delta\|^2T+4\alpha^2\|b\|^2\int_0^t \|\psi_z^\gamma(s)\|^2\,ds, \quad (90)
$$

and

$$
\gamma\int_0^t \|\sigma(s,\phi_z^\gamma(s))\|^2_{L^2(H,\ell^2)}ds \leq 2T\sum_{k=1}^{\infty}\|h_k\|^2_{L^\infty(0,T;\ell^2)}+4\alpha^2\|\delta\|^2T+4\alpha^2\|\delta\|^2\int_0^t \|\phi_z^\gamma(s)\|^2ds. \quad (91)
$$

For the last two terms of (87), by the Burkholder-Davis-Gundy inequality, (90) and (91), we get for $\gamma\in(0,1)$,

$$
\begin{aligned}
&2\sqrt{\gamma}\,\mathbb{E}\left[\sup_{0\leq s\leq t}\int_0^s \langle\psi_z^\gamma(r),\sigma(r,\psi_z^\gamma(r))\,dW(r)\rangle\right] \\
&\leq 6\,\mathbb{E}\left(\int_0^t \|\psi_z^\gamma(s)\|^2\|\sigma(s,\psi_z^\gamma(s))\|^2_{L_2(H,\ell^2)}\,ds\right)^{1/2} \\
&\leq 6\,\mathbb{E}\left[\sup_{0\leq s\leq t}\|\psi_z^\gamma(s)\|\left(\int_0^t \|\sigma(s,\psi_z^\gamma(s))\|^2_{L_2(H,\ell^2)}\,ds\right)^{1/2}\right] \\
&\leq \frac{1}{4}\,\mathbb{E}\left[\sup_{0\leq s\leq t}\|\psi_z^\gamma(s)\|^2\right]+36\,\mathbb{E}\left[\int_0^t \|\sigma(s,\psi_z^\gamma(s))\|^2_{L_2(H,\ell^2)}\,ds\right] \\
&\leq \frac{1}{4}\,\mathbb{E}\sup_{0\leq s\leq t}\|\psi_z^\gamma(s)\|^2+72T\sum_{k=1}^{\infty}\|h_k\|^2_{L^\infty(0,T;\ell^2)}+144a^2\|\delta\|^2T+144a^2\|\delta\|^2\,\mathbb{E}\left[\int_0^t \|\psi_z^\gamma(s)\|^2\,ds\right],
\end{aligned}
$$

and similarly,

$$
2\sqrt{\gamma}\,\mathbb{E}\left[\sup_{0\leq s\leq t}\int_0^s \langle\phi_z^\gamma(r),\sigma(r,\phi_z^\gamma(r))\,dW(r)\rangle\right]
$$

$$\leq \frac{1}{4}\mathbb{E}\sup_{0\leq s\leq t}\|\phi_z^\gamma(s)\|^2 + 72T\sum_{k=1}^{\infty}\|h_k\|^2_{L^\infty(0,T;\ell^2)} + 144a^2\|\delta\|^2T + 144a^2\|\delta\|^2\,\mathbb{E}\left[\int_0^t \|\phi_z^\gamma(s)\|^2\,ds\right], \tag{92}$$

By (87)-(92), we find that there exists a positive number C independent of such that for all $t \in [0,T]$,

$$\begin{aligned}
&\mathbb{E}\left[\sup_{0\leq s\leq t}\left(\|\psi_z^\gamma(t)\|^2 + \|\phi_z^\gamma(t)\|^2\right)\right]\\
\leq& CT + C\int_0^t \mathbb{E}\left[\sup_{0\leq r\leq s}\left(\|\psi_z^\gamma(r)\|^2 + \|\phi_z^\gamma(r)\|^2\right)\right]\mathrm{d}s\\
&+2C\int_0^t\left(\sum_{k=1}^{\infty}\|h_k\|^2_{L^\infty(0,T;\ell^2)} + \|f_1(s)\| + \|f_2(s)\|\right)ds
\end{aligned} \tag{93}$$

Applying Gronwall's lemma and (9) we have for all $t\in[0,T]$,

$$\mathbb{E}\left[\sup_{0\leq s\leq t}\left(\|\psi_z^\gamma(s)\|^2 + \|\phi_z^\gamma(s)\|^2\right)\right] \leq C_2\,\mathrm{e}^{C_1 t}, \tag{94}$$

where $C_2 > 0$, $C_1 > 0$.

Then (84) follows from (94) for $t = T$.

