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Positive Eigenvalues for the Adjoint of an Increasing Operator

Nguyen Hoang Tuan[1,2], **Vo Viet Tri**[3] **and Nguyen Anh Tuan**[4,5*]

1. Department of Applied Mathematics, Faculty of Applied Science, Ho Chi Minh City University of Technology, 268 Ly Thuong Kiet Street, Dien Hong Ward, Ho Chi Minh City, Vietnam
2. Vietnam National University Ho Chi Minh City, Linh Xuan Ward, Ho Chi Minh City, Vietnam
3. Division of Applied Mathematics, Thu Dau Mot University, Ho Chi Minh City, Vietnam
4. Division of Applied Mathematics, Science and Technology Advanced Institute, Van Lang University, Ho Chi Minh City, Vietnam
5. Faculty of Applied Technology, Van Lang School of Technology, Van Lang University, Ho Chi Minh City, Vietnam

*Corresponding author

Abstract

This paper develops a general framework for investigating the eigenvalue problem of the extended adjoint mapping $\Phi^* : E_b^* \to E_b^*$ associated with a possibly nonlinear operator $\Phi : E \to E$. By employing the theory of relative topological degree for condensing mappings with respect to a measure of noncompactness, we establish sufficient conditions ensuring the existence of positive eigenvalues of Φ^*. The extended dual space E_b^* is strictly larger than the classical dual E^*, and the two coincide only when E is finite-dimensional. In this special case, our results recover the classical eigenvalue theory for linear compact operators. The framework introduced here extends the topological degree approach to a broader nonlinear setting and provides a foundation for further applications to convex processes and control problems.

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1 Introduction

Let E be a normed vector space over the field $\mathbb{K}$ ($\mathbb{K} = \mathbb{C}$ or $\mathbb{R}$), and let E^* denote its dual space, that is, the space of all continuous linear functionals from E into $\mathbb{K}$.

Suppose E and F are normed spaces over the same field $\mathbb{K}$. For a continuous linear operator

Contact: Nguyen Hoang Tuan nhtuan.sdh231@hcmut.edu.vn; Vo Viet Tri trivv@tdmu.edu.vn; Nguyen Anh Tuan nguyenanhtuan@vlu.edu.vn

$\Phi : E \to F$, its adjoint operator $\Phi^* : F^* \to E^*$ is defined by

$$\Phi^*(f) = f \circ \Phi, \quad f \in F^*. \tag{1.1}$$

With the definition (1.1), in the case $E = F$, if $(\lambda, u) \in \mathbb{K} \times (E \setminus \{0\})$ satisfies $\Phi(u) = \lambda u$, then λ is called an eigenvalue of Φ, and u is the corresponding eigenvector. The eigenvalues of Φ and Φ^* are related by

$$(\lambda\Phi)^* = \overline{\lambda}\,\Phi^*, \quad \lambda \in \mathbb{K}. \tag{1.2}$$

Several studies have examined the eigenvalues of Φ through its adjoint operator. For instance, in [6, 10, 13], the authors extended the definition (1.1) to the case of multivalued mappings $\Phi : E \to \mathcal{P}(E) \setminus \{\varnothing\}$ (where $\mathcal{P}(E)$ denotes the family of all subsets of E), and defined its adjoint mapping as

$$\Phi^*(f) = \big\{g \in E^* : g(x) \le f(y),\, y \in \Phi(x),\, x \in E\big\}, \quad f \in E^*. \tag{1.3}$$

Under the order relation on E induced by a cone P with nonempty interior, and assuming the invariance of P under Φ, the authors of [5] established the existence of positive eigenvalues for Φ^*.

The theory of relative topological degree provides an effective tool to study eigenvalues of single-valued or multivalued condensing mappings with respect to a measure of noncompactness, including the compact case as a particular example. Representative works in this direction include [4, 5, 8, 9, 10, 11, 12, 13]. From this perspective, the adjoint operator Φ^* is advantageous because it is compact with respect to the weak topology, according to the Banach–Alaoglu theorem.

Motivated by these considerations, we extend the notion of the adjoint operator to a broader class of bounded and positively homogeneous mappings (which includes, but is not limited to, bounded linear operators). In particular, we construct the extended adjoint mapping $\Phi^* : E_b^* \to E_b^*$ associated with a possibly nonlinear operator $\Phi : E \to E$. The extended dual space E_b^* strictly contains the classical dual E^*, and the two coincide only when E is finite-dimensional. This extension allows one to study adjoint-type mappings beyond the linear setting.

