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Weak solutions of Dirichlet discrete nonlinear problems in a two-dimensional Hilbert space

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Abstract

In this paper we prove the existence of at least one weak solution of a discrete nonlinear Dirichlet boundary-value problem in a two-dimensional Hilbert space. The main existence results based on variational approach, specially minimization methods.

Keywords: discrete boundary value problem, critical point, two-dimensional discrete Hilbert space, weak solution, variational approach
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1 Introduction

Let $T_1 \geq 2$ and $T_2 \geq 2$ are two positive integer, $\mathbb{N}[1,T]$ be the discrete interval given by $\{1,2,...,T\}$. Define the foward difference operator $\Delta u(k,h) = u(k+1,h+1) - u(k,h), (k,h) \in \mathbb{N}[1,T_1] \times \mathbb{N}[1,T_2]$. This paper is concerned with the study of the existence of solutions for the anisotropic nonlinear discrete boundary value problem,

$$\begin{cases} -M(K(u(k,h)))\left((\Delta a(k-1,h-1,\Delta u(k-1,h-1))) + |u(k,h)|^{p(k,h)-2}u(k,h)\right) \\ = f(k,h), (k,h) \in \mathbb{N}[1,T_1] \times \mathbb{N}[1,T_2], \\ u(k,h) = 0, \ (k,h) \in \Gamma, \end{cases} \tag{1}$$

where

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$$K(u)=\sum_{k=1}^{T_1+1}\sum_{h=1}^{T_2+1}A(k-1,h-1,\Delta u(k-1,h-1))+\sum_{k=1}^{T_1}\sum_{h=1}^{T_2}\left(\frac{1}{p(k,h)}|u(k,h)|^{p(k,h)}\right). \tag{2}$$

$\Gamma=(\{0,T_1+1\}\times\mathbb{N}[0,T_2+1])\cup(\mathbb{N}[0,T_1+1]\times\{0,T_2+1\})$ is the boundary of the domain $\mathbb{N}[1,T_1]\times\mathbb{N}[1,T_2]$. The mappings a and f are defined by

$$a:\mathbb{N}[1,T_1]\times\mathbb{N}[1,T_2]\times\mathbb{R}\longrightarrow\mathbb{R} \quad \text{and} \quad f:\mathbb{N}[1,T_1]\times\mathbb{N}[1,T_2]\times\mathbb{R}\longrightarrow\mathbb{R}.$$

For the function $p:\mathbb{N}[1,T_1]\times\mathbb{N}[1,T_2]\longrightarrow(1,\infty)$ denote

$$p^-=\min_{k\in\mathbb{N}[1,T_1]}\left(\min_{h\in\mathbb{N}[1,T_2]}p(k,h)\right)\quad p^+=\max_{k\in\mathbb{N}[1,T_1]}\left(\max_{h\in\mathbb{N}[1,T_2]}p(k,h)\right).$$

The study of discrete boundary value problems has captured special attention in the last years. For recent progress in discrete problems, we refer to [1, 5, 10, 11, 14, 15, 16, 17, 18] and the other references therein.

For example, based on the method of minimization, in [9], Ibrango and al. proved the existence and uniqueness of solution to the two-dimensional following Dirichlet problem,

$$\begin{cases}-\Delta(a(k-1,h-1,\Delta u(k-1,h-1)))=f(k,h),\ (k,h)\in\mathbb{N}[1,T_1]\times\mathbb{N}[1,T_2],\\ u(k,h)=0,\ (k,h)\in\Gamma.\end{cases} \tag{3}$$

and established existence of at a least one weak solutions to the problem,

$$\begin{cases}-\Delta(a(k-1,h-1,\Delta u(k-1,h-1)))=f(k,h,u(k,h)),\ (k,h)\in\mathbb{N}[1,T_1]\times\mathbb{N}[1,T_2],\\ u(k,h)=0,\ (k,h)\in\Gamma.\end{cases} \tag{4}$$

A particularly case of problem (4) was studied in [7] , where the authors deal with the existence of multiple solutions to the following p-Laplacian problem, based on three critical points theorem established by Ricerri (see, [3, 4])

$$\Delta_1(\phi_p(\Delta_1(k-1,h)))+\Delta_2(\phi_p(\Delta_2(k,h-1)))+\lambda f(k,h,u(k,h))=0, \tag{5}$$

for any $(k,h)\in\mathbb{N}[1,T_1]\times\mathbb{N}[1,T_2]$ with

$$\Delta_1 u(k,h)=u(k+1,h)-u(k,h) \text{ and } \Delta_2 u(k,h)=u(k,h+1)-u(k,h).$$

ϕ_p is the p-Laplacian operator given by $\phi_p(s)=|s|^{p-2}s$, $1<p<\infty$ and $f(k,h,.)\in C(\mathbb{R},\mathbb{R})$ for all $(k,h)\in\mathbb{N}[1,T_1]\times\mathbb{N}[1,T_2]$.