□

In the following lemma, we verify the condition (ii)

Lemma 4.5. *Suppose that (7)-(9) hold, and $\{z^\gamma\} \subset \mathcal{A}_N$ for some $N < \infty$. If $\{z^\gamma\}$ converges in distribution to z as S_N-valued random variables, then $\mathcal{G}^\gamma\left(W + \gamma^{-\frac{1}{2}}\int_0^t z^\gamma(t)dt\right)$ converges to $\mathcal{G}^0\left(\int_0 z(t)dt\right)$ in $C([0,T],\ell^2\times\ell^2)$ in distribution.*

Proof. Let $(\psi_{z^\gamma}^\gamma, \phi_{z^\gamma}^\gamma) = \mathcal{G}^\gamma\left(W + \gamma^{-\frac{1}{2}}\int_0 z^\gamma(t)\,dt\right)$. Then by Lemma 4.4 we see that $(\psi_{z^\gamma}^\gamma, \phi_{z^\gamma}^\gamma)$is the solution of the system

$$\begin{cases}
\mathrm{d}\psi_{z^\gamma}^\gamma(t) = \left(d_1A\psi_{z^\gamma}^\gamma(t) - a_1\psi_{z^\gamma}^\gamma(t) + b_1F(\psi_{z^\gamma}^\gamma,\phi_{z^\gamma}^\gamma) - b_2G(\psi_{z^\gamma}^\gamma(t))\right)\mathrm{d}t\\
\qquad\qquad + f_1(t)\,\mathrm{d}t + \sqrt{\gamma}\,\sigma(t,\psi_{z^\gamma}^\gamma(t))\,\mathrm{d}W(t) + \sigma(t,\psi_{z^\gamma}^\gamma(t))z^\gamma(t)\,\mathrm{d}t, \quad t>0,\\
\mathrm{d}\phi_{z^\gamma}^\gamma(t) = \left(d_2A\phi_{z^\gamma}^\gamma(t) - a_2\phi_{z^\gamma}^\gamma(t) - b_1F(\psi_{z^\gamma}^\gamma,\phi_{z^\gamma}^\gamma) + b_2G(\psi_{z^\gamma}^\gamma(t))\right)\mathrm{d}t\\
\qquad\qquad + f_2(t)\,\mathrm{d}t + \sqrt{\gamma}\,\sigma(t,\phi_{z^\gamma}^\gamma(t))\,\mathrm{d}W(t) + \sigma(t,\phi_{z^\gamma}^\gamma(t))z^\gamma(t)\,\mathrm{d}t, \quad t>0,
\end{cases} \tag{95}$$

with initial data

$$(\psi_{z^\gamma}^\gamma(0), \phi_{z^\gamma}^\gamma(0)) = (\psi_0,\phi_0) \in \ell^2\times\ell^2, \tag{96}$$

Let $(\psi_z,\phi_z) = \mathcal{G}^0\left(\int_0 z(t)dt\right)$. Then (ψ_z,ϕ_z) is the solution of (21)-(22). Now we only need to prove that $(\psi_{z^\gamma}^\gamma,\phi_{z^\gamma}^\gamma)$ converges to (ψ_z,ϕ_z) in $C([0,T],\ell^2\times\ell^2)$ in distribution as $\gamma\to 0$. To that end, we first establish the convergence of $(\psi_{z^\gamma}^\gamma - \psi_{z^\gamma}, \phi_{z^\gamma}^\gamma - \phi_{z^\gamma})$ with $(\psi_{z^\gamma},\phi_{z^\gamma}) = \mathcal{G}^0\left(\int_0 z^\gamma(t)\,dt\right)$.

By (21)-(22) we have

$$\begin{cases} \dfrac{\mathrm{d}\psi_{z^\gamma}(t)}{\mathrm{d}t} = -d_1 A\psi_{z^\gamma}(t) - a_1\psi_{z^\gamma}(t) + b_1(\psi_{z^\gamma})^{2q}(t)\phi_{z^\gamma}(t) - b_2(\psi_{z^\gamma})^{2q+1}(t) + f_1(t) \\ \qquad\qquad + \sigma(t,\psi_{z^\gamma}(t))z^\gamma(t), \\ \dfrac{\mathrm{d}\phi_{z^\gamma}(t)}{\mathrm{d}t} = -d_2 A\phi_{z^\gamma}(t) - a_2\phi_{z^\gamma}(t) - b_1(\psi_{z^\gamma})^{2q}(t)\phi_{z^\gamma}(t) + b_2(\psi_{z^\gamma})^{2q+1}(t) + f_2(t) \\ \qquad\qquad + \sigma(t,\phi_{z^\gamma}(t))z^\gamma(t). \end{cases} \tag{97}$$

with initial data

$$(\psi_{z^\gamma}(0), \phi_{z^\gamma}(0)) = (\psi_0, \phi_0) \in \ell^2 \times \ell^2. \tag{98}$$

