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A priori strategy for pseudo-parabolic equations by hybrid regularization method

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Abstract

In this study, we address the simultaneous recovery of the source term and the initial value in a time-fractional diffusion equation, a problem inherently characterized by its ill-posed nature. To overcome this challenge, we propose a hybrid regularization method, offering robust solutions and providing precise estimations for both the source term and the initial value.

Keywords: time-fractional diffusion equation, inverse source term, initial value problem, hybrid regularization method, priori estimation
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1 Introduction

In this paper, we consider the fractional diffusion equation as follows

$$\begin{cases} {}_{CH}\mathbb{D}^{\alpha}_{b,t}(1+k\mathcal{L})\theta(x,t)+\mathcal{L}^s\theta(x,t)=\mathcal{F}(x)\mathcal{G}(t), & x\in\mathcal{D}, t\in(b,T], 0<\alpha<1,\ b>1,\\ \theta(x,b)=\mathcal{A}(x), & x\in\mathcal{D},\\ \theta(x,t)=0, & x\in\partial\mathcal{D}, t\in(b,T],\\ \theta(x,t_0)=\mathcal{B}(x), & x\in\mathcal{D}, t_0\in(b,T],\\ \theta(x,T)=\mathcal{C}(x), & x\in\mathcal{D}, \end{cases} \tag{1}$$

where $\mathcal{C}(x)$ is the terminal data, where $0<\mathcal{G}_1\le\mathcal{G}(t)\le\mathcal{G}_2$, and ${}_{CH}\mathbb{D}^{\alpha}_{b,t}$ is the Caputo-Hadamard fractional derivative of the order α defined by,

$$ {}_{CH}\mathbb{D}^{\alpha}_{b,t}\theta(x,t)=\frac{1}{\Gamma(1-\alpha)}\int_b^t\left(\log\frac{t}{z}\right)^{-\alpha}\Upsilon\theta(x,z)dz,\ \ 0<\alpha<1, \tag{2}$$

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where $\Upsilon = t\frac{d}{dt}, \Gamma(x)$ is a Gamma function. For $s > 1$ is a constant, $\mathcal{L}$ is the second-order differential operator and L^s is the fractional-order differential operator with the domain $H = L^2((0,\pi),\mathbb{R})$, and satisfy

$$\mathcal{L}\theta(x) = -\theta^{(2)}(x) + \varepsilon\theta^{(2)}(\pi - x), \quad 0 < x < \pi,$$
$$\mathcal{L}^s\theta(x) = -\theta^{(2s)}(x) + \varepsilon\theta^{(2s)}(\pi - x), \quad 0 < x < \pi. \tag{3}$$

It is accessible to prove that $\mathcal{L}, \mathcal{L}^s$ are self-adjoint operators. For all $|\varepsilon| < 1, \varepsilon \in \mathbb{R}$, nonlocal problem (1.3) has the following eigenvalues, see [12] for the solution procedure

$$\lambda_{2m+1} = (1-\varepsilon)(2k+1)^2, k \in \mathbb{N}, \quad \lambda_{2m} = (1+\varepsilon)(2k)^2, k \in \mathbb{N}^*, \tag{4}$$

and the corresponding system of eigenfunctions

$$e_{2m+1}(x) = \sqrt{\frac{2}{\pi}}\sin((2m+1)x), m \in \mathbb{N}, \quad e_{2m}(x) = \sqrt{\frac{2}{\pi}}\sin(2mx), m \in \mathbb{N}^*, \tag{5}$$

where $\mathbb{N}$ represents a group of natural number and $\mathbb{N}^*$ represents a group of positive integers.

Fractional derivatives extend the classical concept of differentiation by allowing for non-integer, or "fractional," orders of differentiation. Unlike integer-order derivatives, which measure the instantaneous rate of change of a function at a specific point, fractional derivatives account for a memory effect, meaning they depend not only on the function's current value but also on its entire history over an interval. This memory-dependent characteristic enables fractional derivatives to more accurately represent complex systems, particularly in fields where memory and hereditary effects are critical. Several formulations exist to define fractional derivatives, with the Riemann-Liouville, Caputo, and Grünwald-Letnikov derivatives being among the most widely used. Each definition has distinct advantages and is chosen based on the specific needs of the application. For example, the Caputo derivative is often preferred in physical contexts, as it facilitates the incorporation of initial conditions more straightforwardly.

The Caputo-Hadamard derivative represents a hybrid form of fractional derivatives, integrating features from both the Caputo and Hadamard approaches. As a component of fractional calculus—a mathematical domain that generalizes the concept of derivatives to non-integer orders—it has attracted considerable interest due to its ability to address functions with growth characteristics and initial condition requirements that traditional integer-order derivatives cannot adequately handle. This derivative proves particularly effective in modeling systems characterized by fractional-order dynamics and specific boundary conditions, especially when the processes involved are influenced by prior states or exhibit non-local behavior. Distinct from the Caputo and Riemann-Liouville derivatives, the Caputo-Hadamard derivative employs a logarithmic scaling akin to the Hadamard derivative, as opposed to the power-law scaling typically used in other fractional derivatives. Such a feature makes it especially suitable for problems where the domain of the function exhibits scale-invariance or follows a logarithmic framework.

Applications of the Caputo-Hadamard derivative span diverse fields, including physics, engineering, and biology, where it is instrumental in describing phenomena such as population dynamics, heat conduction in heterogeneous materials, and certain types of anomalous diffusion. Its similarity to the Caputo derivative enables the convenient incorporation of initial conditions, which is advantageous for solving fractional differential equations in practical scenarios. By capturing complex behaviors influenced by memory and specific growth patterns, the Caputo-Hadamard derivative has emerged as a valuable tool in the expanding discipline of fractional calculus, see in [5, 16, 25].

Fractional derivatives in time are commonly employed to model particle sticking and trapping phenomena. When fractional derivatives are applied to the spatial diffusion term, they capture the long-range power-law jump behavior of moving particles in space. As a result, such equations are particularly effective in simulating complex transport and diffusion mechanisms. These types of inverse problems have garnered significant attention in the research community. For example, Dou and Hon [3] studied a backward problem involving space–time fractional diffusion equations and introduced a fundamental kernel-based method. Feng et al. [4] investigated numerical approaches for both forward and backward problems of time–space fractional diffusion equations, proposing the QBV method, also known as the non-local boundary value method, for the backward problem. Additionally, Wang et al. [19] developed a fractional Tikhonov regularization method for solving time–space fractional backward heat equations, although their work was limited to the a priori case. For further methods addressing such problems, see references [7, 8, 10, 23].

Extensive research has been conducted on the inverse problems associated with time-fractional diffusion equations, resulting in the development of numerous methodologies. For instance, Yang et al. [17] employed a quasi-reversibility regularization technique to recover the initial value of a fractional time diffusion equation with an inhomogeneous source. Wang et al. [15] applied the Tikhonov regularization method to address the initial value problem for time-fractional diffusion equations within general bounded domains. Liu et al. [6] utilized the strong maximum principle for fractional diffusion equations to solve an inverse source problem. Similarly, Ismailov [26] adopted the eigenfunction expansion method to address an inverse source problem in a time-fractional diffusion equation with nonlocal boundary conditions. Zhang [22] proposed the truncated method to identify an unknown source in a time-fractional diffusion equation, while Xiong [13] introduced an optimal regularization approach for an inverse heat conduction problem related to time-fractional diffusion equations. Additional findings concerning homogeneous backward problems can be found in [14, 18, 21, 24]. Liu et al. [20] also investigated a time-backward problem for one-dimensional time-fractional partial differential equations, introducing a quasi-reversibility method to solve the issue. Wei et al. [2] explored an inverse time-dependent source problem for a time-fractional diffusion equation. Furthermore, Cheng et al. [9] examined a time-fractional inverse diffusion problem under Hölder-type source conditions and proposed an optimal regularization method. Additional advancements, including finite element methods, uniqueness and existence analyses, and finite difference methods, are detailed in references [1, 11].

In this study, we propose a hybrid regularization method to effectively address the simultaneous recovery of the source term and initial value in a time-fractional diffusion equation. When the source term $\mathcal{F}(x)$ and the initial value $\mathcal{A}(x)$ are known, we can figure out $\theta(x,t)$ by boundary conditions, this is the direct problem. But now $\mathcal{F}(x)$ and $\mathcal{A}(x)$ are unknown and need to be determined. We want to identify both the source term $\mathcal{F}(x)$ and the initial value $\mathcal{A}(x)$ by using the data $\mathcal{B}(x)$ and the final data $\mathcal{C}(x)$. In fact, the measurements are noise-contaminated inevitably, we remark the measurements with errors as $\mathcal{B}^{\varepsilon}(x)$ and $\mathcal{C}^{\varepsilon}(x)$ and satisfy

$$\|\mathcal{B}^{\varepsilon} - \mathcal{B}\|_2 \leq \varepsilon, \quad \|\mathcal{C}^{\varepsilon} - \mathcal{C}\|_2 \leq \varepsilon, \tag{6}$$

Our manuscript is structured to guide the reader through a progressive exploration of the key concepts and results. Section 2 lays the foundational groundwork by introducing the necessary preliminaries for our study. In Section 3, we address the resolution of Problem (1) and establish the result of conditional stability. Moving forward, Section 4 delves into the development of a hybrid regularization method. Section 5 focuses on presenting convergent error estimations associated with source terms, while Section 6 concludes with a detailed examination of the convergent error estimations for the initial value.

2 Preliminaries and inverse source problem

Before we introduce the main results of our works, some preliminary materials are given. Let us recall the spectral problem for the fractional Laplace operator on the bounded domain $\mathcal{D}$ as follows

$$\begin{cases} (-\Delta)^s \varphi_m(x) = \lambda_m^s e_m(x), & x \in \mathcal{D}, \forall m \in \mathbb{N}, \\ e_m(x) = 0, & x \in \partial\mathcal{D}, \forall m \in \mathbb{N}, \end{cases} \tag{7}$$

where the sequence of positive eigenvalues $\{\lambda_m\}_{m\in\mathbb{N}}$ satisfy

$$\lambda_1^s \leq \lambda_2^s \leq \cdots \leq \lambda_m^s \nearrow \infty, \tag{8}$$

whose corresponding set of real eigenfunctions $\{e_m\}_{m\in N}$ is orthogonal and complete.

Definition 2.1. The Hilbert scale space $\mathbb{H}^\sigma$, $(\sigma > 0)$ defined by

$$\mathbb{H}^\sigma = \Big\{\varphi \in \mathbb{L}^2 : \sum_{m=0}^{\infty} \lambda_{2m+1}^{2\sigma} \big|\langle \varphi, e_{2m+1}\rangle\big|^2 + \sum_{m=1}^{\infty} \lambda_{2m}^{2\sigma} \big|\langle \varphi, e_{2m}\rangle\big|^2 < \infty\Big\}, \tag{9}$$

$\mathbb{H}^\sigma$ is shown to be a Hilbert space.

$$\max\Big\{\|\mathcal{F}\|_{\mathbb{H}^\sigma}, \|\mathcal{A}\|_{\mathbb{H}^\sigma}\Big\} \leq \mathcal{O}, \quad \sigma > 0.$$

where $\mathcal{O} > 0$ is a constant. The conditional stability of problem (1) in the following theorem 3.1.

Lemma 2.2. *If $\lambda > 0$, then the following equation holds*

$$\int_c^\infty e^{-s\log\frac{t}{c}} \big(\log(t/c)\big)^{\beta k+\gamma-1} E_{\beta,\gamma}^{(k)}\big(\pm\lambda\big(\log(t/c)\big)^\beta\big)\frac{dt}{t} = \frac{k! s^{\beta-\gamma}}{\big(s^\beta \mp \lambda\big)^{k+1}}, \mathrm{Re}(s) > |\lambda|^{\frac{1}{\beta}},$$

where $E_{\beta,\gamma}^{(m)}(y) := \dfrac{d^m}{dy^m} E_{\beta,\gamma}(y)$. The Lemma 2.2 means that the Laplace transformation of

$$\Big(\log\frac{t}{c}\Big)^{\beta k+\gamma-1} E_{\beta,\gamma}^{(k)}\big(\pm\lambda\big(\log\frac{t}{c}\big)^\beta\big) \text{ is } \frac{k! s^{\beta-\gamma}}{(s^\beta \mp \lambda)^{k+1}}.$$

Lemma 2.3. *For $0 < \alpha < 1, z > 0$, we have $0 \leq E_{\alpha,1}(-z) < 1$. Moreover, $E_{\alpha,1}(-z)$ is completely monotonic, that is,*

$$(-1)^n \frac{d^n}{dz^n} E_{\alpha,1}(-z) \geq 0, z \geq 0,$$

Lemma 2.4. *Assume that $0 < \alpha_0 < \alpha_1 < 1$. Then, there exist constants $\mathcal{Z}_\pm > 0$, depending only on α_0, α_1 such that for all $\alpha \in [\alpha_0, \alpha_1]$, we obtain*

$$\frac{\mathcal{Z}_-}{\Gamma(1-\alpha)} \frac{1}{1-x} \leq E_{\alpha,1}(x) \leq \frac{\mathcal{Z}_+}{\Gamma(1-\alpha)} \frac{1}{1-x}, \forall x \leqslant 0.$$

Lemma 2.5. *[12] For any $0 < \alpha < 1, t > 0$, there is $0 < E_{\alpha,1} < 1$. Additionally, $E_{\alpha,1}(-t)$ is a strict decreasing function, that is*

$$(-1)^n \frac{d^n}{dt^n} E_{\alpha,1}(-t) \geq 0$$

Lemma 2.6. *[12] For the constants $\sigma > 0, \gamma > 0$, and $0 < \lambda_1 < \varsigma$, we have*

$$\mathcal{J}_1(\varsigma) = \frac{\gamma \varsigma^{1-\sigma}}{\gamma\varsigma + 1} \le \begin{cases} \mathcal{C}_1 \gamma^{\sigma}, & 0 < \sigma < 1, \\ \mathcal{C}_2 \gamma, & \sigma \ge 1. \end{cases}$$

where $\mathcal{C}_1 = 2^{1-\sigma}\sigma^{\sigma}(1-\sigma)^{1-\sigma}, \mathcal{C}_2 = \lambda_1^{1-\sigma}$.