By employing the theory of relative topological degree for condensing mappings, we derive sufficient conditions ensuring the existence of positive eigenvalues of Φ^*. In the finite-dimensional case, where the weak and norm topologies coincide, our results reduce to the classical eigenvalue theory for linear compact operators. The framework developed in this paper thus provides a unified and generalized approach that bridges linear and nonlinear eigenvalue problems and offers a foundation for future applications to convex processes and control systems.

The remainder of this paper is organized as follows. Section 2 recalls several fundamental concepts and results concerning ordered normed spaces, and the relative topological degree for condensing mappings. In Section 3, we construct the extended adjoint mapping $\Phi^* : E_b^* \to E_b^*$ associated with a possibly nonlinear operator $\Phi : E \to E$, and we establish the main existence results for its positive eigenvalues by employing the relative topological degree. Finally, we provide concluding remarks and discuss potential extensions of the proposed framework to convex processes and control problems.

2 Preliminaries

In the remainder of this paper, unless otherwise specified, we assume that E is a Banach space over the real field, equipped with the norm denoted by $\|\cdot\|$. On E, we consider the order induced by a closed, convex cone P with vertex at the origin that is, P is a closed convex subset of E satisfying $\lambda P \subset P$ for all $\lambda > 0$, $P \cap (-P) = \{\mathbf{0}\}$ (where $\mathbf{0}$ denotes the zero vector), and

$$x \leq_E y \ \Leftrightarrow\ y - x \in P.$$

The cone P is said to be *normal* if there exists a constant $N > 0$ such that $\mathbf{0} \leq_E x \leq_E y$ implies $\|x\| \leq N\|y\|$. In what follows, we always assume that P is a normal cone with $N = 1$.

2.1 Extended dual space E_b^*

Definition 2.1. Let E be a normed space ordered by a normal cone P. A mapping $\Phi : E \to E$ is said to be:

1. *Bounded* if there exists a constant $c > 0$ such that $\|\Phi(x)\| \leq c\|x\|$ for all $x \in E$.
2. *Positively homogeneous* if $\Phi(\lambda x) = \lambda\Phi(x)$ for all $\lambda \geq 0$.
3. *Monotone increasing* if $\mathbf{0} \leq_E x \leq_E y$ implies $\mathbf{0} \leq_E \Phi(x) \leq_E \Phi(y)$ for all $x, y \in E$.

We denote by $\mathcal{H}_b(E)$ the set of all continuous, bounded, positively homogeneous, and monotone increasing mappings from E into itself.

It is clear that $\mathcal{H}_b(E)$ is a vector space under the usual operations of addition and scalar multiplication. We define

$$\|\Phi\|_{\mathcal{H}_b(E)} = \inf\Big\{k > 0 : \|\Phi(x)\| \leq k\|x\| \text{ for all } x \in E\Big\}, \quad \Phi \in \mathcal{H}_b(E).$$

Similarly to the case of continuous linear operators from E into E, one can show that the mapping $\Phi \mapsto \|\Phi\|_{\mathcal{H}_b(E)}$ is a norm on $\mathcal{H}_b(E)$, and that

$$\begin{aligned}\|\Phi\|_{\mathcal{H}_b(E)} &= \sup\left\{\frac{\|\Phi(u)\|}{\|u\|} : u \in E \setminus \{\mathbf{0}\}\right\}\\ &= \sup\Big\{\|\Phi(u)\| : u \in E,\ \|u\| \leq 1\Big\}, \quad \Phi \in \mathcal{H}_b(E).\end{aligned}$$

As a consequence,

$$\|\Phi(x)\| \leq \|\Phi\|_{\mathcal{H}_b(E)}\|x\| \quad \text{for all } x \in E,\ \Phi \in \mathcal{H}_b(E).$$

We denote by E_b^* the set of all continuous, bounded, positively homogeneous, and monotone increasing real-valued functions on E (in the sense of Definition 2.1). Analogously, we equip E_b^* with the norm

$$\|f\|_* = \inf\Big\{c > 0 : |f(x)| \leq c\|x\| \ \forall x \in E\Big\}, \quad f \in E_b^*,$$

and obtain results analogous to those for continuous linear functionals on E.

For each $x \in E$, define the mapping $J_x : E_b^* \to \mathbb{R}$ by

$$J_x(f) = f(x), \quad f \in E_b^*.$$

Clearly, J_x is a continuous linear functional on E_b^*. On E_b^*, we denote by $\sigma_* := \sigma(E_b^*, E)$ the weak-$(*)$ topology induced by the family $\{J_x : x \in E\}$, and by $\sigma = \sigma(E_b^*, E_b^{**})$ the weak topology induced by all continuous linear functionals from $(E_b^*, \|\cdot\|_*)$ into $\mathbb{R}$. If τ denotes the topology on E_b^* induced by the norm $\|\cdot\|_*$, then we have

$$\sigma_* \subset \sigma \subset \tau. \tag{2.1}$$

We recall a characteristic property of convergence in (E_b^*, σ_*).