Discrete problems of Kirchhoff type, model biological, mechanical or physical phenomena in which the tension, stiffness, or resistance depends globally on the state of the system. For instance in [6], the $p(k)$-Laplacian problem of Kirchhoff type subjected to potential boundary values conditions was studied by Dianda and Ouaro.

$$\begin{cases}-M(A(k-1,\Delta u(k-1)))\Delta(a(k-1,\Delta u(k-1)))\in\lambda\partial F(k,u(k)),\ \ k\in\mathbb{Z}[1,T]\\ (a(0,\Delta u(0)),-a(T,\Delta u(T))\in\partial j(u(0);u(T+1)),\end{cases} \tag{6}$$

where λ is a positive real parameter, $\mathbb{Z}[a,b]$ is the discrete interval, $u(k) \in \mathbb{R}$ for all $k \in \mathbb{Z}$ and ∂j denote the subdifferential of j, where, for $z \in \mathbb{R} \times \mathbb{R}$, the set ∂j is defined by

$$\partial j(z) = \{\zeta \in \mathbb{R} \times \mathbb{R} : j(\xi) - j(z) \geq (\zeta \mid \xi - z), \text{ for all } \xi \in \mathbb{R} \times \mathbb{R}\}. \tag{7}$$

They proved the existence and multiplicity of solutions to (6) . To the best of our knowledge, while most previous studies have focused on boundary value problems, only a few have addressed discrete problems of the Kirchhoff type. Motivated by these findings, we aim to explore the existence of multiple solutions for a discrete Kirchhoff-type problem subject to Dirichlet boundary conditions, using critical point theory. We are interested in investigating nonlinear discrete boundary value problems in two-dimensional Hilbert space of Kirchhoff Type which generalize the study of difference equations in two- dimensional.

In this paper, we replaced polynomial growth condition (see, Theorem 3.3 in [16]), by the assumption $M_0 := \inf_{t \geq 0} M(t)$ where $M : (0, \infty) \longrightarrow (0, \infty)$ is continuous and non-decreasing.

The paper is organized as follows. In Section 2, we give some basic definitions and preliminary facts which will be used throughout the following sections. The main existence result is stated and proved in Section 3. In the last section, we discuss some extensions.

2 Some Preliminary results

We consider the $(T_1 \times T_2)$-dimensional hilbert space (see,[2])

$$\mathcal{H} := \{u : \mathbb{N}[0, T_1 + 1] \times \mathbb{N}[0, T_2 + 1] \longrightarrow \mathbb{R} \quad \text{such that} \quad u(k,h) = 0 \quad \forall (k,h) \in \Gamma\}$$

with the inner product

$$\langle u, v \rangle = \sum_{k=1}^{T_1} \sum_{h=1}^{T_2} u(k,h) v(k,h)$$

endowed with the norm

$$\|u\| = \left(\sum_{k=1}^{T_1} \sum_{h=1}^{T_2} |u(k,h)|^2 \right)^{\frac{1}{2}}.$$

However we introduce another norm on the space $\mathcal{H}$, namely

$$|u|_m = \left(\sum_{k=1}^{T_1} \sum_{h=1}^{T_2} |u(k,h)|^m \right)^{\frac{1}{m}}, \forall m \geq 2.$$

Due to the equivalence of $\|.\|$ and $|.|_m$ there exist constant $C_2 \geq C_1 > 0$ such that

$$C_1 \|u\| \leq |u|_m \leq C_2 \|u\|, \qquad \forall u \in \mathcal{H}. \tag{8}$$

For the data f and a we impose the following conditions :

$$a(k,h,.) : \mathbb{R} \longrightarrow \mathbb{R} \text{ is continuous } \forall (k,h) \in \mathbb{N}[1, T_1] \times \mathbb{N}[1, T_2]$$

and there exists a mapping $A : \mathbb{N}[1, T_1] \times \mathbb{N}[1, T_2] \times \mathbb{R} \longrightarrow \mathbb{R}$ which satisfies

$$a(k,h,x) = \frac{\partial}{\partial x} A(k,h,x) \text{ and } A(k,h,0) = 0 \quad \forall (k,h) \in \mathbb{N}[1, T_1] \times \mathbb{N}[1, T_2]. \tag{9}$$

We also assume that there exists a positive constant C_3 such that

$$|a(k,h,x)| \leq C_3 \left(1 + |x|^{p(k,h)-1}\right) \tag{10}$$

and

$$|x|^{p(k,h)} \leq a(k,h,x)x \leq p(k,h)A(k,h,x), \quad \forall x \in \mathbb{R}. \tag{11}$$

The following relations hold true for all $(k,h) \in \mathbb{N}[1,T_1] \times \mathbb{N}[1,T_2]$ with $x \neq y$:

$$(a(k,h,x) - a(k,h,y))(x-y) > 0, \forall x,y \in \mathbb{R}. \tag{12}$$

We assume that the function $M : (0,\infty) \longrightarrow (0,\infty)$ is continuous and non-decreasing and there exists $M_0 > 0$ such that

$$M_0 = \inf_{t \geq 0} M(t). \tag{13}$$

Example 2.1. We can give the following function satisfies assumptions (9) − (12).