Lemma 2.7. *[12] For any $\lambda_m \ge \lambda_1 > 0$, and $s > 1$, then there exist positive constants $\mathcal{Z}_1$ and $\mathcal{Z}_2$ depending only on α, t_0, λ_1, such that*

$$\frac{\mathcal{Z}_1}{\lambda_m^{s-1}} \le E_{\alpha,1}\Big(- \frac{\lambda_m^s}{1 + k\lambda_m} \big(\log \frac{t_0}{b}\big)^{\alpha} \Big) \le \frac{\mathcal{Z}_2}{\lambda_m^{s-1}},$$

where $\mathcal{Z}_1 = \frac{\mathcal{Z}_-}{1+\Gamma(1-\alpha)\frac{\lambda_1}{1+k\lambda_1}(\log\frac{t_0}{b})^{\alpha}}, \mathcal{Z}_2 = \frac{\mathcal{Z}_+}{1+\Gamma(1+\alpha)k^{-1}(\log\frac{t_0}{b})^{\alpha}}$.

Proof. For all $0 < \alpha < 1, t_0 \in (b, T], s > 1$ and $\lambda_m > \lambda_1 > 0$, we have

$$\begin{aligned} E_{\alpha,1}\Big(- \frac{\lambda_m^s}{1 + k\lambda_m} \big(\log \frac{t_0}{b}\big)^{\alpha} \Big) &\le \frac{\mathcal{Z}_+}{1 + \Gamma(1+\alpha)\frac{\lambda_m^s}{1+k\lambda_m}\big(\log\frac{t_0}{b}\big)^{\alpha}} \\ &\le \frac{\mathcal{Z}_+}{1 + \Gamma(1+\alpha)\frac{\lambda_m^{s-1}}{k}\big(\log\frac{t_0}{b}\big)^{\alpha}} = \frac{\mathcal{Z}_2}{\lambda_m^{s-1}}, \end{aligned} \tag{10}$$

$$\begin{aligned} &E_{\alpha,1}\Big(- \frac{\lambda_m^s}{1 + k\lambda_m} \big(\log \frac{t_0}{b}\big)^{\alpha} \Big) \ge \frac{\mathcal{Z}_-}{1 + \Gamma(1-\alpha)\frac{\lambda_m^s}{1+k\lambda_m}\big(\log\frac{t_0}{b}\big)^{\alpha}} \\ &\ge \frac{\mathcal{Z}_-}{1 + \Gamma(1-\alpha)\frac{\lambda_m^{s-1}}{\frac{1}{\lambda_1}+k}(\log\frac{t_0}{b})^{\alpha}} \ge \frac{\mathcal{Z}_-}{\lambda_m^{s-1} + \Gamma(1-\alpha)\frac{\lambda_m^{s-1}}{\frac{1}{\lambda_1}+k}(\log\frac{t_0}{b})^{\alpha}} = \frac{\mathcal{Z}_1}{\lambda_m^{s-1}}. \end{aligned}$$

This Lemma is proven. □

Lemma 2.8. *[12] For any $\lambda_m \ge \lambda_1 > 0, s > 1$, then there exist positive constants $\mathcal{Z}_3$ and $\mathcal{Z}_4$ depending only on α, T, λ_1, such that*

$$\frac{\mathcal{Z}_3}{\lambda_m^{s-1}} \le E_{\alpha,1}\Big(- \lambda_m^s \mu_m \big(\log \frac{T}{b}\big)^{\alpha} \Big) \le \frac{\mathcal{Z}_4}{\lambda_m^{s-1}},$$

where $\mathcal{Z}_3 = \frac{\mathcal{Z}_-}{1+\Gamma(1-\alpha)\frac{\lambda_1}{1+k\lambda_1}(\log\frac{T}{b})^{\alpha}}, \mathcal{Z}_4 = \frac{\mathcal{Z}_+}{1+\Gamma(1+\alpha)k^{-1}(\log\frac{T}{b})^{\alpha}}$.

Proof. This Lemma is proven similar to Lemma 2.7. □

Lemma 2.9. *Assume that there exist positive constants $\mathcal{G}_1, \mathcal{G}_2$ such that $\mathcal{G}_1 \le |\mathcal{G}(t)| \le \mathcal{G}_2 \ \forall t \in [0, T]$. Let choose $\varepsilon \in \left(0, \frac{\mathcal{G}_1}{4}\right)$, we have*

$$\frac{\mathcal{G}_1}{4} \le |\mathcal{G}^{\varepsilon}(t)| \le \mathcal{P}(\mathcal{G}_1, \mathcal{G}_2).$$

Here, $\mathcal{G}^{\varepsilon}$ is the measured data of $\mathcal{G}$.

Lemma 2.10. *[5] For any λ_m^s that satisfies $0 < \lambda_1^s \leq \lambda_m^s$, $s > 1$, and $\mu_m = (1 + k\lambda_m)^{-1}$, there is a positive number $\mathcal{Z}_5$ that depends on α, b, T, and λ_1 such that*

$$\frac{\mathcal{Z}_{5,T}}{\lambda_m^{s-1}} \leq \mu_m \int_b^T \big(\log \frac{T}{z}\big)^{\alpha-1} E_{\alpha,\alpha}\big(-\lambda_m^s \mu_m \big(\log \frac{T}{z}\big)^{\alpha}\big)\mathcal{G}(z)\frac{dz}{z} \leq \frac{\mathcal{G}_2}{\lambda_m^{s-1}}, \tag{11}$$

where $\mathcal{Z}_{5,T} = \mathcal{G}_1\mathcal{C}_T = \mathcal{G}_1\big(1 - E_{\alpha,1}\big(-k^{-1}\lambda_1^{s-1}\big(\log \frac{T}{b}\big)^{\alpha}\big)\big)$, and (18) holds if we substitute $T = t_0$, then we will replace $\mathcal{Z}_{5,T} = \mathcal{Z}_{5,t_0}$. We have

$$\frac{\mathcal{Z}_{5,t_0}}{\lambda_m^{s-1}} \leq \mu_m \int_b^{t_0} \big(\log \frac{t_0}{z}\big)^{\alpha-1} E_{\alpha,\alpha}\big(-\lambda_m^s \mu_m \big(\log \frac{t_0}{z}\big)^{\alpha}\big)\mathcal{G}(z)\frac{dz}{z} \leq \frac{\mathcal{G}_2}{\lambda_m^{s-1}}.$$

Lemma 2.11. *We assume that (12) holds, for any λ_m^s that satisfies $0 < \lambda_1^s \leq \ldots \leq \lambda_m^s$, there is a normal number dependent on $\alpha, T, t_0, \lambda_1^s$, and there is a positive number $\mathcal{Z}_i, i = \overline{6, 15}$ such that*

$$\begin{aligned} \frac{\mathcal{Z}_6}{\lambda_m^{2s-2}} &\leq E_{\alpha,1}\Big(-\frac{\lambda_m^s}{1+k\lambda_m}(\log \frac{t_0}{b})^{\beta}\Big)\mu_m \int_b^T \mathcal{X}_m^{\alpha}(T,z)\mathcal{G}(z)dz \leq \frac{\mathcal{Z}_7}{\lambda_m^{2s-2}}, \text{ and} \\ \frac{\mathcal{Z}_8}{\lambda_m^{2s-2}} &\leq E_{\alpha,1}\Big(-\frac{\lambda_m^s}{1+k\lambda_m}(\log \frac{T}{b})^{\beta}\Big)\mu_m \int_b^{t_0} \mathcal{X}_m^{\alpha}(t_0,z)\mathcal{G}(z)dz \leq \frac{\mathcal{Z}_9}{\lambda_m^{2s-2}}, \end{aligned} \tag{12}$$

whereby $\mathcal{Z}_6 = \mathcal{Z}_1\mathcal{Z}_{5,T}$, $\mathcal{Z}_7 = \mathcal{Z}_2\mathcal{G}_2$, $\mathcal{Z}_8 = \mathcal{Z}_3\mathcal{Z}_{5,t_0}$ and $\mathcal{Z}_9 = \mathcal{Z}_4\mathcal{G}_2$, then we get

$$\frac{\mathcal{Z}_{10}}{\lambda_m^{2s-2}} \leq \mathcal{Y}_m^{\alpha}(b, t_0, T, \mathcal{X}, \mathcal{G}) \leq \frac{\mathcal{Z}_{11}}{\lambda_m^{2s-2}}, \tag{13}$$

where $\mathcal{Z}_{10} = \mathcal{Z}_6 - \mathcal{Z}_9$, $\mathcal{Z}_{11} = \mathcal{Z}_7 - \mathcal{Z}_8$, with $\mathcal{Y}_m^{\alpha}(b, t_0, T, \mathcal{X}, \mathcal{G})$ is defined in (17). From (13), it gives

$$\frac{\mathcal{Z}_{12}}{\lambda_m^{2s-2}} \leq \mathcal{Y}_m^{\alpha}(b, t_0, T, \mathcal{X}, \mathcal{G}^{\varepsilon}) \leq \frac{\mathcal{Z}_{13}}{\lambda_m^{2s-2}},$$

whereby $\mathcal{Z}_{12} = 4\mathcal{Z}_1\mathcal{G}_1\mathcal{C}_T - \mathcal{Z}_4\mathcal{P}(\mathcal{G}_1, \mathcal{G}_2)$, and $\mathcal{Z}_{13} = \mathcal{Z}_2\mathcal{P}(\mathcal{G}_1, \mathcal{G}_2) - \mathcal{Z}_3 4\mathcal{G}_1\mathcal{C}_{t_0}$. And finally, we have

$$\frac{\mathcal{Z}_{14}}{\lambda_m^{2s-2}} \leq \mathcal{Y}_m^{\alpha}(b, t_0, T, \mathcal{X}) \leq \frac{\mathcal{Z}_{15}}{\lambda_m^{2s-2}}, \tag{14}$$

whereby $\mathcal{Z}_{15} = \mathcal{Z}_2 - \mathcal{Z}_3\mathcal{C}_{t_0}$, and $\mathcal{Z}_{14} = \mathcal{Z}_1\mathcal{C}_T - \mathcal{Z}_4$. From the estimate (14), we have conclude

$$\Big(\frac{\mathcal{Z}_{14}}{\mathcal{Z}_{13}}\Big) \leq \frac{\mathcal{Y}_m^{\alpha}(b, t_0, T, \mathcal{X})}{\mathcal{Y}_m^{\alpha}(b, t_0, T, \mathcal{X}, \mathcal{G}^{\varepsilon})} \leq \Big(\frac{\mathcal{Z}_{15}}{\mathcal{Z}_{12}}\Big).$$

Lemma 2.12. *For $s > 1$, for any λ_m^s that satisfies $0 < \lambda_1^s \leq \lambda_m^s$, and $\mu_m = (1 + k\lambda_m)^{-1}$, we have*

$$\begin{aligned} \frac{4\mathcal{G}_1\mathcal{Z}_1\mathcal{C}_T}{\mathcal{Z}_{13}} &\leq \frac{\mu_m \int\limits_b^T \mathcal{X}_m^{\alpha}(T,z)\mathcal{G}^{\varepsilon}(z)dz}{\mathcal{Y}_m^{\alpha}(b, t_0, T, \mathcal{X}, \mathcal{G}^{\varepsilon})} \leq \frac{\mathcal{Z}_2\mathcal{P}(\mathcal{G}_1, \mathcal{G}_2)}{\mathcal{Z}_{12}}, \\ \frac{4\mathcal{G}_1\mathcal{Z}_1\mathcal{C}_{t_0}}{\mathcal{Z}_{13}} &\leq \frac{\mu_m \int\limits_b^{t_0} \mathcal{X}_m^{\alpha}(t_0,z)\mathcal{G}^{\varepsilon}(z)dz}{\mathcal{Y}_m^{\alpha}(b, t_0, T, \mathcal{X}, \mathcal{G}^{\varepsilon})} \leq \frac{\mathcal{Z}_2\mathcal{P}(\mathcal{G}_1, \mathcal{G}_2)}{\mathcal{Z}_{12}}. \end{aligned}$$

whereby

$$\begin{aligned} \mathcal{Y}_m^{\alpha}(b, t_0, T, \mathcal{X}, \mathcal{G}^{\varepsilon}) = \Big(& E_{\alpha,1}\big(-\lambda_m^s \mu_m \big(\log \frac{t_0}{b}\big)^{\alpha}\big)\mu_m \int_b^T \mathcal{X}_m^{\alpha}(T,z)\mathcal{G}^{\varepsilon}(z)dz \\ &- E_{\alpha,1}\big(-\lambda_m^s \mu_m \big(\log \frac{T}{b}\big)^{\alpha}\big)\mu_m \int_b^{t_0} \mathcal{X}_m^{\alpha}(t_0,z)\mathcal{G}^{\varepsilon}(z)dz\Big). \end{aligned}$$