By (95)-(98) we have for any $t > 0$,

$$\begin{cases} \dfrac{d}{dt}\left(\psi^\gamma_{z^\gamma}(t) - \psi_{z^\gamma}(t)\right) = -d_1 A\left(\psi^\gamma_{z^\gamma}(t) - \psi_{z^\gamma}(t)\right) + b_1\left(F(\psi^\gamma_{z^\gamma}(t),\phi^\gamma_{z^\gamma}(t)) - F(\psi_{z^\gamma}(t),\phi_{z^\gamma}(t))\right) \\ \qquad - a_1\left(\psi^\gamma_{z^\gamma}(t) - \psi_{z^\gamma}(t)\right) - b_2\left(G(\psi^\gamma_{z^\gamma}) - G(\psi_{z^\gamma})\right) \\ \qquad + \left(\sigma(t,\psi^\gamma_{z^\gamma}(t))z^\gamma(t) - \sigma(t,\psi_{z^\gamma}(t))z^\gamma(t)\right) \\ \qquad + \sqrt{\gamma}\,\sigma(t,\psi^\gamma_{z^\gamma}(t))\,dW(t), \\ \dfrac{d}{dt}\left(\phi^\gamma_{z^\gamma}(t) - \phi_{z^\gamma}(t)\right) = -d_2 A\left(\phi^\gamma_{z^\gamma}(t) - \phi_{z^\gamma}(t)\right) - b_1\left(F(\psi^\gamma_{z^\gamma}(t),\phi^\gamma_{z^\gamma}(t)) - F(\psi_{z^\gamma}(t),\phi_{z^\gamma}(t))\right) \\ \qquad - a_2\left(\phi^\gamma_{z^\gamma}(t) - \phi_{z^\gamma}(t)\right) + b_2\left(G(\phi^\gamma_{z^\gamma}) - G(\phi_{z^\gamma})\right) \\ \qquad + \left(\sigma(t,\phi^\gamma_{z^\gamma}(t))z^\gamma(t) - \sigma(t,\phi_{z^\gamma}(t))z(t)\right) \\ \qquad + \sqrt{\gamma}\,\sigma(t,\phi^\gamma_{z^\gamma}(t))\,dW(t), \end{cases} \tag{99}$$

with initial data

$$\psi^\gamma_{z^\gamma}(0) - \psi_{z^\gamma}(0) = 0, \;\; \phi^\gamma_{z^\gamma}(0) - \phi_{z^\gamma}(0) = 0. \tag{100}$$

By (99)-(100) and Itô's formula, we have

$$\begin{aligned} & d\|\psi^\gamma_{z^\gamma}(t) - \psi_{z^\gamma}(t)\|^2 \\ = & -2d_1\langle A(\psi^\gamma_{z^\gamma}(t) - \psi_{z^\gamma}(t)), \psi^\gamma_{z^\gamma}(t) - \psi_{z^\gamma}(t)\rangle\, dt \\ & + 2b_1\langle F(\psi^\gamma_{z^\gamma}(t),\phi^\gamma_{z^\gamma}(t)) - F(\psi_{z^\gamma}(t),\phi_{z^\gamma}(t)), \psi^\gamma_{z^\gamma}(t) - \psi_{z^\gamma}(t)\rangle\, dt \\ & - 2b_2\langle G(\psi^\gamma_{z^\gamma}(t)) - G(\psi_{z^\gamma}(t)), \psi^\gamma_{z^\gamma}(t) - \psi_{z^\gamma}(t)\rangle\, dt - 2a_1\|\psi^\gamma_{z^\gamma}(t) - \psi_{z^\gamma}(t)\|^2\, dt \\ & + 2\left\langle \sigma(t,\psi^\gamma_{z^\gamma}(t))z^\gamma(t) - \sigma(t,\psi_{z^\gamma}(t))z^\gamma(t), \psi^\gamma_{z^\gamma}(t) - \psi_{z^\gamma}(t)\right\rangle\, dt \\ & + \gamma\|\sigma(t,\psi^\gamma_{z^\gamma}(t))\|^2_{L_2(H,\ell^2)}\, dt \\ & + 2\sqrt{\gamma}\left\langle \psi^\gamma_{z^\gamma}(t) - \psi_{z^\gamma}(t), \sigma(t,\psi^\gamma_{z^\gamma}(t))\right\rangle\, dW(t), \end{aligned} \tag{101}$$

and

$$\begin{aligned} & d\|\phi^\gamma_{z^\gamma}(t) - \phi_{z^\gamma}(t)\|^2 \\ = & -2d_2\langle A(\phi^\gamma_{z^\gamma}(t) - \phi_{z^\gamma}(t)), \phi^\gamma_{z^\gamma}(t) - \phi_{z^\gamma}(t)\rangle\, dt \\ & - 2b_1\langle F(\psi^\gamma_{z^\gamma}(t),\phi^\gamma_{z^\gamma}(t)) - F(\psi_{z^\gamma}(t),\phi_{z^\gamma}(t)), \phi^\gamma_{z^\gamma}(t) - \phi_{z^\gamma}(t)\rangle\, dt \\ & + 2b_2\langle G(\psi^\gamma_{z^\gamma}(t)) - G(\psi_{z^\gamma}(t)), \phi^\gamma_{z^\gamma}(t) - \phi_{z^\gamma}(t)\rangle\, dt - 2a_2\|\phi^\gamma_{z^\gamma}(t) - \phi_{z^\gamma}(t)\|^2\, dt \\ & + 2\left\langle \sigma(t,\phi^\gamma_{z^\gamma}(t))z_1(t) - \sigma(t,\phi_{z^\gamma}(t))z_2(t), \phi^\gamma_{z^\gamma}(t) - \phi_{z^\gamma}(t)\right\rangle\, dt \end{aligned}$$

$$+\gamma\|\sigma(t,\phi^{\gamma}_{z^{\gamma}}(t))\|^2_{L_2(H,\ell^2)}\,dt+2\sqrt{\gamma}\langle\phi^{\gamma}_{z^{\gamma}}(t)-\phi_{z^{\gamma}}(t),\sigma(t,\phi^{\gamma}_{z^{\gamma}}(t))\rangle\,dW(t). \tag{102}$$