Lemma 2.2. *Let $\{f_\alpha\} \subset E_b^*$ and $f \in E_b^*$. Then $\{f_\alpha\}$ converges to f in (E_b^*, σ_*) if and only if, for every $x \in E$, one has $f_\alpha(x) \to f(x)$ in $\mathbb{R}$.*

Lemma 2.3. *Let $\Phi \in \mathcal{H}_b(E)$ and $f \in E_b^*$. Then $f \circ \Phi \in E_b^*$ and*

$$|f \circ \Phi(x)| \leq \|f\|_* \|\Phi\|_{\mathcal{H}_b(E)} \|x\|, \quad x \in E. \tag{2.2}$$

Proof. It is clear that $f \circ \Phi$ is positively homogeneous. Since $\Phi \in \mathcal{H}_b(E)$ and $f \in E_b^*$, the composition $f \circ \Phi$ is continuous from E into $\mathbb{R}$. For each $x \in E$, we have $|f(\Phi(x))| \leq \|f\|_* \|\Phi(x)\|$ and $\|\Phi(x)\| \leq \|\Phi\|_{\mathcal{H}_b(E)} \|x\|$, which implies that $f \circ \Phi$ is bounded. The monotonicity of $f \circ \Phi$ follows from the monotonicity of f and Φ. Hence $f \circ \Phi \in E_b^*$, and (2.2) holds. □

From Lemma 2.3, we may introduce the following definition.

Definition 2.4. For $\Phi \in \mathcal{H}_b(E)$, define the mapping $\Phi^* : E_b^* \to E_b^*$ by

$$\Phi^*(f) = f \circ \Phi, \quad f \in E_b^*.$$

The mapping Φ^* is called the *adjoint* of Φ. Moreover, we have $(\lambda\Phi)^* = \lambda\Phi^*$ for all $\lambda \in \mathbb{R}$.

Remark 2.5. The present concept is essentially different from the corresponding one considered in Ref. [13]. In that reference, the adjoint operator is defined as a certain set of bounded linear functionals, while in this paper, the adjoint operator is defined as a single positively homogeneous mapping. Hence, these two notions are not special cases of each other.

Lemma 2.6. *Let $\Phi \in \mathcal{H}_b(E)$. Then $\Phi^* : (E_b^*, \sigma_*) \to (E_b^*, \sigma_*)$ is continuous. Consequently, $\Phi^* : (E_b^*, \|\cdot\|_*) \to (E_b^*, \sigma_*)$ is also continuous.*

Proof. Suppose that a net $\{f_\alpha\}$ converges to f in (E_b^*, σ_*). We need to show that $\{\Phi^*(f_\alpha)\}$ converges to $\Phi^*(f)$ in (E_b^*, σ_*). For any $x \in E$, by Lemma 2.2, we have $|f_\alpha \circ \Phi(x) - f \circ \Phi(x)| \to 0$. Setting $\xi = \Phi(x)$, we see that this is equivalent to $|f_\alpha(\xi) - f(\xi)| \to 0$, which follows directly from the assumption $f_\alpha \to f$ in (E_b^*, σ_*). The second assertion follows from the decomposition $\Phi^* = \Phi^* \circ I$, where I is the identity map from (E_b^*, τ) into (E_b^*, σ_*), together with the inclusion (2.1). □

We now present a result analogous to the Banach–Alaoglu theorem.

Proposition 2.7. *The closed unit ball $B^* = \{f \in E_b^* : \|f\|_* \leq 1\}$ in $(E_b^*, \|\cdot\|_*)$ is compact in the space (E_b^*, σ_*).*

Proof. For each $x \in E$, define the set $\mathbb{R}_x = \{\lambda \in \mathbb{R} : |\lambda| \leq \|x\|\}$. Then $\mathbf{P} = \prod_{x \in E} \mathbb{R}_x$ is compact in the product topology τ (that is, τ is the weakest topology on $\mathbf{P}$ making each projection $\varphi_x : \mathbf{P} \to \mathbb{R}_x$, $\varphi_x(f) = f(x)$, continuous, where $|f(x)| \leq \|x\|$). Each element of $\mathbf{P}$ is a function $f : E \to \mathbb{R}$ satisfying $|f(x)| \leq \|x\|$ for all $x \in E$, so that $B^* = \mathbf{P} \cap E_b^*$.