i. $A(k,h,x) = \dfrac{1}{p(k,h)}\left((1+|x|^2)^{\frac{p(k,h)}{2}} - 1\right)$ where $a(k,h,x) = (1+|x|^2)^{\frac{p(k,h)-2}{2}} x$,
$(k,h) \in \mathbb{N}[1,T_1] \times \mathbb{N}[1,T_2],\ x \in \mathbb{R}$.

ii. $A(k,h,x) = \dfrac{1}{p(k,h)}|x|^{p(k,h)}$ where $a(k,h,x) = |x|^{p(k,h)-2}x$,
$(k,h) \in \mathbb{N}[1,T_1] \times \mathbb{N}[1,T_2],\ x \in \mathbb{R}$.

We point out the following auxiliary result.

Lemma 2.2. *(i) For any function $u \in \mathcal{H}$ with $\|u\| > 1$, there exist constants C_4, $C_5 > 0$ such that*

$$\sum_{k=1}^{T_1+1}\sum_{h=1}^{T_2+1} |\Delta u(k-1,h-1)|^{p(k-1,h-1)} \geq C_4\|u\|^{p^-} - C_5. \tag{14}$$

(ii) For any function $u \in \mathcal{H}$ there exists constants $C_6,\ C_7 > 0$ such that

$$\sum_{k=1}^{T_1+1}\sum_{h=1}^{T_2+1} |\Delta u(k-1,h-1)|^{p(k-1,h-1)} \leq C_6\|u\|^{p^+} + C_7. \tag{15}$$

Proof. (i) Using the same arguments of Lemma 2.2 in [9] we obtain (i).
(ii). For any $u \in \mathcal{H}$, let

$$v_h : \mathbb{N}[0,T_1+1] \longrightarrow \mathbb{R},\ k \mapsto v_h(k) = u(k,h)$$

and

$$q_h : \mathbb{N}[0,T_1] \longrightarrow (1,\infty),\ k \mapsto q_h(k) = p(k,h)\ \text{, with } h \text{ fixed in } \mathbb{N}[0,T_2+1].$$

According to Lemma 1 in [8] we have

$$\sum_{k=1}^{T_1+1} |\Delta v_h(k-1)|^{q_h(k-1)} \leq (T_1+1)\|v_h\|^{q_h^+} + T_1 + 1.$$

Then there exist constants $C_6, C_7 > 0$ such that

$$\sum_{k=1}^{T_1+1}\sum_{h=1}^{T_2+1} |\Delta u(k-1,h-1)|^{p(k-1,h-1)} \leq C_6\|u\|^{p^+} + C_7.$$

□

Theorem 2.3. *[13] Let X be reflexive Banach space. If a functional $J \in C^1(X, \mathbb{R})$ is weakly lower semi-continuous and coercive, i.e $\lim_{\|u\| \to \infty} J(u) = \infty$, then there exists $\bar{u}$ such that*

$$J(\bar{u}) = \inf_{u \in X} J(u)$$

and $\bar{u}$ is also a critical point of J, i.e. $J'(\bar{u}) = 0$. Moreover, if J is strictly convex, then critical point is unique.

3 Existence of weak solutions

In this section, we investigate the existence of weak solutions of (1).

Theorem 3.1. *Suppose that assumptions (9) − (13) hold. Then there is at least one weak solution for problem (1).*

Definition 3.2. A function $u \in \mathcal{H}$ is a weak solution of problem (1) if

$$M(K(u)) \left(\sum_{k=1}^{T_1+1} \sum_{h=1}^{T_2+1} a(k-1, h-1, \Delta u(k-1, h-1)) \Delta v(k-1, h-1) + \sum_{k=1}^{T_1} \sum_{h=1}^{T_2} |u(k,h)|^{p(k,h)-2} u(k,h) v(k,h) \right) = \sum_{k=1}^{T_1} \sum_{h=1}^{T_2} f(k,h) v(k,h) \tag{16}$$

for any $v \in \mathcal{H}$.

The energy functional corresponding to problem (1) is defined as $\Phi : \mathcal{H} \longrightarrow \mathbb{R}$, by the formula

$$\Phi(u) = \widehat{M}(K(u)) - \sum_{k=1}^{T_1} \sum_{h=1}^{T_2} f(k,h) u(k,h)$$

where $K(u) = \sum_{k=1}^{T_1+1} \sum_{h=1}^{T_2+1} A(k-1, h-1, \Delta u(k-1, h-1)) + \sum_{k=1}^{T_1} \sum_{h=1}^{T_2} \left(\frac{1}{p(k,h)} |u(k,h)|^{p(k,h)} \right)$ where $(k,h) \in \mathbb{N}[1, T_1] \times \mathbb{N}[1, T_2]$ and $\widehat{M}(x) = \int_0^x M(s) ds$; $x \in \mathbb{R}$.