By (5)-(6) and (101)-(102) we have

$$\begin{aligned}
&\|\psi^{\gamma}_{z^{\gamma}}(t)-\psi_{z^{\gamma}}(t)\|^2+\|\phi^{\gamma}_{z^{\gamma}}(t)-\phi_{z^{\gamma}}(t)\|^2\\
\le& C\int_0^t\left(\|\psi^{\gamma}_{z^{\gamma}}(s)-\psi_{z^{\gamma}}(s)\|^2+\|\phi^{\gamma}_{z^{\gamma}}(s)-\phi_{z^{\gamma}}(s)\|^2\right)ds\\
&+2\left\langle\sigma(t,\psi^{\gamma}_{z^{\gamma}}(t))z^{\gamma}(t)-\sigma(t,\psi_{z^{\gamma}}(t))z^{\gamma}(t),\psi^{\gamma}_{z^{\gamma}}(t)-\psi_{z^{\gamma}}(t)\right\rangle dt\\
&+\gamma\|\sigma(t,\psi^{\gamma}_{z^{\gamma}}(t))\|^2_{L_2(H,\ell^2)}\,dt+2\sqrt{\gamma}\left\langle\psi^{\gamma}_{z^{\gamma}}(t)-\psi_{z^{\gamma}}(t),\sigma(t,\psi^{\gamma}_{z^{\gamma}}(t))\right\rangle dW(t)\\
&+2\left\langle\sigma(t,\phi^{\gamma}_{z^{\gamma}}(t))z_1(t)-\sigma(t,\phi_{z^{\gamma}}(t))z_2(t),\phi^{\gamma}_{z^{\gamma}}(t)-\phi_{z^{\gamma}}(t)\right\rangle dt\\
&+\gamma\|\sigma(t,\phi^{\gamma}_{z^{\gamma}}(t))\|^2_{L_2(H,\ell^2)}\,dt+2\sqrt{\gamma}\left\langle\phi^{\gamma}_{z^{\gamma}}(t)-\phi_{z^{\gamma}}(t),\sigma(t,\phi^{\gamma}_{z^{\gamma}}(t))\right\rangle dW(t).
\end{aligned} \tag{103}$$

For a fixed $M>0$, define a stopping time

$$\tau^{\gamma}=\begin{cases}\inf\left\{t\ge 0:\|\psi^{\gamma}_{z^{\gamma}}(t)\|\vee\|\phi^{\gamma}_{z^{\gamma}}(t)\|\ge M\right\}\wedge T,\\ +\infty\quad\text{if }\left\{t\ge 0:\|\psi^{\gamma}_{z^{\gamma}}(t)\|\vee\|\phi^{\gamma}_{z^{\gamma}}(t)\|\ge M\right\}=\emptyset.\end{cases} \tag{104}$$

By (103)-(104) we have, for all $t\in[0,T]$,

$$\begin{aligned}
&\sup_{0\le s\le t}\left(\|\psi^{\gamma}_{z^{\gamma}}(s\wedge\tau^{\gamma})-\psi_{z^{\gamma}}(s\wedge\tau^{\gamma})\|^2+\|\phi^{\gamma}_{z^{\gamma}}(s\wedge\tau^{\gamma})-\phi_{z^{\gamma}}(s\wedge\tau^{\gamma})\|^2\right)\\
\le& C\int_0^{t\wedge\tau^{\gamma}}\left(\|\psi^{\gamma}_{z^{\gamma}}(s)-\psi_{z^{\gamma}}(s)\|^2+\|\phi^{\gamma}_{z^{\gamma}}(s)-\phi_{z^{\gamma}}(s)\|^2\right)ds\\
&+2\int_0^{t\wedge\tau^{\gamma}}\left\langle\sigma(t,\psi^{\gamma}_{z^{\gamma}}(s))z^{\gamma}(s)-\sigma(t,\psi_{z^{\gamma}}(s))z^{\gamma}(s),\psi^{\gamma}_{z^{\gamma}}(s)-\psi_{z^{\gamma}}(s)\right\rangle ds\\
&+\gamma\int_0^{t\wedge\tau^{\gamma}}\|\sigma(s,\psi^{\gamma}_{z^{\gamma}}(s))\|^2_{L_2(H,\ell^2)}\,ds\\
&+2\sqrt{\gamma}\sup_{0\le s\le t}\left|\int_0^{s\wedge\tau^{\gamma}}\left\langle\psi^{\gamma}_{z^{\gamma}}(r)-\psi_{z^{\gamma}}(r),\sigma(s,\psi^{\gamma}_{z^{\gamma}}(r))\right\rangle dW(r)\right|\\
&+2\int_0^{t\wedge\tau^{\gamma}}\left\langle\sigma(t,\phi^{\gamma}_{z^{\gamma}}(s))z^{\gamma}(s)-\sigma(t,\phi_{z^{\gamma}}(s))z^{\gamma}(s),\phi^{\gamma}_{z^{\gamma}}(s)-\phi_{z^{\gamma}}(s)\right\rangle ds\\
&+\gamma\int_0^{t\wedge\tau^{\gamma}}\|\sigma(s,\phi^{\gamma}_{z^{\gamma}}(s))\|^2_{L_2(H,\ell^2)}\,ds\\
&+2\sqrt{\gamma}\sup_{0\le s\le t}\left|\int_0^{s\wedge\tau^{\gamma}}\left\langle\phi^{\gamma}_{z^{\gamma}}(r)-\phi_{z^{\gamma}}(r),\sigma(r,\phi^{\gamma}_{z^{\gamma}}(r))\right\rangle dW(r)\right|\\
=:&\sum_{j=1}^{5}I_j.
\end{aligned} \tag{105}$$