Proof. Using Lemma 2.10, Lemma 2.11 and simple transformations. □

Lemma 2.13. *For any $\lambda_m^s \geq \lambda_1^s > 0$, we have assume that $\underline{\mathcal{B}} \leq \mathcal{B} \leq \overline{\mathcal{B}}$, and $\underline{\mathcal{C}} \leq \mathcal{C} \leq \overline{\mathcal{C}}$, and then there exist positive constants $\mathcal{Z}_{16}, \mathcal{Z}_{17}, \mathcal{Z}_{18}$ and $\mathcal{Z}_{19}$ such that*

$$\frac{\mathcal{Z}_{16}}{\lambda_m^{s-1}} \leq \langle \mathcal{B}, e_m \rangle \mu_m \int_b^T \mathcal{X}_m^\alpha(T,z)\mathcal{G}(z)dz \leq \frac{\mathcal{Z}_{17}}{\lambda_m^{s-1}}, \quad \frac{\mathcal{Z}_{18}}{\lambda_m^{s-1}} \leq \langle \mathcal{C}, e_m \rangle \mu_m \int_b^{t_0} \mathcal{X}_m^\alpha(t_0,z)\mathcal{G}(z)dz \leq \frac{\mathcal{Z}_{19}}{\lambda_m^{s-1}}. \tag{15}$$

where $\mathcal{Z}_{16} = \underline{\mathcal{B}}\mathcal{Z}_{5,T}$, $\mathcal{Z}_{17} = \mathcal{G}_2\overline{\mathcal{B}}$, $\mathcal{Z}_{18} = \underline{\mathcal{C}}\mathcal{Z}_{5,t_0}$, $\mathcal{Z}_{19} = \mathcal{G}_2\overline{\mathcal{C}}$. And from (15), we have a corollary

$$\frac{\underline{\mathcal{B}}\mathcal{C}_T}{\lambda_m^{s-1}} \leq \langle \mathcal{B}, e_m \rangle \mu_m \int_b^T \mathcal{X}_m^\alpha(T,z)dz \leq \frac{\overline{\mathcal{B}}}{\lambda_m^{s-1}}, \quad \frac{\underline{\mathcal{C}}\mathcal{C}_{t_0}}{\lambda_m^{s-1}} \leq \langle \mathcal{C}, e_m \rangle \mu_m \int_b^{t_0} \mathcal{X}_m^\alpha(t_0,z)dz \leq \frac{\overline{\mathcal{C}}}{\lambda_m^{s-1}}.$$

Proof. Using the Lemmas 2.10, 2.11 and the boundedness of the functions $\mathcal{B}$ and $\mathcal{C}$. □

3 THE SOLUTION, THE ILL-POSED ANALYSIS, AND THE RESULT OF CONDITIONAL STABILITY

The solution of problem (1) is obtained by using the separated variable method, the Laplace transformation, and the inverse Laplace transformation of Mittag-Leffler function

$$\begin{aligned}
\theta(x,t) &= \sum_{m=0}^{\infty} \langle \theta_1(t), e_{2m+1} \rangle e_{2m+1}(x) + \sum_{m=1}^{\infty} \langle \theta_2(t), e_{2m} \rangle e_{2m}(x), \\
\mathcal{A}(x) &= \sum_{m=0}^{\infty} \langle \mathcal{A}_1, e_{2m+1} \rangle e_{2m+1}(x) + \sum_{m=1}^{\infty} \langle \mathcal{A}_2, e_{2m} \rangle e_{2m}(x), \\
\mathcal{F}(x) &= \sum_{m=0}^{\infty} \langle \mathcal{F}_1, e_{2m+1} \rangle e_{2m+1}(x) + \sum_{m=1}^{\infty} \langle \mathcal{F}_2, e_{2m} \rangle e_{2m}(x).
\end{aligned}$$

One has

$$\begin{aligned}
\theta(x,t) = \sum_{m=0}^{\infty} \Bigg(& E_{\alpha,1}\Big(-\lambda_{2m+1}^s \mu_{2m+1} \big(\log \frac{t}{b}\big)^\alpha \Big) \langle \mathcal{A}_1, e_{2m+1} \rangle \\
& + \langle \mathcal{F}_1, e_{2m+1} \rangle \mu_{2m+1} \int_b^t \big(\log \frac{t}{z}\big)^{\alpha-1} E_{\alpha,\alpha}\big(-\lambda_{2m+1}^s \mu_{2m+1} \big(\log \frac{t}{z}\big)^\alpha \big) \mathcal{G}(z) \frac{dz}{z} \Bigg) e_{2m+1}(x) \\
& + \sum_{m=1}^{\infty} \Bigg(E_{\alpha,1}\big(-\lambda_{2m}^s \mu_{2m} \big(\log \frac{t}{b}\big)^\alpha \big) \langle \mathcal{A}_2, e_{2m} \rangle \\
& + \langle \mathcal{F}_2, e_{2m} \rangle \mu_{2m} \int_b^t \big(\log \frac{t}{z}\big)^{\alpha-1} E_{\alpha,\alpha}\big(-\lambda_{2m}^s \mu_{2m} \big(\log \frac{t}{z}\big)^\alpha \big) \mathcal{G}(z) \frac{dz}{z} \Bigg) e_{2m}(x),
\end{aligned} \tag{16}$$

Letting $t = t_0$, we have

$$\begin{aligned}
\mathcal{B}(x) = \sum_{m=0}^{\infty} \Bigg(& E_{\alpha,1}\Big(-\lambda_{2m+1}^s \mu_{2m+1} \big(\log \frac{t_0}{b}\big)^\alpha \Big) \langle \mathcal{A}_1, e_{2m+1} \rangle \\
& + \langle \mathcal{F}_1, e_{2m+1} \rangle \mu_{2m+1} \int_b^{t_0} \big(\log \frac{t_0}{z}\big)^{\alpha-1} E_{\alpha,\alpha}\big(-\lambda_{2m+1}^s \mu_{2m+1} \big(\log \frac{t_0}{z}\big)^\alpha \big) \mathcal{G}(z) \frac{dz}{z} \Bigg) e_{2m+1}(x) \\
+ \sum_{m=1}^{\infty} \Bigg(& E_{\alpha,1}\big(-\lambda_{2m}^s \mu_{2m} \big(\log \frac{t_0}{b}\big)^\alpha \big) \langle \mathcal{A}_2, e_{2m} \rangle \\
& + \langle \mathcal{F}_2, e_{2m} \rangle \mu_{2m} \int_b^{t_0} \big(\log \frac{t_0}{z}\big)^{\alpha-1} E_{\alpha,\alpha}\big(-\lambda_{2m}^s \mu_{2m} \big(\log \frac{t_0}{z}\big)^\alpha \big) \mathcal{G}(z) \frac{dz}{z} \Bigg) e_{2m}(x),
\end{aligned}$$

Next, letting $t=T$, one has

$$\begin{aligned}\mathcal{C}(x)=&\sum_{m=0}^{\infty}\Big(E_{\alpha,1}\Big(-\lambda_{2m+1}^{s}\mu_{2m+1}\big(\log\frac{T}{b}\big)^{\alpha}\Big)\langle\mathcal{A}_1,e_{2m+1}\rangle\\&+\langle\mathcal{F}_1,e_{2m+1}\rangle\mu_{2m+1}\int_b^T\big(\log\frac{t}{z}\big)^{\alpha-1}E_{\alpha,\alpha}\Big(-\lambda_{2m+1}^{s}\mu_{2m+1}\big(\log\frac{t}{z}\big)^{\alpha}\Big)\mathcal{G}(z)\frac{dz}{z}\Big)e_{2m+1}(x)\\=&\sum_{m=1}^{\infty}\Big(E_{\alpha,1}\big(-\lambda_{2m}^{s}\mu_{2m}\big(\log\frac{T}{b}\big)^{\alpha}\big)\langle\mathcal{A}_2,e_{2m}\rangle\\&+\langle\mathcal{F}_2,e_{2m}\rangle\mu_{2m}\int_b^T\big(\log\frac{t}{z}\big)^{\alpha-1}E_{\alpha,\alpha}\big(-\lambda_{2m}^{s}\mu_{2m}\big(\log\frac{t}{z}\big)^{\alpha}\big)\mathcal{G}(z)\frac{dz}{z}\Big)e_{2m}(x).\end{aligned}$$

Here, by denoting

$$\begin{aligned}\mathcal{X}_m^{\alpha}(t,z)&=\frac{1}{z}\big(\log\frac{t}{z}\big)^{\alpha-1}E_{\alpha,\alpha}\big(-\lambda_m^{s}\mu_m\Big(\log\frac{t}{z}\big)^{\alpha}\Big),\\\mathcal{Y}_m^{\alpha}(b,t_0,T,\mathcal{X},\mathcal{G})&=E_{\alpha,1}\big(-\lambda_m^{s}\mu_m\big(\log\frac{t_0}{b}\big)^{\alpha}\big)\mu_m\int_b^T\mathcal{X}_m^{\alpha}(T,z)\mathcal{G}(z)dz\\&\quad-E_{\alpha,1}\big(-\lambda_m^{s}\mu_m\big(\log\frac{T}{b}\big)^{\alpha}\big)\mu_m\int_b^{t_0}\mathcal{X}_m^{\alpha}(t_0,z)\mathcal{G}(z)dz.\end{aligned}\tag{17}$$

Combining (16) to (17), one has

$$\begin{aligned}&\mathcal{A}(x)=\\&\sum_{m=0}^{\infty}\left(\frac{\langle\mathcal{B}_1,e_{2m+1}\rangle\mu_{2m+1}\int_b^T\mathcal{X}_{2m+1}^{\alpha}(T,z)\mathcal{G}(z)dz}{\mathcal{Y}_{2m+1}^{\alpha}(b,t_0,T,\mathcal{X},\mathcal{G})}-\frac{\langle\mathcal{C}_1,e_{2m+1}\rangle\mu_{2m+1}\int_b^{t_0}\mathcal{X}_{2m+1}^{\alpha}(t_0,z)\mathcal{G}(z)dz}{\mathcal{Y}_{2m+1}^{\alpha}(b,t_0,T,\mathcal{X},\mathcal{G})}\right)e_{2m+1}(x)\\&+\sum_{m=1}^{\infty}\left(\frac{\langle\mathcal{B}_2,e_{2m}\rangle\mu_{2m}\int_b^T\mathcal{X}_{2m}^{\alpha}(T,z)\mathcal{G}(z)dz}{\mathcal{Y}_{2m}^{\alpha}(b,t_0,T,\mathcal{X},\mathcal{G})}-\frac{\langle\mathcal{C}_2,e_{2m}\rangle\mu_{2m}\int_b^{t_0}\mathcal{X}_{2m}^{\alpha}(t_0,z)\mathcal{G}(z)dz}{\mathcal{Y}_{2m}^{\alpha}(b,t_0,T,\mathcal{X},\mathcal{G})}\right)e_{2m}(x).\end{aligned}\tag{18}$$

$$\begin{aligned}&\mathcal{F}(x)=\\&\sum_{m=0}^{\infty}\left(\frac{\langle\mathcal{C}_1,e_{2m+1}\rangle E_{\alpha,1}\big(-\lambda_{2m+1}^{s}\mu_{2m+1}\big(\log\frac{t_0}{b}\big)^{\alpha}\big)}{\mathcal{Y}_{2m+1}^{\alpha}(b,t_0,T,\mathcal{X},\mathcal{G})}-\frac{\langle\mathcal{B}_1,e_{2m+1}\rangle E_{\alpha,1}\big(-\lambda_{2m+1}^{s}\mu_{2m+1}\big(\log\frac{T}{b}\big)^{\alpha}\big)}{\mathcal{Y}_{2m+1}^{\alpha}(b,t_0,T,\mathcal{X},\mathcal{G})}\right)e_{2m+1}(x)\\&+\sum_{m=1}^{\infty}\left(\frac{\langle\mathcal{C}_2,e_{2m}\rangle E_{\alpha,1}\big(-\lambda_{2m}^{s}\mu_{2m}\big(\log\frac{t_0}{b}\big)^{\alpha}\big)}{\mathcal{Y}_{2m}^{\alpha}(b,t_0,T,\mathcal{X},\mathcal{G})}-\frac{\langle\mathcal{B}_2,e_{2m}\rangle E_{\alpha,1}\big(-\lambda_{2m}^{s}\mu_{2m}\big(\log\frac{T}{b}\big)^{\alpha}\big)}{\mathcal{Y}_{2m}^{\alpha}(b,t_0,T,\mathcal{X},\mathcal{G})}\right)e_{2m}(x),\end{aligned}\tag{19}$$

Now, using some memorized notation (18) and (19), we obtain

$$\begin{aligned}\mathcal{A}&=\mathcal{J}_1^{-1}\mathcal{B}_1+\mathcal{J}_2^{-1}\mathcal{C}_1+\mathcal{J}_3^{-1}\mathcal{B}_2+\mathcal{J}_4^{-1}\mathcal{C}_2,\\\mathcal{F}&=\mathcal{J}_{11}^{-1}\mathcal{C}_1+\mathcal{J}_{22}^{-1}\mathcal{B}_1+\mathcal{J}_{33}^{-1}\mathcal{C}_2+\mathcal{J}_{44}^{-1}\mathcal{B}_2,\end{aligned}$$

where $\mathcal{J}_i(i=1,2,3,4,11,22,33,44)$ are self-adjoint operators. When $m\to\infty$, then $\lambda_{2m+1},\lambda_{2m}\to\infty$, from the above Lemmas, it yields that $\mathcal{J}_i\to\infty(i=1,2,3,4,11,22,33,44)$, thus problem (1) is non-well-posed.

Theorem 3.1. *Suppose the priori bound condition (9) holds, then we have*

$$\|\mathcal{F}\|_2 \le \sqrt{2}\mathcal{Z}_{10}^{-\frac{\tau}{2(\tau+1)}} \mathcal{O}^{\frac{1}{\tau+1}} \left(\mathcal{Z}_2^2\lambda_1^{-2q}\|\mathcal{C}\|_2^2 + \mathcal{Z}_4^2\lambda_1^{-2q}\|\mathcal{B}\|_2^2\right)^{\frac{\tau}{2(\tau+1)}},$$
$$\|\mathcal{A}\|_2 \le \mathcal{Z}_{10}^{\dagger}\left(\|\mathcal{B}\|_2^2 + \|\mathcal{C}\|_2^2\right)^{\frac{\tau}{2(\tau+1)}} \mathcal{O}^{\frac{1}{\tau+1}}.$$

where $\tau, q = s-1 > 0$, $\mathcal{Z}_{10}^{\dagger} = 2\lambda_1^{(1-s)\frac{\tau}{2(\tau+1)}} \mathcal{G}_2^{\frac{1}{2}}$.