By definition, Φ is continuous, Gâteaux differentiable and its Gâteaux derivative at u yields the following expression:

$$\langle \Phi'(u), v \rangle = M(K(u)) \left(\sum_{k=1}^{T_1+1} \sum_{h=1}^{T_2+1} a(k-1, h-1, \Delta u(k-1, h-1)) \Delta v(k-1, h-1) + \sum_{k=1}^{T_1} \sum_{h=1}^{T_2} |u(k,h)|^{p(k,h)-2} u(k,h) v(k,h) \right) - \sum_{k=1}^{T_1} \sum_{h=1}^{T_2} f(k,h) v(k,h) \tag{17}$$

for all $v \in \mathcal{H}$.
Assume that u is a critical point of Φ, i.e $\langle \Phi'(u), v \rangle = 0$ for all $v \in \mathcal{H}$.
Consequently, the critical points of Φ in $\mathcal{H}$ are exactly the solutions of the problem (1).

Proposition 3.3. *The functional $\Phi : \mathcal{H} \longrightarrow \mathbb{R}$, is coercive and bounded from below.*

Proof. Let $\|u\| \geq 1$, recall that

$$\Phi(u) = \widehat{M}(K(u)) - \sum_{k=1}^{T_1}\sum_{h=1}^{T_2} f(k,h)u(k,h). \tag{18}$$

From (13), we obtain

$$\widehat{M}(K(u)) \geq M_0 \int_0^{K(u)} \mathrm{d}t \geq M_0 K(u). \tag{19}$$

We deduce that

$$\Phi(u) \quad \geq \quad M_0 K(u) - \sum_{k=1}^{T_1}\sum_{h=1}^{T_2} f(k,h)u(k,h).$$

Using (13) and (14), we obtain

$$\begin{aligned}
\Phi(u) \quad &\geq \quad M_0 \sum_{k=1}^{T_1+1}\sum_{h=1}^{T_2+1} A(k-1,h-1,\Delta u(k-1,h-1)) \\
&\quad +M_0 \sum_{k=1}^{T_1}\sum_{h=1}^{T_2}\left(\frac{1}{p(k,h)}|u(k,h)|^{p(k,h)}\right) - \sum_{k=1}^{T_1}\sum_{h=1}^{T_2} f(k,h)u(k,h) \\
&\geq \quad M_0\left(\sum_{k=1}^{T_1+1}\sum_{h=1}^{T_2+1} A(k-1,h-1,\Delta u(k-1,h-1))\right) - \sum_{k=1}^{T_1}\sum_{h=1}^{T_2} f(k,h)u(k,h) \\
&\geq \quad \sum_{k=1}^{T_1+1}\sum_{h=1}^{T_2+1} \frac{M_0}{p(k-1,h-1)}|\Delta u(k-1,h-1)|^{p(k-1,h-1)} \\
&\quad -\left(\sum_{k=1}^{T_1}\sum_{h=1}^{T_2}|f|^2\right)^{\frac{1}{2}}\left(\sum_{k=1}^{T_1}\sum_{h=1}^{T_2}|u|^2\right)^{\frac{1}{2}} \\
\Phi(u) \quad &\geq \quad \sum_{k=1}^{T_1+1}\sum_{h=1}^{T_2+1} \frac{M_0}{p(k-1,h-1)}|\Delta u(k-1,h-1)|^{p(k-1,h-1)} - \|f\|\|u\|.
\end{aligned}$$

Finally, one has

$$\Phi(u) \geq \frac{M_0}{p^+}\left(C_4\|u\|^{p^-} - C_5\right) - \|f\|\|u\|.$$

Since $p^- > 1$, then Φ is coercive.
For the boundedness, for all u in $\mathcal{H}$ such that $\|u\| \leq 1$. From the inequality

$$\Phi(u) \quad \geq \quad \frac{M_0}{p^+}\left(C_4\|u\|^{p^-} - C_5\right) - \|f\|\|u\|$$

we deduce that

$$\Phi(u) \quad \geq \quad -\frac{M_0}{p^+}C_5 - \|f\|\|u\| > -\infty.$$

Then, Φ is bounded from below. □

Proposition 3.4. *The functional Φ is weakly lower semi-continuous.*

Proof. For any $u \in \mathcal{H}$, let

$$\begin{aligned} K(u) &= \sum_{k=1}^{T_1+1}\sum_{h=1}^{T_2+1} A(k-1,h-1,\Delta u(k-1,h-1)) \\ &\quad + \sum_{k=1}^{T_1}\sum_{h=1}^{T_2}\left(\frac{1}{p(k,h)}\,|u(k,h)|^{p(k,h)}\right) \\ K(u) &= L(u)+I(u), \quad \text{where} \end{aligned}$$

$$L(u) = \sum_{k=1}^{T_1+1}\sum_{h=1}^{T_2+1} A(k-1,h-1,\Delta u(k-1,h-1)) \quad \text{and}$$