Next, we estimate each term on the right-hand side of (105). For the first term

$$I_1\le C\int_0^t\sup_{0\le r\le t}\left(\|\psi^{\gamma}_{z^{\gamma}}(r\wedge\tau^{\gamma})-\psi^{\gamma}(r\wedge\tau^{\gamma})\|^2+\|\phi^{\gamma}_{z^{\gamma}}(r\wedge\tau^{\gamma})-\phi_{z^{\gamma}}(r\wedge\tau^{\gamma})\|^2\right)ds. \tag{106}$$

By Lemma 3.4, there exists $C_1=C_1(N)$ such that $\mathbb{P}$-almost surely,

$$\sup_{\gamma\in(0,1]}\sup_{t\in[0,T]}\left(\|\psi_{z^{\gamma}}(t)\|+\|\phi_{z^{\gamma}}(t)\|\right)\le C_1. \tag{107}$$

By (7), (104) and (107), there exists $C_2 = C_2(N, M) > 0$ such that

$$
\begin{aligned}
I_2 \le & \int_0^{t\wedge\tau^\gamma} \|\sigma(t,\psi^\gamma_{z^\gamma}(s))z^\gamma(s) - \sigma(t,\psi_{z^\gamma}(s))z^\gamma(s)\| \|\psi^\gamma_{z^\gamma}(s) - \psi_{z^\gamma}(s)\| \, ds \\
\le & C_2\|\delta\| \int_0^{t\wedge\tau^\gamma} \|z^\gamma(s)\|_H \|\psi^\gamma_{z^\gamma}(s) - \psi_{z^\gamma}(s)\|^2 \, ds \\
\le & C_2\|\delta\| \int_0^t \|z^\gamma(s)\|_H \sup_{0\le r\le s} \|\psi^\gamma_{z^\gamma}(r\wedge\tau^\gamma) - \psi_{z^\gamma}(r\wedge\tau^\gamma)\|^2 \, ds.
\end{aligned}
\tag{108}
$$

Similarly, we have

$$
I_4 \le C_2\|\delta\| \int_0^t \|z^\gamma(s)\|_H \sup_{0\le r\le s} \|\phi^\gamma_{z^\gamma}(r\wedge\tau^\gamma) - \phi_{z^\gamma}(r\wedge\tau^\gamma)\|^2 \, ds. \tag{109}
$$

It follows from (105)-(106), (108) and (109) that for all $t \in [0, T]$, $\mathbb{P}$-almost surely,

$$
\begin{aligned}
& \sup_{0\le s\le t} \left(\|\psi^\gamma_{z^\gamma}(s\wedge\tau^\gamma) - \psi_{z^\gamma}(s\wedge\tau^\gamma)\|^2 + \|\phi^\gamma_{z^\gamma}(s\wedge\tau^\gamma) - \phi_{z^\gamma}(s\wedge\tau^\gamma)\|^2 \right) \\
& \le C \int_0^t \sup_{0\le r\le s} \left(\|\psi^\gamma_{z^\gamma}(r\wedge\tau^\gamma) - \psi_{z^\gamma}(r\wedge\tau^\gamma)\|^2 + \|\phi^\gamma_{z^\gamma}(r\wedge\tau^\gamma) - \phi_{z^\gamma}(r\wedge\tau^\gamma)\|^2 \right) ds \\
& \quad + C_2\|\delta\| \int_0^t \|z^\gamma(s)\|_H \sup_{0\le r\le s} \|\psi^\gamma_{z^\gamma}(r\wedge\tau^\gamma) - \psi_{z^\gamma}(r\wedge\tau^\gamma)\|^2 ds \\
& \quad + C_2\|\delta\| \int_0^t \|z^\gamma(s)\|_H \sup_{0\le r\le s} \|\phi^\gamma_{z^\gamma}(r\wedge\tau^\gamma) - \phi_{z^\gamma}(r\wedge\tau^\gamma)\|^2 ds \\
& \quad + \gamma \int_0^{T\wedge\tau^\gamma} \|\sigma(s,\psi^\gamma_{z^\gamma}(s))\|^2_{L_2(H,\ell^2)} ds \\
& \quad + \gamma \int_0^{T\wedge\tau^\gamma} \|\sigma(s,\phi^\gamma_{z^\gamma}(s))\|^2_{L_2(H,\ell^2)} ds \\
& \quad + 2\sqrt{\gamma} \sup_{0\le s\le t} \left| \int_0^{s\wedge\tau^\gamma} \left\langle \psi^\gamma_{z^\gamma}(r) - \psi_{z^\gamma}(r), \sigma(r,\psi^\gamma_{z^\gamma}(r)) \right\rangle dW(r) \right| \\
& \quad + 2\sqrt{\gamma} \sup_{0\le s\le t} \left| \int_0^{s\wedge\tau^\gamma} \left\langle \phi^\gamma_{z^\gamma}(r) - \phi_{z^\gamma}(r), \sigma(r,\phi^\gamma_{z^\gamma}(r)) \right\rangle dW(r) \right|.
\end{aligned}
$$