Proof. Using (19) and Hölder inequality, we obtain

$$\|\mathcal{F}\|_2^2 = \sum_{m=0}^{\infty} \left|\langle \mathcal{F}_1, e_{2m+1}\rangle\right|^2 + \sum_{m=1}^{\infty} \left|\langle \mathcal{F}_2, e_{2m}\rangle\right|^2$$
$$= \sum_{m=0}^{\infty}\left(\frac{\langle \mathcal{C}_1, e_{2m+1}\rangle E_{\alpha,1}\big(-\lambda_{2m+1}^s\mu_{2m+1}\big(\log\frac{t_0}{b}\big)^{\alpha}\big)}{\mathcal{Y}_{2m+1}^{\alpha}(b,t_0,T,\mathcal{X},\mathcal{G})} - \frac{\langle \mathcal{B}_1, e_{2m+1}\rangle E_{\alpha,1}\big(-\lambda_{2m+1}^s\mu_{2m+1}\big(\log\frac{T}{b}\big)^{\alpha}\big)}{\mathcal{Y}_{2m+1}^{\alpha}(b,t_0,T,\mathcal{X},\mathcal{G})}\right)^2$$
$$+ \sum_{m=1}^{\infty}\left(\frac{\langle \mathcal{C}_2, e_{2m}\rangle E_{\alpha,1}\big(-\lambda_{2m}^s\mu_{2m}\big(\log\frac{t_0}{b}\big)^{\alpha}\big)}{\mathcal{Y}_{2m}^{\alpha}(b,t_0,T,\mathcal{X},\mathcal{G})} - \frac{\langle \mathcal{B}_2, e_{2m}\rangle E_{\alpha,1}\big(-\lambda_{2m}^s\mu_{2m}\big(\log\frac{T}{b}\big)^{\alpha}\big)}{\mathcal{Y}_{2m}^{\alpha}(b,t_0,T,\mathcal{X},\mathcal{G})}\right)^2$$
$$:= \mathcal{I}_1 + \mathcal{I}_2.$$

and $\mathcal{I}_1$

$$= \sum_{m=0}^{\infty}\frac{\Big[\langle \mathcal{C}_{1m}, e_{2m+1}\rangle E_{\alpha,1}\big(-\lambda_{2m+1}^s\mu_{2m+1}\big(\log\frac{t_0}{b}\big)^{\alpha}\big) - \langle \mathcal{B}_1, e_{2m+1}\rangle E_{\alpha,1}\big(-\lambda_{2m+1}^s\mu_{2m+1}\big(\log\frac{T}{b}\big)^{\alpha}\big)\Big]^{\frac{2}{\tau+1}}}{\Big|\mathcal{Y}_{2m+1}^{\alpha}(b,t_0,T,\mathcal{X},\mathcal{G})\Big|^2}$$
$$\times\Big[\langle \mathcal{C}_1, e_{2m+1}\rangle E_{\alpha,1}\big(-\lambda_{2m+1}^s\mu_{2m+1}\big(\log\frac{t_0}{b}\big)^{\alpha}\big) - \langle \mathcal{B}_{1m}, e_{2m+1}\rangle E_{\alpha,1}\big(-\lambda_{2m+1}^s\mu_{2m+1}\big(\log\frac{T}{b}\big)^{\alpha}\big)\Big]^{\frac{2\tau}{\tau+1}}$$
$$\le \left[\sum_{m=0}^{\infty}\left[\frac{\Big(\langle \mathcal{C}_1, e_{2m+1}\rangle E_{\alpha,1}\big(-\lambda_{2m+1}^s\mu_{2m+1}\big(\log\frac{t_0}{b}\big)^{\alpha}\big) - \langle \mathcal{B}_1, e_{2m+1}\rangle E_{\alpha,1}\big(-\lambda_{2m+1}^s\mu_{2m+1}\big(\log\frac{T}{b}\big)^{\alpha}\big)\Big)}{|\mathcal{Y}_{2m+1}^{\alpha}(b,t_0,T,\mathcal{X},\mathcal{G})|}\right]^2\right]^{\frac{1}{\tau+1}}$$
$$\times\left[2\sum_{m=0}^{\infty}\left(E_{\alpha,1}\big(-\lambda_{2m+1}^s\mu_{2m+1}(\log\frac{t_0}{b})^{\alpha}\big)\big|\langle \mathcal{C}_1, e_{2m+1}\rangle\big|\right)^2\right.$$
$$\left.+2\sum_{m=0}^{\infty}\left(E_{\alpha,1}\left(-\lambda_{2m+1}^s\mu_{2m+1}(\log\frac{T}{b})^{\alpha}\right)\big|\langle \mathcal{B}_1, e_{2m+1}\rangle\big|\right)^2\right]^{\frac{\tau}{\tau+1}}$$
$$\le \left[\sum_{m=0}^{\infty}\mathcal{Z}_{10}^{-\tau}\lambda_{2m+1}^{(2s-2)\tau}\big|\langle \mathcal{F}_1, e_{2m+1}\rangle\big|^2\right]^{\frac{1}{\tau+1}}$$
$$\times\left[2\sum_{m=0}^{\infty}\left(\mathcal{Z}_2\lambda_{2m+1}^{1-s}\big|\langle \mathcal{C}_1, e_{2m+1}\rangle\big|\right)^2 + 2\sum_{m=0}^{\infty}\left(\mathcal{Z}_4\lambda_{2m+1}^{1-s}\big|\langle \mathcal{B}_{1m}, e_{2m+1}\rangle\big|\right)^2\right]^{\frac{\tau}{\tau+1}}$$
$$\le \left[\sum_{m=0}^{\infty}\mathcal{Z}_{10}^{-\tau}\lambda_{2m+1}^{2q\tau}\big|\langle \mathcal{F}_1, e_{2m+1}\rangle\big|^2\right]^{\frac{1}{\tau+1}}$$
$$\times\left[2\sum_{m=0}^{\infty}\left(\mathcal{Z}_2\lambda_{2m+1}^{-q}\big|\langle \mathcal{C}_1, e_{2m+1}\rangle\big|\right)^2 + 2\sum_{m=0}^{\infty}\left(\mathcal{Z}_4\lambda_{2m+1}^{-q}\big|\langle \mathcal{B}_1, e_{2m+1}\rangle\big|\right)^2\right]^{\frac{\tau}{\tau+1}}$$
$$\le 2^{\frac{\tau}{\tau+1}}\mathcal{Z}_{10}^{-\frac{\tau}{\tau+1}}\mathcal{O}^{\frac{2}{\tau+1}}\left(\mathcal{Z}_2^2\lambda_1^{-2q}\|\mathcal{C}_1\|^2 + \mathcal{Z}_4^2\lambda_1^{-2q}\|\mathcal{B}_1\|^2\right)^{\frac{\tau}{\tau+1}}.$$

Similarly,

$$\mathcal{I}_2 \le 2^{\frac{\tau}{\tau+2}}\mathcal{Z}_{10}^{-\frac{\tau}{\tau+1}}\mathcal{O}^{\frac{2}{\tau+1}}\left(\mathcal{Z}_2^2\lambda_1^{-2q}\|\mathcal{C}\|_2^2 + \mathcal{Z}_4^2\lambda_1^{-2q}\|\mathcal{B}\|_2^2\right)^{\frac{\tau}{\tau+1}}.$$

Thus,

$$\|\mathcal{F}\|_2 \le \sqrt{2}\mathcal{Z}_{10}^{-\frac{\tau}{2(\tau+1)}}\mathcal{O}^{\frac{1}{\tau+1}}\left(\mathcal{Z}_2^2\lambda_1^{-2q}\|\mathcal{C}\|^2+\mathcal{Z}_4^2\lambda_1^{-2q}\|\mathcal{B}\|^2\right)^{\frac{\tau}{2(\tau+1)}}.$$

A similar method can be used to obtain the conditional stability of the initial value:

$$\|\mathcal{A}\|_2 \le \mathcal{Z}_{10}^{\dagger}\left(\|\mathcal{B}\|_2^2+\|\mathcal{C}\|_2^2\right)^{\frac{\tau}{2(\tau+1)}}\mathcal{O}^{\frac{1}{\tau+1}}.$$

Then we complete the proof. □

4 A hybrid regularization method and convergent rates

In this section, we mainly apply the quasi-inverse and quasi-initial combined regularization methods to solve the problem (1). Furthermore, the convergence estimates by the priori regularization parameter selection rule and the posteriori regularization parameter selection rule are given respectively.

4.1 Regularization solutions

Since the inverse problem is ill-posed, we use a regularization method combining quasi-inverse and quasi-initial value to address it. Let $\theta_\gamma^\varepsilon(x,t)$ denote the solution of the following regularization problem:

$$\begin{cases} {}_{CH}\mathbb{D}_{b,t}^{\alpha}(1+k\mathcal{L})\theta_\gamma^\varepsilon(x,t)+\mathcal{L}^s\theta_\gamma^\varepsilon(x,t)=\mathcal{F}^\varepsilon(x)\mathcal{G}^\varepsilon(t)+\gamma\mathcal{L}\mathcal{F}^\varepsilon(x)\mathcal{G}^\varepsilon(t), & x\in\mathcal{D}, t\in(b,T], 0<\gamma<1,\ b>1,\\ \theta_\gamma^\varepsilon(x,b)=\mathcal{A}^\varepsilon(x)+\gamma\mathcal{L}^\varsigma\mathcal{A}^\varepsilon(x), & x\in\mathcal{D},\\ \theta_\gamma^\varepsilon(0,t)=\theta_\gamma^\varepsilon(\pi,t)=0, & t\in(b,T],\\ \theta_\gamma^\varepsilon(x,t_0)=\mathcal{B}^\varepsilon(x), & x\in\mathcal{D}, t_0\in(b,T]\\ \theta_\gamma^\varepsilon(x,T)=\mathcal{C}^\varepsilon(x), & x\in\mathcal{D}, \end{cases} \tag{20}$$

where $\gamma>0$ is a regularization parameter. Using the separation of variables method and the Laplace transform of the MittagLeffler function, we obtain $\theta_\gamma^\varepsilon(x,t)$ as follows

$$\begin{aligned} \theta_\gamma^\varepsilon(x,t)=&\sum_{m=0}^{\infty}\Bigg(E_{\alpha,1}\big(-\lambda_{2m+1}^s\mu_{2m+1}(\log\frac{t}{b})^\alpha\big)(1+\gamma\lambda_{2m+1}^\sigma)\langle\mathcal{A}_1^\varepsilon,e_{2m+1}\rangle\\ &+\langle\mathcal{F}_1^\varepsilon,e_{2m+1}\rangle(1+\gamma\lambda_{2m+1})\mu_{2m+1}\int_b^t\big(\log\frac{t}{z}\big)^{\alpha-1}E_{\alpha,\alpha}\big(-\lambda_{2m+1}^s\mu_{2m+1}(\log\frac{t}{z})^\alpha\big)\mathcal{G}^\varepsilon(z)\frac{dz}{z}\Bigg)e_{2m+1}(x)\\ &+\sum_{m=1}^{\infty}\Bigg(E_{\alpha,1}\big(-\lambda_{2m}^s\mu_{2m}(\log\frac{t}{b})^\alpha\big)(1+\gamma\lambda_{2m}^\sigma)\langle\mathcal{A}_2^\varepsilon,e_{2m}\rangle\\ &+\langle\mathcal{F}_2^\varepsilon,e_{2m}\rangle(1+\gamma\lambda_{2m})\mu_{2m}\int_b^t\big(\log\frac{t}{z}\big)^{\alpha-1}E_{\alpha,\alpha}\big(-\lambda_{2m}^s\mu_{2m}(\log\frac{t}{z})^\alpha\big)\mathcal{G}^\varepsilon(z)\frac{dz}{z}\Bigg)e_{2m}(x), \end{aligned}$$

and letting $t=t_0$, we have

$$\begin{aligned} \mathcal{B}_\gamma^\varepsilon(x)=&\sum_{m=0}^{\infty}\Bigg(E_{\alpha,1}\big(-\lambda_{2m+1}^s\mu_{2m+1}(\log\frac{t_0}{b})^\alpha\big)(1+\gamma\lambda_{2m+1}^\sigma)\langle\mathcal{A}_1^\varepsilon,e_{2m+1}\rangle\\ &+\langle\mathcal{F}_1^\varepsilon,e_{2m+1}\rangle(1+\gamma\lambda_{2m+1})\mu_{2m+1}\int_b^{t_0}\big(\log\frac{t_0}{z}\big)^{\alpha-1}E_{\alpha,\alpha}\big(-\lambda_{2m+1}^s\mu_{2m+1}(\log\frac{t_0}{z})^\alpha\big)\mathcal{G}^\varepsilon(z)\frac{dz}{z}\Bigg)e_{2m+1}(x)\\ &+\sum_{m=1}^{\infty}\Bigg(E_{\alpha,1}\big(-\lambda_{2m}^s(\log\frac{t_0}{b})^\alpha\mu_{2m}\big)(1+\gamma\lambda_{2m}^\sigma)\langle\mathcal{A}_2^\varepsilon,e_{2m}\rangle\\ &+\langle\mathcal{F}_2^\varepsilon,e_{2m}\rangle(1+\gamma\lambda_{2m})\mu_{2m}\int_b^{t_0}\big(\log\frac{t_0}{z}\big)^{\alpha-1}E_{\alpha,\alpha}\big(-\lambda_{2m}^s\mu_{2m}(\log\frac{t_0}{z})^\alpha\big)\mathcal{G}^\varepsilon(z)\frac{dz}{z}\Bigg)e_{2m}(x). \end{aligned}$$

and letting $t = T$, it gives

$$\begin{aligned}
\mathcal{C}_\gamma^\varepsilon(x) &= \sum_{m=0}^{\infty} \Big(E_{\alpha,1}\big(-\lambda_{2m+1}^s \mu_{2m+1} (\log \frac{T}{b})^\alpha\big)(1+\gamma\lambda_{2m+1}^\sigma)\langle \mathcal{A}_1^\varepsilon, e_{2m+1}\rangle \\
&+ \langle \mathcal{F}_1^\varepsilon, e_{2m+1}\rangle (1+\gamma\lambda_{2m+1})\mu_{2m+1} \int_b^T \big(\log \frac{T}{z}\big)^{\alpha-1} E_{\alpha,\alpha}\big(-\lambda_{2m+1}^s \mu_{2m+1} (\log \frac{T}{z})^\alpha\big)\mathcal{G}^\varepsilon(z)\frac{dz}{z}\Big) e_{2m+1}(x) \\
&+ \sum_{m=1}^{\infty} \Big(E_{\alpha,1}\big(-\lambda_{2m}^s \mu_{2m} (\log \frac{T}{b})^\alpha\big)(1+\gamma\lambda_{2m}^\sigma)\langle \mathcal{A}_2^\varepsilon, e_{2m}\rangle \\
&+ \langle \mathcal{F}_2^\varepsilon, e_{2m}\rangle (1+\gamma\lambda_{2m})\mu_{2m} \int_b^T \big(\log \frac{T}{z}\big)^{\alpha-1} E_{\alpha,\alpha}\big(-\lambda_{2m}^s \mu_{2m} (\log \frac{T}{z})^\alpha\big)\mathcal{G}^\varepsilon(z)\frac{dz}{z}\Big) e_{2m}(x).
\end{aligned}$$