$$I(u) = \sum_{k=1}^{T_1}\sum_{h=1}^{T_2}\frac{1}{p(k,h)}|u(k,h)|^{p(k,h)}.$$

From (12), we have L is convex. Then, the functional $K-I$ is convex. Thus, it is enough to show that K is lower semi-continuous. Let us fix $u \in \mathcal{H}$ and $\epsilon > 0$. According to the convexity of the functional L, we have
$L(v) \geq L(u) + < L'(u), v-u >$, for any $v \in \mathcal{H}$. Therefore

$$\begin{aligned} (K-I)(v) &\geq (K-I)(u) + < (K-I)'(u), v-u > \\ K(v) &\geq K(u) + < L'(u), v-u > + I(v) - I(u). \end{aligned}$$

We have

$$\begin{aligned} & L'(u)(v-u) \\ = & \sum_{k=1}^{T_1+1}\sum_{h=1}^{T_2+1} a(k-1,h-1,\Delta u(k-1,h-1))(\Delta v(k-1,h-1)-\Delta u(k-1,h-1)) \\ \geq & -\sum_{k=1}^{T_1+1}\sum_{h=1}^{T_2+1} |a(k-1,h-1,\Delta u(k-1,h-1))||\Delta v(k-1,h-1)-\Delta u(k-1,h-1)| \end{aligned}$$

$$L'(u)(v-u) \geq -\sum_{k=1}^{T_1+1}\sum_{h=1}^{T_2+1} |a(k-1,h-1,\Delta u(k-1,h-1))||v(k,h)-u(k,h)|$$
$$-\sum_{k=1}^{T_1+1}\sum_{h=1}^{T_2+1} |a(k-1,h-1,\Delta u(k-1,h-1))||v(k-1,h-1)-u(k-1,h-1)|.$$

From [9], we have

$$L'(u)(v-u) \geq -\left(1+2\left(\sum_{k=1}^{T_1+1}\sum_{h=1}^{T_2+1}|a(k-1,h-1,\Delta u(k-1,h-1))|^2\right)\right)^{\frac{1}{2}}\|v-u\|.$$

Thanks to the inequality $||a|-|b|| \leq |a-b|$, $\forall a, b \in \mathbb{R}$, setting $a = |v(k,h)|^{p(k,h)}$ and $b =$

$|u(k,h)|^{p(k,h)}$, we obtain

$$\begin{aligned}
I(v)-I(u) &= \sum_{k=1}^{T_1}\sum_{h=1}^{T_2}\frac{1}{p(k,h)}|v(k,h)|^{p(k,h)} - \sum_{k=1}^{T_1}\sum_{h=1}^{T_2}\frac{1}{p(k,h)}|u(k,h)|^{p(k,h)} \\
&\geq \sum_{k=1}^{T_1}\sum_{h=1}^{T_2}\frac{1}{p^-}\left(|v(k,h)|^{p(k,h)} - |u(k,h)|^{p(k,h)}\right) \\
&\geq -\sum_{k=1}^{T_1}\sum_{h=1}^{T_2}\frac{1}{p^-}||v(k,h)|^{p(k,h)} - |u(k,h)|^{p(k,h)}| \\
I(v)-I(u) &\geq -\frac{1}{p^-}\sum_{k=1}^{T_1}\sum_{h=1}^{T_2}\left|(v(k,h))^{p(k,h)} - (u(k,h))^{p(k,h)}\right|.
\end{aligned}$$

There exists a positive constant $\beta > 1$ such that

$$\left|(v(k,h))^{p(k,h)} - (u(k,h))^{p(k,h)}\right| \leq \beta|v(k,h)-u(k,h)|.$$

We obtain

$$I(v)-I(u) \geq -\frac{\beta}{p^-}\sum_{k=1}^{T_1}\sum_{h=1}^{T_2}|v(k,h)-u(k,h)|.$$

By Cauchy-Shwarz inequality, we obtain

$$\begin{aligned}
I(v)-I(u) &\geq -\frac{\beta}{p^-}\left(\sum_{k=1}^{T_1}\sum_{h=1}^{T_2}|1|^2\right)^{\frac{1}{2}}\left(\sum_{k=1}^{T_1}\sum_{h=1}^{T_2}|v-u|^2\right)^{\frac{1}{2}} \\
I(v)-I(u) &\geq -\frac{\beta}{p^-}\sqrt{T_1\times T_2}\|v-u\|.
\end{aligned}$$