Moreover, we get

$$
\begin{aligned}
& \sup_{0\le s\le t} \left(\|\psi^\gamma_{z^\gamma}(s\wedge\tau^\gamma) - \psi_{z^\gamma}(s\wedge\tau^\gamma)\|^2 + \|\phi^\gamma_{z^\gamma}(s\wedge\tau^\gamma) - \phi_{z^\gamma}(s\wedge\tau^\gamma)\|^2 \right) \\
\le & \int_0^t (C + C_2\|\delta\|\|z^\gamma(s)\|_H) \sup_{0\le r\le s} \left(\|\psi^\gamma_{z^\gamma}(r\wedge\tau^\gamma) - \psi_{z^\gamma}(r\wedge\tau^\gamma)\|^2 + \|\psi^\gamma_{z^\gamma}(r\wedge\tau^\gamma) - \phi_{z^\gamma}(r\wedge\tau^\gamma)\|^2 \right) ds \\
& + \gamma \int_0^{T\wedge\tau^\gamma} \|\sigma(s,\psi^\gamma_{z^\gamma}(s))\|^2_{L_2(H,\ell^2)} ds \\
& + \gamma \int_0^{T\wedge\tau^\gamma} \|\sigma(s,\phi^\gamma_{z^\gamma}(s))\|^2_{L_2(H,\ell^2)} ds \\
& + 2\sqrt{\gamma} \sup_{0\le s\le t} \int_0^{s\wedge\tau^\gamma} \langle \psi^\gamma_{z^\gamma}(r) - \psi_{z^\gamma}(r), \sigma(s,\psi^\gamma_{z^\gamma}(r)) dW(r) \rangle \\
& + 2\sqrt{\gamma} \sup_{0\le s\le t} \int_0^{s\wedge\tau^\gamma} \langle \phi^\gamma_{z^\gamma}(r) - \phi_{z^\gamma}(r), \sigma(s,\phi^\gamma_{z^\gamma}(r)) dW(r) \rangle.
\end{aligned}
$$

By Gronwall's lemma, we obtain

$$\begin{aligned}
&\sup_{0\le s\le t}\left(\|\psi^{\gamma}_{z^\gamma}(s\wedge\tau^\gamma)-\psi_{z^\gamma}(s\wedge\tau^\gamma)\|^2+\|\phi^{\gamma}_{z^\gamma}(s\wedge\tau^\gamma)-\phi_{z^\gamma}(s\wedge\tau^\gamma)\|^2\right)\\
\le& e^{\int_0^t (C+C_2\|\delta\|\|z^\gamma(s)\|_H)ds}b\\
\le& e^{\left(CT+C_2T^{\frac12}N^{\frac12}\|\delta\|\right)}b\\
=:& C_3 b,
\end{aligned}\tag{110}$$

where

$$\begin{aligned}
b:=&\gamma\int_0^{T\wedge\tau^\gamma}\|\sigma(s,\psi^{\gamma}_{z^\gamma}(s))\|^2_{L_2(H,\ell^2)}ds+\gamma\int_0^{T\wedge\tau^\gamma}\|\sigma(s,\phi^{\gamma}_{z^\gamma}(s))\|^2_{L_2(H,\ell^2)}ds\\
&+2\sqrt{\gamma}\sup_{0\le s\le t}\left|\int_0^{s\wedge\tau^\gamma}\langle\psi^{\gamma}_{z^\gamma}(r)-\psi_{z^\gamma}(r),\sigma(s,\psi^{\gamma}_{z^\gamma}(r))dW(r)\rangle\right|\\
&+2\sqrt{\gamma}\sup_{0\le s\le t}\left|\int_0^{s\wedge\tau^\gamma}\langle\phi^{\gamma}_{z^\gamma}(r)-\phi_{z^\gamma}(r),\sigma(s,\phi^{\gamma}_{z^\gamma}(r))dW(r)\rangle\right|.
\end{aligned}\tag{111}$$

By (11) we have

$$\begin{aligned}
&\gamma\int_0^{T\wedge\tau^\gamma}\|\sigma(s,\psi^{\gamma}_{z^\gamma}(s))\|^2_{L_2(H,\ell^2)}ds\\
\le&\int_0^{T\wedge\tau^\gamma}\left(2\sum_{k=1}^{\infty}\|h_k(s)\|^2_{L^\infty(0,T;\ell^2)}+4\alpha^2\|\delta\|^2(1+\|\psi^{\gamma}_{z^\gamma}(s)\|^2)\right)ds\\
\le& 2\gamma T\sum_{k=1}^{\infty}\|h_k(s)\|^2_{L^\infty(0,T;\ell^2)}+4\gamma\alpha^2\|\delta\|^2T(1+M^2).
\end{aligned}\tag{112}$$