Similar in setting (17), we have

$$\begin{aligned}
\mathcal{Y}_m^\alpha(\sigma,\gamma,b,t_0,T,\mathcal{X},\mathcal{G}^\varepsilon) &= (1+\gamma\lambda_m^\varsigma)(1+\gamma\lambda_m)\Big(E_{\alpha,1}\big(-\lambda_m^s\mu_m\big(\log\frac{t_0}{b}\big)^\alpha\big)\mu_m \int_b^T \mathcal{X}_m^\alpha(T,z)\mathcal{G}^\varepsilon(z)dz \\
&- E_{\alpha,1}\big(-\lambda_m^s\mu_m\big(\log\frac{T}{b}\big)^\alpha\big)\mu_m \int_b^{t_0} \mathcal{X}_m^\alpha(t_0,z)\mathcal{G}^\varepsilon(z)dz\Big).
\end{aligned}$$

$$\begin{aligned}
&\mathcal{A}_\gamma^\varepsilon(x) \\
&= \sum_{m=0}^{\infty} \frac{(1+\gamma\lambda_{2m+1})\Big(\langle \mathcal{B}_1^\varepsilon, e_{2m+1}\rangle \mu_{2m+1}\int_b^T \mathcal{X}_{2m+1}^\alpha(T,z)\mathcal{G}^\varepsilon(z)dz - \langle \mathcal{C}_1^\varepsilon, e_{2m+1}\rangle \mu_{2m+1}\int_b^{t_0} \mathcal{X}_{2m+1}^\alpha(t_0,z)\mathcal{G}^\varepsilon(z)dz\Big)}{\mathcal{Y}_{2m+1}^\alpha(\sigma,\gamma,b,t_0,T,\mathcal{X},\mathcal{G}^\varepsilon)} e_{2m+1}(x) \\
&+ \sum_{m=1}^{\infty} \frac{(1+\gamma\lambda_{2m})\Big(\langle \mathcal{B}_2^\varepsilon, e_{2m}\rangle \mu_{2m}\int_b^T \mathcal{X}_{2m}^\alpha(T,z)\mathcal{G}^\varepsilon(z)dz - \langle \mathcal{C}_2^\varepsilon, e_{2m}\rangle \mu_{2m}\int_b^{t_0} \mathcal{X}_{2m}^\alpha(t_0,z)\mathcal{G}^\varepsilon(z)dz\Big)}{\mathcal{Y}_{2m}^\alpha(\sigma,\gamma,b,t_0,T,\mathcal{X},\mathcal{G}^\varepsilon)} e_{2m}(x), \text{ and}
\end{aligned} \tag{21}$$

$$\begin{aligned}
&\mathcal{F}_\gamma^\varepsilon(x) \\
&= \sum_{m=0}^{\infty} \frac{(1+\gamma\lambda_{2m+1}^\varsigma)\Big(\langle \mathcal{C}_1^\varepsilon, e_{2m+1}\rangle E_{\alpha,1}(-\lambda_{2m+1}^s\mu_{2m+1}(\log\frac{t_0}{b})^\alpha) - \langle \mathcal{B}_1^\varepsilon, e_{2m+1}\rangle E_{\alpha,1}(-\lambda_{2m+1}^s\mu_{2m+1}(\log\frac{T}{b})^\alpha)\Big)}{\mathcal{Y}_{2m+1}^\alpha(\sigma,\gamma,b,t_0,T,\mathcal{X},\mathcal{G}^\varepsilon)} e_{2m+1}(x) \\
&+ \sum_{m=1}^{\infty} \frac{(1+\gamma\lambda_{2m}^\varsigma)\Big(\langle \mathcal{C}_2^\varepsilon, e_{2m}\rangle E_{\alpha,1}(-\lambda_{2m}^s\mu_{2m}(\log\frac{t_0}{b})^\alpha) - \langle \mathcal{B}_2^\varepsilon, e_{2m}\rangle E_{\alpha,1}(-\lambda_{2m}^s\mu_{2m}(\log\frac{T}{b})^\alpha)\Big)}{\mathcal{Y}_{2m}^\alpha(\sigma,\gamma,b,t_0,T,\mathcal{X},\mathcal{G}^\varepsilon)} e_{2m}(x).
\end{aligned} \tag{22}$$

5 Convergent error estimations of source term

5.1 The priori convergent rate

The rate at which the regularization solution of the source term converges to the exact solution can be obtained under the priori regularization parameter selection rule.

Theorem 5.1. *For $1 < s < 2$, and let $\mathcal{F}$ be given by (19) and $\mathcal{F}_\gamma^\varepsilon$ be given by (22). Suppose that the priori bound condition (10) and the noise assumptions (6) hold, then by choosing* $\gamma = \begin{cases} \big(\frac{\varepsilon}{\mathcal{O}}\big)^{\frac{1}{\sigma+1}}, & 0 < \sigma < 1 \\ \big(\frac{\varepsilon}{\mathcal{O}}\big)^{\frac{1}{2}}, & \sigma \geq 1, \end{cases}$

then, it gives

$$\|\mathcal{F}_\gamma^\varepsilon - \mathcal{F}\|_2 \leq \begin{cases} \left(\sqrt{\mathcal{C}_1^2 + \mathcal{C}_3^2} + \mathcal{C}_5\right)\mathcal{O}^{\frac{1}{\sigma+1}}\varepsilon^{\frac{\sigma}{\sigma+1}}, & 0 < \sigma < 1, \\ \left(\sqrt{\mathcal{C}_2^2 + \mathcal{C}_4^2} + \mathcal{C}_5\right)\mathcal{O}^{\frac{1}{2}}\varepsilon^{\frac{1}{2}}, & \sigma \geq 1. \end{cases}$$

where $\mathcal{C}_5 = \left(\frac{1}{\lambda_1^{2-s}}\left(\frac{\mathcal{Z}_2^* + \mathcal{Z}_3^*}{\mathcal{Z}_{12}} + \frac{\mathcal{Z}_2^{**} + \mathcal{Z}_3^{**}}{\mathcal{Z}_{12}}\right) + 2\max\left\{\frac{\mathcal{Z}_{15}^*}{\mathcal{Z}_{12}^*}, \frac{\mathcal{Z}_{15}^{**}}{\mathcal{Z}_{12}^{**}}\right\}\|\mathcal{F}\|_2\right)$.

Proof. Put $\mathcal{J}_{2m+1}^{\varsigma} = (1 + \gamma\lambda_{2m+1}^{\varsigma})^{-1}$, and $\mathcal{J}_{2m}^{\varsigma} = (1 + \gamma\lambda_{2m}^{\varsigma})^{-1}$, we get

$$\begin{aligned}
&\|\mathcal{F}_\gamma^\varepsilon - \mathcal{F}_\gamma\|_2 \\
&= \Bigg\| \sum_{m=0}^{\infty} \mathcal{J}_{2m+1}\Bigg(\frac{\langle \mathcal{C}_1^\varepsilon - \mathcal{C}_1, e_{2m+1}\rangle E_{\alpha,1}(-\lambda_{2m+1}^s \mu_{2m+1}(\log\frac{t_0}{z})^\alpha)}{\mathcal{Y}_{2m+1}^\alpha(b, t_0, T, \mathcal{X}, \mathcal{G}^\varepsilon)} \\
&\qquad - \frac{\langle \mathcal{B}_1^\varepsilon - \mathcal{B}_1, e_{2m+1}\rangle E_{\alpha,1}(-\lambda_{2m+1}^s \mu_{2m+1}(\log\frac{T}{z})^\alpha)}{\mathcal{Y}_{2m+1}^\alpha(b, t_0, T, \mathcal{X}, \mathcal{G}^\varepsilon)}\Bigg) e_{2m+1}(x) \\
&\quad + \sum_{m=1}^{\infty} \mathcal{J}_{2m}\Bigg(\frac{\langle \mathcal{C}_2^\varepsilon - \mathcal{C}_2, e_{2m}\rangle E_{\alpha,1}(-\lambda_{2m}^s \mu_{2m}(\log\frac{t_0}{z})^\alpha)}{\mathcal{Y}_{2m}^\alpha(b, t_0, T, \mathcal{X}, \mathcal{G}^\varepsilon)} - \frac{\langle \mathcal{B}_2^\varepsilon - \mathcal{B}_2, e_{2m}\rangle E_{\alpha,1}(-\lambda_{2m}^s \mu_{2m}(\log\frac{T}{z})^\alpha)}{\mathcal{Y}_{2m}^\alpha(b, t_0, T, \mathcal{X}, \mathcal{G}^\varepsilon)}\Bigg) e_{2m}(x) \\
&\quad + \sum_{m=0}^{\infty} \mathcal{J}_{2m+1}\varepsilon \frac{\mathcal{Y}_{2m+1}^\alpha(b, t_0, T, \mathcal{X})}{\mathcal{Y}_{2m+1}^\alpha(b, t_0, T, \mathcal{X}, \mathcal{G}^\varepsilon)} \\
&\qquad \times \frac{\left(\langle \mathcal{C}_1, e_{2m+1}\rangle E_{\alpha,1}(-\lambda_{2m+1}^s \mu_{2m+1}(\log\frac{t_0}{b})^\alpha) - \langle \mathcal{B}_1, e_{2m+1}\rangle E_{\alpha,1}(-\lambda_{2m+1}^s \mu_{2m+1}(\log\frac{T}{b})^\alpha)\right)}{\mathcal{Y}_{2m+1}^\alpha(b, t_0, T, \mathcal{X}, \mathcal{G})} \\
&+ \sum_{m=1}^{\infty} \mathcal{J}_{2m}\varepsilon \frac{\mathcal{Y}_{2m}^\alpha(b, t_0, T, \mathcal{X})}{\mathcal{Y}_{2m}^\alpha(b, t_0, T, \mathcal{X}, \mathcal{G}^\varepsilon)} \frac{\left(\langle \mathcal{C}_2, e_{2m}\rangle E_{\alpha,1}(-\lambda_{2m}^s \mu_{2m}(\log\frac{t_0}{b})^\alpha) - \langle \mathcal{B}_2, e_{2m}\rangle E_{\alpha,1}(-\lambda_{2m}^s \mu_{2m}(\log\frac{T}{b})^\alpha)\right)}{\mathcal{Y}_{2m}^\alpha(b, t_0, T, \mathcal{X}, \mathcal{G})}\Bigg\|_2
\end{aligned} \tag{23}$$

From (23), it gives

$$\begin{aligned}
\|\mathcal{F}_\gamma^\varepsilon - \mathcal{F}_\gamma\|_2 \leq &\left(\sum_{m=0}^{\infty} \mathcal{J}_{2m+1}^2 \left(\frac{E_{\alpha,1}\left(-\lambda_{2m+1}^s \mu_{2m+1}\left(\log\frac{t_0}{b}\right)^\alpha\right)|\langle \mathcal{C}_1^\varepsilon - \mathcal{C}_1, e_{2m+1}\rangle|}{\mathcal{Y}_{2m+1}^\alpha(b, t_0, T, \mathcal{X}, \mathcal{G}^\varepsilon)}\right)^2\right)^{\frac{1}{2}} \\
&+ \left(\sum_{m=0}^{\infty} \mathcal{J}_{2m+1}^2 \left(\frac{E_{\alpha,1}\left(-\lambda_{2m+1}^s \mu_{2m+1}\left(\log\frac{T}{b}\right)^\alpha\right)|\langle \mathcal{B}_1^\varepsilon - \mathcal{B}_1, e_{2m+1}\rangle|}{\mathcal{Y}_{2m+1}^\alpha(b, t_0, T, \mathcal{X}, \mathcal{G}^\varepsilon)}\right)^2\right)^{\frac{1}{2}} \\
&+ \left(\sum_{m=1}^{\infty} \mathcal{J}_{2m}^2 \left(\frac{E_{\alpha,1}\left(-\lambda_{2m}^s \mu_{2m}(\log\frac{t_0}{b})^\alpha\right)|\langle \mathcal{C}_2^\varepsilon - \mathcal{C}_2, e_{2m}\rangle|}{\mathcal{Y}_{2m}^\alpha(b, t_0, T, \mathcal{X}, \mathcal{G}^\varepsilon)}\right)^2\right)^{\frac{1}{2}} \\
&+ \left(\sum_{m=1}^{\infty} \mathcal{J}_{2m}^2 \left(\frac{E_{\alpha,1}\left(-\lambda_{2m}^s \mu_{2m}(\log\frac{T}{b})^\alpha\right)|\langle \mathcal{B}_2^\varepsilon - \mathcal{B}_2, e_{2m}\rangle|}{\mathcal{Y}_{2m}^\alpha(b, t_0, T, \mathcal{X}, \mathcal{G}^\varepsilon)}\right)^2\right)^{\frac{1}{2}} \\
&+ \left(\sum_{m=0}^{\infty} \left(\frac{\varepsilon}{\gamma}\right)^2 \left(\frac{\mathcal{Y}_{2m+1}^\alpha(b, t_0, T, \mathcal{X})}{\mathcal{Y}_{2m+1}^\alpha(b, t_0, T, \mathcal{X}, \mathcal{G}^\varepsilon)}\right)^2 \left(\frac{\langle \mathcal{C}_1, e_{2m+1}\rangle E_{\alpha,1}(-\lambda_{2m+1}^s \mu_{2m+1}(\log\frac{t_0}{b})^\alpha)}{\mathcal{Y}_{2m+1}^\alpha(b, t_0, T, \mathcal{X}, \mathcal{G})}\right.\right. \\
&\qquad \left.\left. - \frac{\langle \mathcal{B}_1, e_{2m+1}\rangle E_{\alpha,1}(-\lambda_{2m+1}^s \mu_{2m+1}(\log\frac{T}{b})^\alpha)}{\mathcal{Y}_{2m+1}^\alpha(b, t_0, T, \mathcal{X}, \mathcal{G})}\right)^2\right)^{\frac{1}{2}} \\
&+ \left(\sum_{m=1}^{\infty} \left(\frac{\varepsilon}{\gamma}\right)^2 \left(\frac{\mathcal{Y}_{2m}^\alpha(b, t_0, T, \mathcal{X})}{\mathcal{Y}_{2m}^\alpha(b, t_0, T, \mathcal{X}, \mathcal{G}^\varepsilon)}\right)^2 \left(\frac{\langle \mathcal{C}_2, e_{2m}\rangle E_{\alpha,1}(-\lambda_{2m}^s \mu_{2m}(\log\frac{t_0}{b})^\alpha)}{\mathcal{Y}_{2m}^\alpha(b, t_0, T, \mathcal{X}, \mathcal{G})}\right.\right. \\
&\qquad \left.\left. - \frac{\langle \mathcal{B}_2, e_{2m}\rangle E_{\alpha,1}(-\lambda_{2m}^s \mu_{2m}(\log\frac{T}{b})^\alpha)}{\mathcal{Y}_{2m}^\alpha(b, t_0, T, \mathcal{X}, \mathcal{G})}\right)^2\right)^{\frac{1}{2}}
\end{aligned} \tag{24}$$