We deduce that

$$\begin{aligned}
K(v) &\geq K(u) + < L'(u), v-u > + I(v) - I(u) \\
&\geq K(u) - \left(1 + 2\left(\sum_{k=1}^{T_1+1}\sum_{h=1}^{T_2+1}|a(k-1,h-1,\Delta u(k-1,h-1)|^2\right)\right)^{\frac{1}{2}} \\
&\quad \times\|v-u\| - \frac{\beta}{p^-}\sqrt{T_1\times T_2}\|v-u\| \\
K(v) &\geq K(u) - \left(\left(1 + 2\sum_{k=1}^{T_1+1}\sum_{h=1}^{T_2+1}|a(k-1,h-1,\Delta u(k-1,h-1)|^2\right)^{\frac{1}{2}}\right)\|v-u\| \\
&\quad + \frac{\beta}{p^-}\sqrt{T_1\times T_2}\|v-u\| \\
K(v) &\geq K(u) - \epsilon, \text{ for all } v\in\mathcal{H} \text{ such that } \|v-u\| \leq \delta = \frac{\epsilon}{C(u,\beta,T_1,T_2)}
\end{aligned}$$

where

$$C(u,\beta,T_1,T_2) = \left(\left(1 + 2\sum_{k=1}^{T_1+1}\sum_{h=1}^{T_2+1}|a(k-1,h-1,\Delta u(k-1,h-1)|^2\right)^{\frac{1}{2}} + \frac{\beta}{p^-}\sqrt{T_1\times T_2}\right).$$

We conclude that the functional K is lower semi-continuous. It's well know that $\hat{M}$ is continuous and non-decreasing. This implies that the functional Φ is also lower semi-continuous. □

Remark 3.5. The functional Φ is proper, weakly lower semi-continuous and coercive on the Hilbert space $\mathcal{H}$. Thanks to the Theorem 2.3 and using the correspondence between critical points of Φ and the weak solution of problem (1), we conclude that Φ admits a minimizer which is a weak solution of problem (1). Then the Theorem 3.1 is satisfied.

4 Extension

In this section, we investigate the existence of weak nontrivial solution to the following anisotropic nonlinear discrete Dirichlet problem,

$$(P):\begin{cases} -M(K(u(k,h)))\left((\Delta a(k-1,h-1,\Delta u(k-1,h-1))) + |u(k,h)|^{p(k,h)-2}u(k,h)\right) \\ = f(k,h,u(k,h)), (k,h) \in \mathbb{N}[1,T_1]\times\mathbb{N}[1,T_2], \\ u(k,h)=0,\ (k,h)\in\Gamma. \end{cases}$$

Let $\gamma : \mathbb{N}[1,T_1]\times\mathbb{N}[1,T_2] \longrightarrow [2,\infty)$, and

$$\gamma^- = \min_{k\in\mathbb{N}[1,T_1]}\left(\min_{h\in\mathbb{N}[1,T_2]}\gamma(k,h)\right) \quad ; \quad \gamma^+ = \max_{k\in\mathbb{N}[1,T_1]}\left(\max_{h\in\mathbb{N}[1,T_2]}\gamma(k,h)\right).$$

For each couple $(k,h)\in\mathbb{N}[1,T_1]\times\mathbb{N}[1,T_2]$, the function $f(k,h,.):\mathbb{R}\longrightarrow\mathbb{R}$ is jointly continuous and there exists constant $C_8>0$ such that

$$|f(k,h,x| \le C_8\left(1+|x|^{\gamma(k,h)-1}\right)$$

We denote

$$F(k,h,x)=\int_0^x f(k,h,s)ds \quad \text{for } (k,h,x)\in\mathbb{N}[1,T_1]\times\mathbb{N}[1,T_2]\times\mathbb{R}$$

and we deduce that there exists a constant $C_9>0$ such that

$$|F(k,h,x)|\le C_9\left(1+|x|^{\gamma(k,h)}\right)$$

Definition 4.1. A function $u\in\mathcal{H}$ is a solution of problem (P) if for any $v\in\mathcal{H}$,

$$M(K(u))\left(\sum_{k=1}^{T_1+1}\sum_{h=1}^{T_2+1}a(k-1,h-1,\Delta u(k-1,h-1))\Delta v(k-1,h-1)+\right.$$
$$\left.\sum_{k=1}^{T_1}\sum_{h=1}^{T_2}|u(k,h)|^{p(k,h)-2}u(k,h)v(k,h)\right)=\sum_{k=1}^{T_1}\sum_{h=1}^{T_2}f(k,h,u(k,h))v(k,h)$$

for any $v\in\mathcal{H}$.

The corresponding energy functional of (P) is

$$T(u)=\widehat{M}(K(u))-\sum_{k=1}^{T_1}\sum_{h=1}^{T_2}F(k,h,u(k,h))$$

where $K(u)=\sum_{k=1}^{T_1+1}\sum_{h=1}^{T_2+1}A(k-1,h-1,\Delta u(k-1,h-1))+\sum_{k=1}^{T_1}\sum_{h=1}^{T_2}\left(\frac{1}{p(k,h)}|u(k,h)|^{p(k,h)}\right)$.