Similarly, we have

$$\gamma\int_0^{T\wedge\tau^\gamma}\|\sigma(s,\phi^{\gamma}_{z^\gamma}(s))\|^2_{L_2(H,\ell^2)}ds\le 2\gamma T\sum_{k=1}^{\infty}\|h_k(s)\|^2_{L^\infty(0,T;\ell^2)}+4\gamma\alpha^2\|\delta\|^2T(1+M^2).\tag{113}$$

Thus

$$\lim_{\gamma\to0}\gamma C_3\int_0^{T\wedge\tau^\gamma}\|\sigma(s,\psi^{\gamma}_{z^\gamma}(s))\|^2_{L_2(H,\ell^2)}ds=0,\quad \mathbb{P}\text{-almost surely.}\tag{114}$$

and

$$\lim_{\gamma\to0}\gamma C_3\int_0^{T\wedge\tau^\gamma}\|\sigma(s,\phi^{\gamma}_{z^\gamma}(s))\|^2_{L_2(H,\ell^2)}ds=0,\quad \mathbb{P}\text{-almost surely.}\tag{115}$$

By (107), (114)-(115) and Doob's inequality, we get

$$\begin{aligned}
&\mathbb{E}\left(\sup_{0\le s\le t}\left|\int_0^{s\wedge\tau^\gamma}\sqrt{\gamma}\left\langle\psi^{\gamma}_{z^\gamma}(r)-\psi_{z^\gamma}(r),\sigma(s,\psi^{\gamma}_{z^\gamma}(r))\right\rangle dW(r)\right|^2\right)\\
\le& 4\gamma\mathbb{E}\left(\int_0^{T\wedge\tau^\gamma}\|\psi^{\gamma}_{z^\gamma}(s)-\psi_{z^\gamma}(s)\|^2\|\sigma(s,\psi^{\gamma}_{z^\gamma}(s))\|^2_{L_2(H,\ell^2)}ds\right)\\
\le& 4\gamma(M+C_1)^2\mathbb{E}\left(\int_0^{T\wedge\tau^\gamma}\|\sigma(s,\psi^{\gamma}_{z^\gamma}(s))\|^2_{L_2(H,\ell^2)}ds\right)
\end{aligned}$$

$$\leq 8\gamma(M+C_1)^2 T\mathbb{E}\left(\sum_{k=1}^{\infty}\|h_k(s)\|^2_{L_\infty(0,T;L_2)}+2\alpha^2\|\delta\|^2(1+M^2)\right), \tag{116}$$

and similarly,

$$\begin{aligned}&\mathbb{E}\left[\sup_{0\leq s\leq t}\int_0^{s\wedge\tau^\gamma}\sqrt{\gamma}\|\phi^\gamma_{z^\gamma}(r)-\phi_{z^\gamma}(r)\|\|\sigma(s,\phi^\gamma_{z^\gamma}(r))\|dW(r)\right]^2\\ &\leq 8\gamma(M+C_1)^2 T\left(\sum_{k=1}^{\infty}\|h_k(s)\|^2_{L_\infty(0,T;L_2)}+2\alpha^2\|\delta\|^2(1+M^2)\right).\end{aligned} \tag{117}$$

Thus we have

$$\lim_{\gamma\to 0}\mathbb{E}\left(\sup_{0\leq s\leq t}\left|\int_0^{s\wedge\tau^\gamma}2C_3\sqrt{\gamma}\langle\psi^\gamma_{z^\gamma}(r)-\psi_{z^\gamma}(r),\sigma(s,\psi^\gamma_{z^\gamma}(r))\rangle dW(r)\right|^2\right)=0, \tag{118}$$

and

$$\lim_{\gamma\to 0}\mathbb{E}\left(\sup_{0\leq s\leq t}\left|\int_0^{s\wedge\tau^\gamma}2C_3\sqrt{\gamma}\langle\phi^\gamma_{z^\gamma}(r)-\phi_{z^\gamma}(r),\sigma(s,\phi^\gamma_{z^\gamma}(r))\rangle dW(r)\right|^2\right)=0, \tag{119}$$

By (110)-(119), we obtain

$$\lim_{\gamma\to 0}\sup_{0\leq s\leq t}\left(\|\psi^\gamma_{z^\gamma}(s\wedge\tau^\gamma)-\psi_{z^\gamma}(s\wedge\tau^\gamma)\|^2+\|\phi^\gamma_{z^\gamma}(s\wedge\tau^\gamma)-\phi_{z^\gamma}(s\wedge\tau^\gamma)\|^2\right)=0,\quad \text{in probability.} \tag{120}$$

On the other hand, by (84), (104) and Chebyshev's inequality, we have for all $\epsilon\in(0,1)$,

$$\begin{aligned}P(\tau^\gamma<T)=&P\left(\sup_{0\leq t\leq T}\left(\|\psi^\gamma_{z^\gamma}(t)\|_H\vee\|\phi^\gamma_{z^\gamma}(t)\|\right)\geq M\right)\\ \leq&\frac{1}{M^2}\mathbb{E}\left[\sup_{0\leq s\leq T}\left(\|\psi^\gamma_{z^\gamma}(t)\|^2+\|\phi^\gamma_{z^\gamma}(t)\|^2\right)\right]\leq\frac{C_4}{M^2},\end{aligned} \tag{121}$$

where $C_4=C_4(T,N)>0$.