It yields

$$\begin{aligned}\|\mathcal{F}_\gamma^\varepsilon-\mathcal{F}_\gamma\|_2 \le& \sup_{m\ge0}\left|\mathcal{J}_{2m+1}\frac{E_{\alpha,1}\big(-\lambda_{2m+1}^s\mu_{2m+1}(\log\frac{t_0}{b})^\alpha\big)}{\mathcal{Y}_{2m+1}^\alpha(b,t_0,T,\mathcal{X},\mathcal{G}^\varepsilon)}\right|\varepsilon+\sup_{m\ge0}\left|\mathcal{J}_{2m+1}\frac{E_{\alpha,1}\big(-\lambda_{2m+1}^s\mu_{2m+1}(\log\frac{T}{b})^\alpha\big)}{\mathcal{Y}_{2m+1}^\alpha(b,t_0,T,\mathcal{X},\mathcal{G}^\varepsilon)}\right|\varepsilon\\
&+\sup_{m\ge1}\left|\mathcal{J}_{2m}\frac{E_{\alpha,1}\big(-\lambda_{2m}^s\mu_{2m}(\log\frac{t_0}{b})^\alpha\big)}{\mathcal{Y}_{2m}^\alpha(b,t_0,T,\mathcal{X},\mathcal{G}^\varepsilon)}\right|\varepsilon+\sup_{m\ge1}\left|\mathcal{J}_{2m}\frac{E_{\alpha,1}\big(-\lambda_{2m}^s\mu_{2m}(\log\frac{T}{b})^\alpha\big)}{\mathcal{Y}_{2m}^\alpha(b,t_0,T,\mathcal{X},\mathcal{G}^\varepsilon)}\right|\varepsilon\\
&+\Big(\frac{\varepsilon}{\gamma}\frac{\mathcal{Z}_{15}^*}{\mathcal{Z}_{12}^*}\Big)\sum_{m=0}^\infty\frac{\langle\mathcal{C}_1,e_{2m+1}\rangle E_{\alpha,1}(-\lambda_{2m+1}^s\mu_{2m+1}(\log\frac{t_0}{b})^\alpha)}{\mathcal{Y}_{2m+1}^\alpha(b,t_0,T,\mathcal{X},\mathcal{G})}\\
&\qquad-\frac{\langle\mathcal{B}_1,e_{2m+1}\rangle E_{\alpha,1}(-\lambda_{2m+1}^s\mu_{2m+1}(\log\frac{T}{b})^\alpha)}{\mathcal{Y}_{2m+1}^\alpha(b,t_0,T,\mathcal{X},\mathcal{G})}\\
&+\Big(\frac{\varepsilon}{\gamma}\frac{\mathcal{Z}_{15}^{**}}{\mathcal{Z}_{12}^{**}}\Big)\sum_{m=1}^\infty\frac{\langle\mathcal{C}_2,e_{2m}\rangle E_{\alpha,1}(-\lambda_{2m}^s\mu_{2m}(\log\frac{t_0}{b})^\alpha)}{\mathcal{Y}_{2m}^\alpha(b,t_0,T,\mathcal{X},\mathcal{G})}\\
&\qquad-\frac{\langle\mathcal{B}_2,e_{2m}\rangle E_{\alpha,1}(-\lambda_{2m}^s\mu_{2m}(\log\frac{T}{b})^\alpha)}{\mathcal{Y}_{2m}^\alpha(b,t_0,T,\mathcal{X},\mathcal{G})},\end{aligned}\tag{25}$$

From (25), we conclude that

$$\begin{aligned}\|\mathcal{F}_\gamma^\varepsilon-\mathcal{F}_\gamma\|_2 \le& \sup_{m\ge0}\Big|\mathcal{J}_{2m+1}\lambda_{2m+1}^{s-1}\frac{\mathcal{Z}_2^*}{\mathcal{Z}_{12}}\Big|\varepsilon+\sup_{m\ge0}\Big|\mathcal{J}_{2m+1}\lambda_{2m+1}^{s-1}\frac{\mathcal{Z}_3^*}{\mathcal{Z}_{12}}\Big|\varepsilon+\sup_{m\ge1}\Big|\mathcal{J}_{2m}\lambda_{2m}^{s-1}\frac{\mathcal{Z}_2^{**}}{\mathcal{Z}_{12}}\Big|\varepsilon\\
&+\sup_{m\ge1}\Big|\mathcal{J}_{2m}\lambda_{2m}^{s-1}\frac{\mathcal{Z}_3^{**}}{\mathcal{Z}_{12}}\Big|\varepsilon+2\frac{\varepsilon}{\gamma}\max\Big\{\frac{\mathcal{Z}_{15}^*}{\mathcal{Z}_{12}^*},\frac{\mathcal{Z}_{15}^{**}}{\mathcal{Z}_{12}^{**}}\Big\}\|\mathcal{F}\|_2\\
\le&\frac{\varepsilon}{\gamma}\left(\frac{1}{\lambda_1^{2-s}}\Big(\frac{\mathcal{Z}_2^*+\mathcal{Z}_3^*}{\mathcal{Z}_{12}}+\frac{\mathcal{Z}_2^{**}+\mathcal{Z}_3^{**}}{\mathcal{Z}_{12}}\Big)+2\max\Big\{\frac{\mathcal{Z}_{15}^*}{\mathcal{Z}_{12}^*},\frac{\mathcal{Z}_{15}^{**}}{\mathcal{Z}_{12}^{**}}\Big\}\|\mathcal{F}\|_2\right).\end{aligned}$$

Then we estimate the second term by Lemma 2.6,

$$\begin{aligned}\|\mathcal{F}_\gamma-\mathcal{F}\|_2^2=&\Bigg\|\sum_{m=0}^\infty\mathcal{J}_{2m+1}\gamma\lambda_{2m+1}\frac{\Big(\langle\mathcal{C}_1,e_{2m+1}\rangle E_{\alpha,1}(-\lambda_{2m+1}^s\mu_{2m+1}(\log\frac{t_0}{b})^\alpha)}{\mathcal{Y}_{2m+1}^\alpha(b,t_0,T,\mathcal{X},\mathcal{G})}\\
&\qquad-\frac{\langle\mathcal{B}_1,e_{2m+1}\rangle E_{\alpha,1}(-\lambda_{2m+1}^s\mu_{2m+1}(\log\frac{T}{b})^\alpha)\Big)}{\mathcal{Y}_{2m+1}^\alpha(b,t_0,T,\mathcal{X},\mathcal{G})}e_{2m+1}(x)\\
&+\sum_{m=1}^\infty\gamma\mathcal{J}_{2m}\lambda_{2m}\frac{\Big(\langle\mathcal{C}_{2m},e_{2m}\rangle E_{\alpha,1}(-\lambda_{2m}^s\mu_{2m}(\log\frac{t_0}{b})^\alpha)}{\mathcal{Y}_{2m}^\alpha(b,t_0,T,\mathcal{X},\mathcal{G})}\\
&\qquad-\frac{\langle\mathcal{B}_2,e_{2m}\rangle E_{\alpha,1}(-\lambda_{2m}^s\mu_{2m}(\log\frac{T}{b})^\alpha)\Big)}{\mathcal{Y}_{2m}^\alpha(b,t_0,T,\mathcal{X},\mathcal{G})}e_{2m}(x)\Bigg\|_2^2\\
\le&\sum_{m=0}^\infty\frac{\lambda_{2m+1}^{2\sigma}}{\lambda_{2m+1}^{2\sigma}}\gamma^2\mathcal{J}_{2m+1}^2\lambda_{2m+1}^2\Bigg(\frac{\Big(\langle\mathcal{C}_1,e_{2m+1}\rangle E_{\alpha,1}(-\lambda_{2m+1}^s\mu_{2m+1}(\log\frac{t_0}{b})^\alpha)}{\mathcal{Y}_{2m+1}^\alpha(b,t_0,T,\mathcal{X},\mathcal{G})}\\
&\qquad-\frac{\langle\mathcal{B}_1,e_{2m+1}\rangle E_{\alpha,1}(-\lambda_{2m+1}^s\mu_{2m+1}(\log\frac{T}{b})^\alpha)\Big)}{\mathcal{Y}_{2m+1}^\alpha(b,t_0,T,\mathcal{X},\mathcal{G})}\Bigg)^2\\
&+\sum_{m=1}^\infty\frac{\lambda_{2m}^{2\sigma}}{\lambda_{2m}^{2\sigma}}\gamma^2\mathcal{J}_m^2\lambda_{2m}^2\Bigg(\frac{\Big(\langle\mathcal{C}_2,e_{2m}\rangle E_{\alpha,1}(-\lambda_{2m}^s\mu_{2m}(\log\frac{t_0}{b})^\alpha)}{\mathcal{Y}_{2m}^\alpha(b,t_0,T,\mathcal{X},\mathcal{G})}\\
&\qquad-\frac{\langle\mathcal{B}_2,e_{2m}\rangle E_{\alpha,1}(-\lambda_{2m}^s\mu_{2m}(\log\frac{T}{b})^\alpha)\Big)}{\mathcal{Y}_{2m}^\alpha(b,t_0,T,\mathcal{X},\mathcal{G})}\Bigg)^2\end{aligned}\tag{26}$$

From (26), this leads to

$$\|\mathcal{F}_\gamma - \mathcal{F}\|_2^2 \le \Big(\big(\sup_{m\ge 0} \mathcal{H}(2m+1)\big)^2 + \big(\sup_{m\ge 1} \mathcal{H}(2m)\big)^2 \Big) \mathcal{O}^2, \tag{27}$$

whereby

$$\mathcal{H}(2m) = \frac{\gamma\lambda_{2m}^{1-\sigma}}{1+\gamma\lambda_{2m}} \le \begin{cases} \mathcal{C}_1\gamma^\sigma, & 0<\sigma<1, \\ \mathcal{C}_2\gamma, & \sigma \ge 1, \end{cases}$$

and we have similar reviews for $\mathcal{H}(2m+1)$, therefore

$$\|\mathcal{F}_\gamma - \mathcal{F}\|_2 \le \begin{cases} \sqrt{\mathcal{C}_1^2+\mathcal{C}_3^2}\,\mathcal{O}\gamma^\sigma, & 0<\sigma<1, \\ \sqrt{\mathcal{C}_2^2+\mathcal{C}_4^2}\,\mathcal{O}\gamma, & \sigma\ge 1. \end{cases}$$

Combining (23) and (26), we choose the regularization parameter γ by

$$\gamma = \begin{cases} \left(\frac{\varepsilon}{\mathcal{O}}\right)^{\frac{1}{\sigma+1}}, & 0<\sigma<1, \\ \left(\frac{\varepsilon}{\mathcal{O}}\right)^{\frac{1}{2}}, & \sigma\ge 1. \end{cases}$$

This implies that

$$\|\mathcal{F}_\gamma^\varepsilon - \mathcal{F}\|_2 \le \begin{cases} \left(\sqrt{\mathcal{C}_1^2+\mathcal{C}_3^2}+\mathcal{C}_5\right)\mathcal{O}^{\frac{1}{\sigma+1}}\varepsilon^{\frac{\sigma}{\sigma+1}}, & 0<\sigma<1, \\ \left(\sqrt{\mathcal{C}_2^2+\mathcal{C}_4^2}+\mathcal{C}_5\right)\mathcal{O}^{\frac{1}{2}}\varepsilon^{\frac{1}{2}}, & \sigma\ge 1. \end{cases}$$

in which

$$\mathcal{C}_5 = \left(\frac{1}{\lambda_1^{2-s}} \left(\frac{\mathcal{Z}_2^* + \mathcal{Z}_3^*}{\mathcal{Z}_{12}} + \frac{\mathcal{Z}_2^{**} + \mathcal{Z}_3^{**}}{\mathcal{Z}_{12}} \right) + 2\max\left\{ \frac{\mathcal{Z}_{15}^*}{\mathcal{Z}_{12}^*}, \frac{\mathcal{Z}_{15}^{**}}{\mathcal{Z}_{12}^{**}} \right\} \|\mathcal{F}\|_2 \right).$$