Proposition 4.2. *Let T the functional of problem (P). One has*

(B_1) *T is of class $C^1(\mathcal{H},\mathbb{R})$ and lower semi-continuous;*
(B_2) *T is coercive;*
(B_3) *T is bounded.*

Proof. For (B_1), by definition, $T \in C^1(\mathcal{H},\mathbb{R})$ and its derivative is given by the following expression

$$\langle T'(u),v\rangle = M(K(u))\left(\sum_{k=1}^{T_1+1}\sum_{h=1}^{T_2+1} a(k-1,h-1,\Delta u(k-1,h-1))\Delta v(k-1,h-1)+\sum_{k=1}^{T_1}\sum_{h=1}^{T_2}|u(k,h)|^{p(k,h)-2}u(k,h)v(k,h)\right)-\sum_{k=1}^{T_1}\sum_{h=1}^{T_2} f(k,h,u(k,h))v(k,h)$$

for any $v \in \mathcal{H}$. Using the same technics in section 3, it is clear that T is weakly lower semi-continuous. For (B_2), from (13) and (4) we have

$$\begin{aligned} T(u) &\geq M_0K(u)-\sum_{k=1}^{T_1}\sum_{h=1}^{T_2}F(k,h)u(k,h)\\ &\geq M_0\left(\sum_{k=1}^{T_1+1}\sum_{h=1}^{T_2+1}A(k-1,h-1,\Delta u(k-1,h-1))\right)\\ &\quad +M_0\sum_{k=1}^{T_1}\sum_{h=1}^{T_2}\left(\frac{1}{p(k,h)}\left|u(k,h)\right|^{p(k,h)}\right)-\sum_{k=1}^{T_1}\sum_{h=1}^{T_2}C_9(1+|u(k,h)|^{\gamma(k,h)}). \end{aligned}$$

For any $u \in \mathcal{H}$, we have $|u(k,h)|^{\gamma(k,h)} \leq |u(k,h)|^{\gamma^+}+|u(k,h)|^{\gamma^-}$.
We obtain

$$\begin{aligned} T(u) &\geq \sum_{k=1}^{T_1+1}\sum_{h=1}^{T_2+1}\frac{M_0}{p(k-1,h-1)}|\Delta u(k-1,h-1)|^{p(k-1,h-1)}\\ &\quad -\sum_{k=1}^{T_1}\sum_{h=1}^{T_2}C_9\left(|u(k,h)|^{\gamma^+}+|u(k,h)|^{\gamma^-}\right)-C_9T_1\times T_2. \end{aligned}$$

According to (8) and (14), we deduce that

$$T(u) \geq \frac{M_0}{p^+}\left(C_4\|u\|^{p^-}-C_5\right)-C_9\left(|u|_{\gamma^+}^{\gamma^+}+|u|_{\gamma^-}^{\gamma^-}\right)-C_9T_1\times T_2.$$

Due to the equivalence of $\|.\|$ and $|.|_m$, from (8), one has

$$T(u) \geq \frac{M_0}{p^+}\left(C_4\|u\|^{p^-}-C_5\right)-C_9\left((C_2)^{\gamma^+}\|u\|^{\gamma^+}+(C_2)^{\gamma^-}\|u\|^{\gamma^-}\right)-C_9T_1\times T_2.$$

Taking $C_0=\max\{(C_2)^{\gamma^+},(C_2)^{\gamma^-}\}$, it follows that

$$T(u) \geq \frac{M_0}{p^+}\left(C_4\|u\|^{p^-}-C_5\right)-C_9C_0\left((\|u\|^{\gamma^+}+\|u\|^{\gamma^-}\right)-C_9T_1\times T_2.$$

Since $p^- > \gamma^+$ and $\|u\| > 1$, $\lim_{\|u\|\to\infty} T(u)=\infty$. Thus, (B_2) is satisfied. For the boundedness (B_3), for all u in $\mathcal{H}$ such that $\|u\| \leq 1$, we have

$$T(u) \geq -\frac{M_0}{p^+}C_5-2C_9C_0-C_9T_1\times T_2 > -\infty.$$

T is bounded from below and (B_3) is satisfied. □

Since $(B_1)-(B_3)$ hold, Then in view of Theorem 2.3, problem (P) has at least one solution.

Now, we show that, the functional T has a nontrivial critical point using variational approach.

Theorem 4.3. *Suppose that* $(B_1)-(B_3)$ *are satisfied and*

(a) there exists a constants $M_1>0$ *such that* $M_1=\sup_{t\geq 0} M(t)$,

(b) there exist constants $C_{10}, C_{11}>0$ *such that* $F(k,h,x)\geq C_{10}|x|^{\alpha}-C_{11}$. *Then, for all* $\alpha>p^+$ *the problem* (P) *has at least one weak nontrivial solution.*