From (121), one has that for all $n>0$,

$$\begin{aligned}&P\left(\|(\psi^\gamma_{z^\gamma}-\psi_{z^\gamma},\phi^\gamma_{z^\gamma}-\phi_{z^\gamma})\|_{C([0,T];H_2)}>\eta\right)\\ \leq&P\left(\sup_{0\leq t\leq T}\left(\|\psi^\gamma_{z^\gamma}(t)-\psi_{z^\gamma}(t)\|^2+\|\phi^\gamma_{z^\gamma}(t)-\phi_{z^\gamma}(t)\|^2\right)>\eta^2,\tau^\gamma=T\right)\\ &+P\left(\sup_{0\leq t\leq T}\left(\|\psi^\gamma_{z^\gamma}(t)-\psi_{z^\gamma}(t)\|^2+\|\phi^\gamma_{z^\gamma}(t)-\phi_{z^\gamma}(t)\|^2\right)>\eta^2,\tau^\gamma<T\right)\\ \leq&P\left(\sup_{0\leq t\leq T}\left(\|\psi^\gamma_{z^\gamma}(t\wedge\tau^\gamma)-\psi_{z^\gamma}(t\wedge\tau^\gamma)\|^2+\|\phi^\gamma_{z^\gamma}(t\wedge\tau^\gamma)-\phi_{z^\gamma}(t\wedge\tau^\gamma)\|^2\right)>\eta^2\right)+\frac{C_4}{M^2}\end{aligned} \tag{122}$$

First taking the limit as $\gamma\to 0$, and then as $M\to\infty$, we get from (120) that

$$\lim_{\gamma\to 0}\left(\psi^\gamma_{z^\gamma}-\psi_{z^\gamma},\phi^\gamma_{z^\gamma}-\phi_{z^\gamma}\right)=0,\quad \text{in } C([0,T],\ell^2\times\ell^2) \text{ in probability.} \tag{123}$$

Since $\{z^\gamma\}$ converges in distribution to z as S_N-valued random elements, by Skorokhod's theorem, there exists a probability space $(\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{\mathbb{P}})$ and S_N-valued random variables $\tilde{z}^\gamma$ and $\tilde{z}$ on $(\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{\mathbb{P}})$ such that $\{\tilde{z}^\gamma\}$ converges to $\tilde{z}$ almost surely in S_N which is equipped with the weak topology, where $\tilde{z}^\gamma$ and $\tilde{z}$ have the same distribution laws as z^γ and z, respectively. It then follows from Lemma 4.2 that

$$(\psi_{\tilde{z}^\gamma}, \phi_{\tilde{z}^\gamma}) \to (\psi_{\tilde{z}}, \phi_{\tilde{z}}), \quad \text{in } C([0,T], \ell^2 \times \ell^2) \text{ in distribution.}$$

which implies

$$(\psi_{z^\gamma}, \phi_{z^\gamma}) \to (\psi_z, \phi_z), \quad \text{in } C([0,T], \ell^2 \times \ell^2) \text{ in distribution.} \tag{124}$$

Then by (123) and (124), we have

$$(\psi^\gamma_{z^\gamma}, \phi^\gamma_{z^\gamma}) \to (\psi_z, \phi_z), \quad \text{in } C([0,T], \ell^2 \times \ell^2) \text{ in distribution,} \tag{125}$$

as desired. □

Now we show the proof of the main result of this paper.

Proof. From Lemmas 4.3 and 4.5, it follows that the conditions (i) and (ii) are fulfilled for the measurable map $\mathcal{G}^\gamma$ in (20) and $\mathcal{G}^0$ in (46) with the rate function I given by (19). Therefore, the family $\{(\psi^\gamma, \phi^\gamma)\}$ satisfies the Laplace principle in $C([0,T], \ell^2 \times \ell^2)$ with rate function I by Proposition 2.5. Then by Proposition 2.4, we know that $\{(\psi^\gamma, \phi^\gamma)\}$ satisfies the large deviation principle in space $C([0,T], \ell^2 \times \ell^2)$ with the same rate function, which concludes the proof. □

Declarations

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Competing Interests

The authors declare that there are no conflicts of interest related to this manuscript. No financial, personal, or professional relationships exist that could inappropriately influence, or be perceived to influence, the work presented in this paper within the past three years. Consequently, the publication of this manuscript will not be impeded by any conflicts of interest.

Ethical Approval

Not applicable.

Authors's Contributions

All authors contributed equally. All the authors read and approved the final manuscript.

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