Then we complete the proof. □

6 Convergent error estimations of the initial value

6.1 The priori convergent rate

Theorem 6.1. *Let $\mathcal{A}$ be given by (18) and $\mathcal{A}_\gamma^\varepsilon(x)$ be given by (21). Suppose the priori bound condition (10) and the noise assumptions (6) hold, then if we choose*

$$\gamma = \begin{cases} \left(\frac{\varepsilon}{\mathcal{O}}\right)^{\frac{1}{\sigma+1}}, & 0<\sigma<1, \\ \left(\frac{\varepsilon}{\mathcal{O}}\right)^{\frac{1}{2}}, & \sigma\ge 1, \end{cases}$$

We get

$$\|\mathcal{A}_\gamma^\varepsilon - \mathcal{A}\|_2 \le \begin{cases} \left(\sqrt{\mathcal{C}_1^2+\mathcal{C}_3^2}\lambda_1^{\sigma(\varsigma-1)}+\mathcal{C}_6\right)\mathcal{O}^{\frac{1}{\sigma+1}}\varepsilon^{\frac{\sigma}{\sigma+1}}, & 0<\sigma<1 \\ \left(\sqrt{\mathcal{C}_2^2+\mathcal{C}_4^2}\lambda_1^{\sigma(\varsigma-1)}+\mathcal{C}_6\right)\mathcal{O}^{\frac{1}{2}}\varepsilon^{\frac{1}{2}}, & \sigma\ge 1. \end{cases}$$

where $\mathcal{C}_6 = \left(\frac{2\mathcal{Z}_2^*}{\mathcal{Z}_{12}^*\lambda_1^\varsigma} + \frac{\overline{\mathcal{B}}+\overline{\mathcal{C}}}{\lambda_1^{\varsigma+s-1}} + 2\frac{1}{\lambda_1^\varsigma}\max\left\{ \frac{\mathcal{Z}_{15}^*}{\mathcal{Z}_{12}^*}, \frac{\mathcal{Z}_{15}^{**}}{\mathcal{Z}_{12}^{**}} \right\} \|\mathcal{F}\|_2 \right)$.

Proof. By the triangle inequality, we have

$$\|\mathcal{A}_\gamma^\varepsilon - \mathcal{A}\|_2 \le \|\mathcal{A}_\gamma^\varepsilon - \mathcal{A}_\gamma\|_2 + \|\mathcal{A}_\gamma - \mathcal{A}\|_2 \tag{28}$$

Then we estimate the second term of (28)

$$
\begin{aligned}
&\|\mathcal{A}_\gamma - \mathcal{A}\|_2^2 \\
= &\Bigg\| \sum_{m=0}^{\infty} \varepsilon^2 \left(\mathcal{J}^{\varsigma}_{2m+1}\right)^2 \frac{\left(\langle \mathcal{B}_1, e_{2m+1}\rangle \mu_{2m+1} \int_b^T \mathcal{X}^{\alpha}_{2m+1}(T,z)\mathcal{G}(z)dz - \langle \mathcal{C}_1, e_{2m+1}\rangle \mu_{2m+1} \int_b^{t_0} \mathcal{X}^{\alpha}_{2m+1}(t_0,z)\mathcal{G}(z)dz\right)}{\mathcal{Y}^{\alpha}_{2m+1}(b,t_0,T,\mathcal{X},\mathcal{G})} e_{2m+1}(x) \\
&+ \sum_{m=1}^{\infty} \gamma \lambda^{\varsigma}_{2m} \mathcal{J}^{\varsigma}_{2m} \frac{\left(\langle \mathcal{B}_1, e_{2m}\rangle \mu_{2m} \int_b^T \mathcal{X}^{\alpha}_{2m}(T,z)\mathcal{G}(z)dz - \langle \mathcal{C}_1, e_{2m}\rangle \mu_{2m} \int_b^{t_0} \mathcal{X}^{\alpha}_{2m}(t_0,z)\mathcal{G}(z)dz\right)}{\mathcal{Y}^{\alpha}_{2m}(b,t_0,T,\mathcal{X},\mathcal{G})} e_{2m}(x) \Bigg\|_2^2
\end{aligned}
\tag{29}
$$

From (30), we get

$$
\begin{aligned}
&\|\mathcal{A}_\gamma - \mathcal{A}\|_2^2 \\
\le &\sum_{m=0}^{\infty} \left(\frac{\gamma \lambda^{\varsigma}_{2m+1}}{1+\gamma\lambda^{\varsigma}_{2m+1}}\right)^2 \left(\frac{\left(\langle \mathcal{B}_1, e_{2m+1}\rangle \mu_{2m+1} \int_b^T \mathcal{X}^{\alpha}_{2m+1}(T,z)\mathcal{G}(z)dz - \langle \mathcal{C}_1, e_{2m+1}\rangle \mu_{2m+1} \int_b^{t_0} \mathcal{X}^{\alpha}_{2m+1}(t_0,z)\mathcal{G}(z)dz\right)}{\mathcal{Y}^{\alpha}_{2m+1}(b,t_0,T,\mathcal{X},\mathcal{G})}\right)^2 \\
&\frac{\lambda^{2\sigma}_{2m+1}}{\lambda^{2\sigma}_{2m+1}} + \sum_{m=1}^{\infty} \left(\frac{\gamma \lambda^{\varsigma}_{2m}}{1+\gamma\lambda^{\varsigma}_{2m}}\right)^2 \left(\frac{\left(\langle \mathcal{B}_1, e_{2m}\rangle \mu_{2m} \int_b^T \mathcal{X}^{\alpha}_{2m}(T,z)\mathcal{G}(z)dz - \langle \mathcal{C}_1, e_{2m}\rangle \mu_{2m} \int_b^{t_0} \mathcal{X}^{\alpha}_{2m}(t_0,z)\mathcal{G}(z)dz\right)}{\mathcal{Y}^{\alpha}_{2m}(b,t_0,T,\mathcal{X},\mathcal{G})}\right)^2 \frac{\lambda^{2\sigma}_{2m}}{\lambda^{2\sigma}_{2m}} \\
\le &\left(\left(\sup_{m\ge 0} \mathcal{H}(2m+1)\right)^2 + \left(\sup_{m\ge 1} \mathcal{H}(2m)\right)^2\right)\mathcal{O}^2,
\end{aligned}
$$

whereby

$$
\mathcal{H}(2m+1) = \frac{\gamma \lambda^{(1-\sigma)\varsigma}_{2m+1}}{1+\gamma\lambda^{\varsigma}_{2m+1}} \lambda^{\sigma(\varsigma-1)}_{2m+1} \le \begin{cases} \mathcal{C}_1 \lambda_1^{\sigma(\varsigma-1)} \gamma^{\sigma}, & 0<\sigma<1, \\ \mathcal{C}_2 \lambda_1^{\sigma(\varsigma-1)} \gamma, & \sigma \ge 1, \end{cases}
$$

$$
\mathcal{H}(2m) = \frac{\gamma \lambda^{(1-\sigma)\varsigma}_{2m}}{1+\gamma\lambda^{\varsigma}_{2m}} \lambda^{\sigma(\varsigma-1)}_{2m} \le \begin{cases} \mathcal{C}_3 \lambda_1^{\sigma(\varsigma-1)} \mu^{\frac{p}{2}}, & 0<\sigma<1, \\ \mathcal{C}_4 \lambda_1^{\sigma(\varsigma-1)} \mu, & \sigma \ge 1. \end{cases}
$$

Then we have

$$
\|\mathcal{A}_\gamma - \mathcal{A}\|_2 \le \begin{cases} \sqrt{\mathcal{C}_1^2 + \mathcal{C}_3^2}\, \lambda_1^{\sigma(\varsigma-1)} \mathcal{O} \gamma^{\sigma}, & 0<\sigma<1, \\ \sqrt{\mathcal{C}_3^2 + \mathcal{C}_4^2}\, \lambda_1^{\sigma(\varsigma-1)} \mathcal{O} \gamma, & \sigma \ge 1. \end{cases} \tag{30}
$$

Secondly, we give an estimate for the first term, one has

$$
\begin{aligned}
&\left\|\mathcal{A}_\gamma^\varepsilon-\mathcal{A}_\gamma\right\|_2\\
&=\Bigg\|\sum_{m=0}^{\infty}(\mathcal{J}_{2m+1})^2\mathcal{J}_{2m+1}^{\varsigma}\Bigg(\frac{\left\langle\mathcal{B}_1^\varepsilon-\mathcal{B}_1,e_{2m+1}\right\rangle\mu_{2m+1}\int_b^T\mathcal{X}_{2m+1}^\alpha(T,z)\mathcal{G}^\varepsilon(z)dz}{\mathcal{Y}_{2m+1}^\alpha(b,t_0,T,\mathcal{X},\mathcal{G}^\varepsilon)}\\
&\qquad-\frac{\left\langle\mathcal{C}_1^\varepsilon-\mathcal{C}_1,e_{2m+1}\right\rangle\mu_{2m+1}\int_b^{t_0}\mathcal{X}_{2m+1}^\alpha(t_0,z)\mathcal{G}^\varepsilon(z)dz}{\mathcal{Y}_{2m+1}^\alpha(b,t_0,T,\mathcal{X},\mathcal{G}^\varepsilon)}\Bigg)e_{2m+1}(x)\\
&\quad+\sum_{m=0}^{\infty}\mathcal{J}_{2m+1}^{\varsigma}\Bigg(\frac{\left\langle\mathcal{B}_1,e_{2m+1}\right\rangle\mu_{2m+1}\int_b^T\mathcal{X}_{2m+1}^\alpha(T,z)(\mathcal{G}^\varepsilon(z)-\mathcal{G}(z))(z)dz}{\mathcal{Y}_{2m+1}^\alpha(b,t_0,T,\mathcal{X},\mathcal{G}^\varepsilon)}\\
&\qquad-\frac{\left\langle\mathcal{C}_1,e_{2m+1}\right\rangle\mu_{2m+1}\int_b^{t_0}\mathcal{X}_{2m+1}^\alpha(t_0,z)(\mathcal{G}^\varepsilon(z)-\mathcal{G}(z))(z)dz}{\mathcal{Y}_{2m+1}^\alpha(b,t_0,T,\mathcal{X},\mathcal{G}^\varepsilon)}\Bigg)e_{2m+1}(x)\\
&\quad+\sum_{m=0}^{\infty}\mathcal{J}_{2m+1}^{\varsigma}\left(\frac{\mathcal{Y}_{2m+1}^\alpha(b,t_0,T,\mathcal{X},\mathcal{G}^\epsilon-\mathcal{G})}{\mathcal{Y}_{2m+1}^\alpha(b,t_0,T,\mathcal{X},\mathcal{G}^\epsilon)}\right)\Bigg(\frac{\left\langle\mathcal{B}_1,e_{2m+1}\right\rangle\mu_{2m+1}\int_b^T\mathcal{X}_{2m+1}^\alpha(T,z)\mathcal{G}(z)dz}{\mathcal{Y}_{2m+1}^\alpha(b,t_0,T,\mathcal{X},\mathcal{G})}\\
&\qquad-\frac{\left\langle\mathcal{C}_1,e_{2m+1}\right\rangle\mu_{2m+1}\int_b^{t_0}\mathcal{X}_{2m+1}^\alpha(t_0,z)\mathcal{G}(z)dz}{\mathcal{Y}_{2m+1}^\alpha(b,t_0,T,\mathcal{X},\mathcal{G})}\Bigg)e_{2m+1}(x)\\
&\quad+\sum_{m=1}^{\infty}\mathcal{J}_{2m}^{\varsigma}\Bigg(\frac{\left\langle\mathcal{B}_2^\varepsilon-\mathcal{B}_2,e_{2m}\right\rangle\mu_{2m+1}\int_b^T\mathcal{X}_{2m}^\alpha(T,z)\mathcal{G}^\varepsilon(z)dz}{\mathcal{Y}_{2m}^\alpha(b,t_0,T,\mathcal{X},\mathcal{G}^\varepsilon)}\\
&\qquad-\frac{\left\langle\mathcal{C}_2^\varepsilon-\mathcal{C}_2,e_{2m}\right\rangle\mu_{2m}\int_b^{t_0}\mathcal{X}_{2m}^\alpha(t_0,z)\mathcal{G}^\varepsilon(z)dz}{\mathcal{Y}_{2m}^\alpha(b,t_0,T,\mathcal{X},\mathcal{G}^\varepsilon)}\Bigg)e_{2m}(x)\\
&\quad+\sum_{m=1}^{\infty}\mathcal{J}_{2m}^{\varsigma}\Bigg(\frac{\left\langle\mathcal{B}_2,e_{2m}\right\rangle\mu_{2m}\int_b^T\mathcal{X}_{2m}^\alpha(T,z)(\mathcal{G}^\varepsilon(z)-\mathcal{G}(z))(z)dz}{\mathcal{Y}_{2m}^\alpha(b,t_0,T,\mathcal{X},\mathcal{G}^\varepsilon)}\\
&\qquad-\frac{\left\langle\mathcal{C}_2,e_{2m}\right\rangle\mu_{2m}\int_b^T\mathcal{X}_{2m}^\alpha(T,z)(\mathcal{G}^\varepsilon(z)-\mathcal{G}(z))(z)dz}{\mathcal{Y}_{2m}^\alpha(b,t_0,T,\mathcal{X},\mathcal{G}^\varepsilon)}\Bigg)e_{2m}(x)\\
&\quad+\sum_{m=1}^{\infty}\mathcal{J}_{2m}^{\varsigma}\left(\frac{\mathcal{Y}_{2m}^\alpha(b,t_0,T,\mathcal{X},\mathcal{G}^\epsilon-\mathcal{G})}{\mathcal{Y}_{2m}^\alpha(b,t_0,T,\mathcal{X},\mathcal{G}^\epsilon)}\right)\Bigg(\frac{\left\langle\mathcal{B}_2,e_{2m}\right\rangle\mu_{2m}\int_b^T\mathcal{X}_{2m}^\alpha(T,z)\mathcal{G}(z)dz}{\mathcal{Y}_{2m}^\alpha(b,t_0,T,\mathcal{X},\mathcal{G})}\\
&\qquad-\frac{\left\langle\mathcal{C}_2,e_{2m}\right\rangle\mu_{2m}\int_b^{t_0}\mathcal{X}_{2m}^\alpha(t_0,z)\mathcal{G}(z)dz}{\mathcal{Y}_{2m}^\alpha(b,t_0,T,\mathcal{X},\mathcal{G})}\Bigg)e_{2m}(x)\Bigg\|_2. \qquad (31)
\end{aligned}
$$