Proof. When $(B_1)-(B_3)$ hold, one has $T\in C^1(\mathcal{H},\mathbb{R})$; weakly lower semi-continuous and coercive on $\mathcal{H}$. Consider $\bar{u}(k,h)\in\mathcal{H}$ a global minimizer of T, a weak solution of problem (P). We show that $\bar{u}$ is nontrivial for all $\alpha>p^+$. Taking $s>1$ be a real and $u\in\mathcal{H}\setminus\{0\}$, we have

$$T(su)=\widehat{M}(K(su))-\sum_{k=1}^{T_1}\sum_{h=1}^{T_2}F(k,h,su(k,h)).$$

From (a), we have $\widehat{M}(K(su))\leq M_1\int_0^{K(su)}\mathrm{d}t\leq M_1K(su)$.
According to (8) − (10) and (b), we deduce that

$$\begin{aligned}
T(su) &\leq M_1(K(su))-\sum_{k=1}^{T_1}\sum_{h=1}^{T_2}F(k,h,su(k,h))\\
&\leq M_1\left(\sum_{k=1}^{T_1+1}\sum_{h=1}^{T_2+1}A(k-1,h-1,\Delta su(k-1,h-1))\right)\\
&\quad +M_1\sum_{k=1}^{T_1}\sum_{h=1}^{T_2}\left(\frac{1}{p(k,h)}|su(k,h)|^{p(k,h)}\right)-\sum_{k=1}^{T_1}\sum_{h=1}^{T_2}(C_{10}|su(k,h)|^{\alpha}-C_{11})
\end{aligned}$$

$$\begin{aligned}
T(su) &\leq M_1\sum_{k=1}^{T_1+1}\sum_{h=1}^{T_2+1}C_3|\Delta su(k-1,h-1)|\\
&\quad +M_1\sum_{k=1}^{T_1+1}\sum_{h=1}^{T_2+1}\frac{C_3}{p(k,h)}|\Delta su(k-1,h-1)|^{p(k-1,h-1)}\\
&\quad +M_1\left(\sum_{k=1}^{T_1}\sum_{h=1}^{T_2}\left(\frac{1}{p(k,h)}|su(k,h)|^{p(k,h)}\right)\right)-\sum_{k=1}^{T_1}\sum_{h=1}^{T_2}(C_{10}|su(k,h)|^{\alpha}-C_{11}).
\end{aligned}$$

We show that $\sum_{k=1}^{T_1+1}\sum_{h=1}^{T_2+1}C_3|\Delta su(k-1,h-1)|\leq 2sC_3\sqrt{(T_1+1)(T_2+1)}\|u\|.$

We have

$$\begin{aligned}
|\Delta su(k-1,h-1)| &= s|u(k,h)-u(k-1,h-1)|\\
|su(k,h)-su(k-1,h-1)| &\leq s|u(k,h)|+s|u(k-1,h-1)|\\
\sum_{k=1}^{T_1+1}\sum_{h=1}^{T_2+1}|\Delta su(k-1,h-1)| &\leq \sum_{k=1}^{T_1+1}\sum_{h=1}^{T_2+1}|su(k,h)|+\sum_{k=1}^{T_1+1}\sum_{h=1}^{T_2+1}|su(k-1,h-1)|\\
\sum_{k=1}^{T_1+1}\sum_{h=1}^{T_2+1}|\Delta su(k-1,h-1)| &\leq s\sqrt{(T_1+1)(T_2+1)}|u|_2+\sum_{k=0}^{T_1}\sum_{h=0}^{T_2}|su(k,h)|\\
\sum_{k=1}^{T_1+1}\sum_{h=1}^{T_2+1}|\Delta su(k-1,h-1)| &\leq 2s\sqrt{(T_1+1)(T_2+1)}|u|_2.
\end{aligned}$$

Then,

$$\sum_{k=1}^{T_1+1}\sum_{h=1}^{T_2+1} C_3|\Delta su(k-1,h-1)| \leq 2sC_3\sqrt{(T_1+1)(T_2+1)}\|u\|.$$

Consequently, we have

$$\begin{aligned}
T(su) &\leq M_1\left(2C_3s\sqrt{(T_1+1)\times(T_2+1)}\|u\|\right)\\
&+M_1\sum_{k=1}^{T_1+1}\sum_{h=1}^{T_2+1}\frac{C_3}{p(k,h)}|\Delta su(k-1,h-1)|^{p(k-1,h-1)}\\
&+\frac{M_1C_0}{p^-}\left(s^{p^+}(\|u\|^{p^+}+s^{p^-}\|u\|^{p^-}\right)-C_{10}C_1s^{\alpha}\|u\|^{\alpha}+C_{12}\\
T(su) &\leq M_1\left(2C_3s\sqrt{(T_1+1)\times T_2+1}\|u\|+\frac{C_3}{p^-}(C_6s^{p^+}\|u\|^{p^+}+C_7)\right)\\
&+\frac{M_1C_0}{p^-}\left(s^{p^+}(\|u\|^{p^+}+s^{p^-}\|u\|^{p^-}\right)-C_{10}C_1s^{\alpha}\|u\|^{\alpha}+C_{12}.
\end{aligned}$$

Since $\alpha > p^+$, for sufficiently large $s > 1$ we conclude that $T(su) < 0$. Thus $T(\bar{u}) < 0$, so from [12], we have $\bar{u}$ is a weak nontrivial solution of problem (P). □

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