From (31), we have the estimate

$$
\begin{aligned}
&\|\mathcal{A}_\gamma^\varepsilon - \mathcal{A}_\gamma\|_2 \\
&\leq \Bigg(\sum_{m=0}^{\infty} \varepsilon^2 (\mathcal{J}_{2m+1}^{\varsigma})^2 \Big(\frac{\mu_{2m+1} \int_b^T \mathcal{X}_{2m+1}^{\alpha}(T,z) \mathcal{G}^\varepsilon(z) dz}{\mathcal{Y}_{2m+1}^{\alpha}(b,t_0,T,\mathcal{X},\mathcal{G}^\varepsilon)} \Big)^2 \Bigg)^{\frac{1}{2}} \\
&\qquad + \Bigg(\sum_{m=0}^{\infty} \varepsilon^2 (\mathcal{J}_{2m+1}^{\varsigma})^2 \Big(\frac{\mu_{2m+1} \int_b^{t_0} \mathcal{X}_{2m+1}^{\alpha}(t_0,z) \mathcal{G}^\varepsilon(z) dz}{\mathcal{Y}_{2m+1}^{\alpha}(b,t_0,T,\mathcal{X},\mathcal{G}^\varepsilon)} \Big)^2 \Bigg)^{\frac{1}{2}} \\
&+ \Bigg(\sum_{m=0}^{\infty} \varepsilon^2 (\mathcal{J}_{2m+1}^{\varsigma})^2 \Big(\frac{|\langle \mathcal{B}_1, e_{2m+1} \rangle| \mu_{2m+1} \int_b^T \mathcal{X}_{2m+1}^{\alpha}(T,z) dz}{\mathcal{Y}_{2m+1}^{\alpha}(b,t_0,T,\mathcal{X},\mathcal{G}^\varepsilon)} \Big)^2 \Bigg)^{\frac{1}{2}} \\
&\qquad + \Bigg(\sum_{m=0}^{\infty} \varepsilon^2 (\mathcal{J}_{2m+1}^{\varsigma})^2 \Big(\frac{|\langle \mathcal{C}_1, e_{2m+1} \rangle| \mu_{2m+1} \int_b^{t_0} \mathcal{X}_{2m+1}^{\alpha}(t_0,z) dz}{\mathcal{Y}_{2m+1}^{\alpha}(b,t_0,T,\mathcal{X},\mathcal{G}^\varepsilon)} \Big)^2 \Bigg)^{\frac{1}{2}} \\
&+ \Bigg(\sum_{m=0}^{\infty} \varepsilon^2 (\mathcal{J}_{2m+1}^{\varsigma})^2 \Big(\frac{\mathcal{Y}_{2m+1}^{\alpha}(b,t_0,T,\mathcal{X})}{\mathcal{Y}_{2m+1}^{\alpha}(b,t_0,T,\mathcal{X},\mathcal{G}^\varepsilon)} \Big)^2 \times \Big(\frac{\langle \mathcal{B}_1, e_{2m+1} \rangle \mu_{2m+1} \int_b^T \mathcal{X}_{2m+1}^{\alpha}(T,z) \mathcal{G}(z) dz}{\mathcal{Y}_{2m+1}^{\alpha}(b,t_0,T,\mathcal{X},\mathcal{G})} \\
&\qquad - \frac{\langle \mathcal{C}_1, e_{2m+1} \rangle \mu_{2m+1} \int_b^{t_0} \mathcal{X}_{2m+1}^{\alpha}(t_0,z) \mathcal{G}(z) dz}{\mathcal{Y}_{2m+1}^{\alpha}(b,t_0,T,\mathcal{X},\mathcal{G})} \Big)^2 \Bigg)^{\frac{1}{2}}, \qquad (32)
\end{aligned}
$$

From (32), we notice that

$$
\begin{aligned}
&\|\mathcal{A}_\gamma^\varepsilon - \mathcal{A}_\gamma\|_2 \\
&\leq \Bigg(\sum_{m=1}^{\infty} \varepsilon^2 (\mathcal{J}_{2m}^{\varsigma})^2 \Big(\frac{\mu_{2m} \int_b^T \mathcal{X}_{2m}^{\alpha}(T,z) \mathcal{G}^\varepsilon(z) dz}{\mathcal{Y}_{2m}^{\alpha}(b,t_0,T,\mathcal{X},\mathcal{G}^\varepsilon)} \Big)^2 \Bigg)^{\frac{1}{2}} + \Bigg(\sum_{m=1}^{\infty} \varepsilon^2 (\mathcal{J}_{2m}^{\varsigma})^2 \Big(\frac{\mu_{2m} \int_b^{t_0} \mathcal{X}_{2m}^{\alpha}(t_0,z) \mathcal{G}^\varepsilon(z) dz}{\mathcal{Y}_{2m}^{\alpha}(b,t_0,T,\mathcal{X},\mathcal{G}^\varepsilon)} \Big)^2 \Bigg)^{\frac{1}{2}} \\
&\qquad + \Bigg(\sum_{m=0}^{\infty} \varepsilon^2 (\mathcal{J}_{2m}^{\varsigma})^2 \Big(\frac{|\langle \mathcal{B}_1, e_{2m} \rangle| \mu_{2m} \int_b^T \mathcal{X}_{2m}^{\alpha}(T,z) dz}{\mathcal{Y}_{2m}^{\alpha}(b,t_0,T,\mathcal{X},\mathcal{G}^\varepsilon)} \Big)^2 \Bigg)^{\frac{1}{2}} \\
&+ \Bigg(\sum_{m=0}^{\infty} \varepsilon^2 (\mathcal{J}_{2m+1}^{\varsigma})^2 \Big(\frac{|\langle \mathcal{C}_1, e_{2m} \rangle| \mu_{2m} \int_b^{t_0} \mathcal{X}_{2m}^{\alpha}(t_0,z) dz}{\mathcal{Y}_{2m}^{\alpha}(b,t_0,T,\mathcal{X},\mathcal{G}^\varepsilon)} \Big)^2 \Bigg)^{\frac{1}{2}} \\
&+ \Bigg(\sum_{m=0}^{\infty} \varepsilon^2 (\mathcal{J}_{2m+1}^{\varsigma})^2 \Big(\frac{\mathcal{Y}_{2m}^{\alpha}(b,t_0,T,\mathcal{X})}{\mathcal{Y}_{2m}^{\alpha}(b,t_0,T,\mathcal{X},\mathcal{G}^\varepsilon)} \Big)^2 \times \Big(\frac{\langle \mathcal{B}_2, e_{2m} \rangle \mu_{2m} \int_b^T \mathcal{X}_{2m}^{\alpha}(T,z) \mathcal{G}(z) dz}{\mathcal{Y}_{2m}^{\alpha}(b,t_0,T,\mathcal{X},\mathcal{G})} \\
&\qquad - \frac{\langle \mathcal{C}_2, e_{2m} \rangle \mu_{2m} \int_b^{t_0} \mathcal{X}_{2m}^{\alpha}(t_0,z) \mathcal{G}(z) dz}{\mathcal{Y}_{2m}^{\alpha}(b,t_0,T,\mathcal{X},\mathcal{G})} \Big)^2 \Bigg)^{\frac{1}{2}}.
\end{aligned}
$$

It yields

$$
\begin{aligned}
&\left\|\mathcal{A}_\gamma^\varepsilon-\mathcal{A}_\gamma\right\|_2\\
&\le \sup_{m\ge 0}\left|\mathcal{J}_{2m+1}^{\varsigma}\frac{\mathcal{Z}_2^*\mathcal{P}(\mathcal{G}_1,\mathcal{G}_2)}{\mathcal{Z}_{12}^*}\right|\varepsilon+\sup_{m\ge 0}\left|\mathcal{J}_{2m+1}^{\varsigma}\frac{\mathcal{Z}_2^*\mathcal{P}(\mathcal{G}_1,\mathcal{G}_2)}{\mathcal{Z}_{12}^*}\right|\varepsilon+\sup_{m\ge 0}\left|\mathcal{J}_{2m+1}^{\varsigma}\frac{\overline{B}}{\lambda_{2m+1}^{s-1}}\right|\varepsilon\\
&+\sup_{m\ge 1}\left|\mathcal{J}_{2m+1}^{\varsigma}\frac{\overline{C}}{\lambda_{2m+1}^{s-1}}\right|\varepsilon+\sup_{m\ge 0}\left|\varepsilon\mathcal{J}_{2m+1}^{\varsigma}\left(\frac{\mathcal{Z}_{15}^*}{\mathcal{Z}_{12}^*}\right)\right.\\
&\qquad\times\left|\sum_{m=0}^{\infty}\frac{\langle\mathcal{B}_1,e_{2m+1}\rangle\mu_{2m+1}\int_b^T\mathcal{X}_{2m+1}^{\alpha}(T,z)\mathcal{G}(z)dz-\langle\mathcal{C}_1,e_{2m+1}\rangle\mu_{2m+1}\int_b^{t_0}\mathcal{X}_{2m+1}^{\alpha}(t_0,z)\mathcal{G}(z)dz}{\mathcal{Y}_{2m+1}^{\alpha}(b,t_0,T,\mathcal{X},\mathcal{G})}\right.\\
&\quad+\sup_{m\ge 1}\left|\mathcal{J}_{2m}^{\varsigma}\frac{\mathcal{Z}_2^*\mathcal{P}(\mathcal{G}_1,\mathcal{G}_2)}{\mathcal{Z}_{12}^*}\right|\varepsilon+\sup_{m\ge 1}\left|\mathcal{J}_{2m}^{\varsigma}\frac{\mathcal{Z}_2^*\mathcal{P}(\mathcal{G}_1,\mathcal{G}_2)}{\mathcal{Z}_{12}^*}\right|\varepsilon+\sup_{m\ge 1}\left|\mathcal{J}_{2m}^{\varsigma}\frac{\overline{B}}{\lambda_{2m}^{s-1}}\right|\varepsilon+\sup_{m\ge 0}\left|\mathcal{J}_{2m}^{\varsigma}\frac{\overline{C}}{\lambda_{2m}^{s-1}}\right|\varepsilon\\
&\quad+\sup_{m\ge 1}\left|\varepsilon\mathcal{J}_{2m}^{\varsigma}\left(\frac{\mathcal{Z}_{15}^{**}}{\mathcal{Z}_{12}^{**}}\right)\right|\sum_{m=1}^{\infty}\frac{\langle\mathcal{B}_2,e_{2m}\rangle\mu_{2m}\int_b^T\mathcal{X}_{2m}^{\alpha}(T,z)\mathcal{G}(z)dz-\langle\mathcal{C}_2,e_{2m}\rangle\mu_{2m}\int_b^{t_0}\mathcal{X}_{2m}^{\alpha}(t_0,z)\mathcal{G}(z)dz}{\mathcal{Y}_{2m}^{\alpha}(b,t_0,T,\mathcal{X},\mathcal{G})},\\
&\le\frac{\varepsilon}{\gamma}\left(\frac{2\mathcal{Z}_2^*}{\mathcal{Z}_{12}^*\lambda_1^{\varsigma}}+\frac{\overline{B}+\overline{C}}{\lambda_1^{\varsigma+s-1}}+2\frac{1}{\lambda_1^{\varsigma}}\max\left\{\frac{\mathcal{Z}_{15}^*}{\mathcal{Z}_{12}^*},\frac{\mathcal{Z}_{15}^{**}}{\mathcal{Z}_{12}^{**}}\right\}\left\|\mathcal{F}\right\|_2\right).
\end{aligned}
$$

The regularization parameter γ by

$$
\gamma=\begin{cases}\left(\dfrac{\varepsilon}{\mathcal{O}}\right)^{\frac{1}{\sigma+1}}, & 0<\sigma<1,\\[2mm] \left(\dfrac{\varepsilon}{\mathcal{O}}\right)^{\frac{1}{2}}, & \sigma\ge 1.\end{cases}\tag{33}
$$

Combining (30) and (33), we have

$$
\left\|\mathcal{A}_\gamma^\varepsilon-\mathcal{A}\right\|_2\le\begin{cases}\left(\sqrt{\mathcal{C}_1^2+\mathcal{C}_3^2\lambda_1^{\sigma(\varsigma-1)}}+\mathcal{C}_6\right)\mathcal{O}^{\frac{1}{\sigma+1}}\varepsilon^{\frac{\sigma}{\sigma+1}} & 0<\sigma<1\\ \left(\sqrt{\mathcal{C}_2^2+\mathcal{C}_4^2\lambda_1^{\sigma(\varsigma-1)}}+\mathcal{C}_6\right)\mathcal{O}^{\frac{1}{2}}\varepsilon^{\frac{1}{2}}, & \sigma\ge 1.\end{cases}
$$

Then we complete the proof. □

7 Conclusion

In this work, we address the problem of simultaneous recovery of the source term and the initial value in a time-fractional diffusion equation, which is not well formulated. With a hybrid regularization method, we propose solutions and provide accurate estimates for both the source term and the initial value, subject to a priori parameter selection rules.

Declarations

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Ethical Approval

Not applicable.

Authors' Contributions

All authors contributed equally. All the authors read and approved the final manuscript.

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