Let σ' and τ' denote the topologies on B^* induced by σ_* and τ, respectively. We show that (B^*, σ') is compact. First, B^* is closed in $(\mathbf{P}, \tau)$. Indeed, if a net $\{f_\alpha\}_\alpha \subset B^*$ converges to f in τ, then for each $x \in E$, by definition of τ, we have $\varphi_x(f_\alpha) \to \varphi_x(f)$, i.e., $f_\alpha(x) \to f(x)$. Since $|f_\alpha(x)| \leq \|x\|$, it follows that $|f(x)| \leq \|x\|$. Moreover, since $\|f_\alpha\|_* \leq 1$, we have $\frac{|f_\alpha(x)|}{\|x\|} \leq 1$ for $x \neq \mathbf{0}$, hence $\|f\|_* \leq 1$, so $f \in B^*$. Therefore, B^* is closed in the compact space $(\mathbf{P}, \tau)$, and thus (B^*, τ') is compact.

It remains to show that $\sigma' = \tau'$. Consider the identity mapping $I : (B^*, \sigma') \to (B^*, \tau')$. We have

$$\begin{aligned}(f_\alpha \to f \text{ in } (B^*, \sigma')) &\Leftrightarrow (f_\alpha(x) \to f(x)\ \forall x \in E) \\ &\Leftrightarrow (f_\alpha \to f \text{ in } (B^*, \tau')),\end{aligned}$$

which yields $\sigma' = \tau'$. □

Corollary 2.8. *A closed and bounded subset of $(E_b^*, \|\cdot\|_*)$ is compact in (E_b^*, σ_*). Consequently, if $\Phi \in \mathcal{H}_b(E)$ and $f \in E_b^*$, then $\Phi^* : (E_b^*, \|\cdot\|_*) \to (E_b^*, \sigma_*)$ is a compact mapping; that is, $\Phi^*(G)$ is relatively compact in (E_b^*, σ_*) whenever G is bounded in $(E_b^*, \|\cdot\|_*)$.*

Proof. Assume that G is a closed and bounded subset of $(E_b^*, \|\cdot\|_*)$. Then there exists $r > 0$ such that

$$G \subset \{f \in E_b^* : \|f\|_* \leq r\} = rB^*.$$

This, together with Proposition 2.7, implies that G is compact in (E_b^*, σ_*).

Now, to prove the second assertion, assume G is bounded in E_b^*. By Lemma 2.3, $f \circ \Phi(G)$ is bounded in (E_b^*, σ_*), and hence there exists $r > 0$ such that $\Phi^*(G) = f \circ \Phi(G) \subset rB^*$. The desired conclusion follows. □

2.2 Relative topological degree for compact mappings

We recall some results concerning the relative topological degree for condensing mappings with respect to a measure of noncompactness. Let X be a Fréchet space, $P \subset X$ a convex set, and $G \subset X$ a bounded open subset of X. Assume that $\mathbb{T} : X \to X$ is a continuous mapping satisfying $\mathbb{T}(P) \subset P$, $\mathbb{T}(G)$ is relatively compact, and $x \notin \mathbb{T}(x)$ for all $x \in (P \cap \partial G)$ (here, ∂G denotes the boundary of G). Then, the relative topological degree of $\mathbb{T}$ in G with respect to P is defined as an integer and denoted by $i_P(\mathbb{T}, G)$. Noting that any compact mapping is condensing with respect to a regular measure of noncompactness, we can employ the following result presented in [2, 3, 4, 13].

Lemma 2.9. [2, Theorem 2.1] *Let $G \subset X$ be open and $\mathbb{T} : X \to X$ be a continuous mapping such that $\mathbb{T}(P) \subset P$ and $\mathbb{T}(G \cap P)$ is relatively compact. Assume that $x \neq \mathbb{T}(x)$ for all $x \in P \cap \partial G$. Then*

(i) If $i_P(\mathbb{T}, G) \neq 0$, then $\mathrm{Fix}(\mathbb{T}) \neq \varnothing$ (where $\mathrm{Fix}(\mathbb{T}) = \{u \in X : x = \mathbb{T}(x)\}$).
(ii) $i_P(\mathbb{H}(1, \cdot), G) = i_P(\mathbb{H}(0, \cdot), G)$ if $\mathbb{H}(G)$ is relatively compact and $x \neq \mathbb{H}(\gamma, x)$ for all $(\gamma, x) \in [0, 1] \times P \cap \partial G$.
(iii) If $G = G_1 \cup G_2 \subset X$ with $G_1 \cap G_2 = \varnothing$ and $x \neq \mathbb{T}(x)$ for all $x \notin P \cup (\partial G_1 \cup \partial G_2)$, then $i_P(\mathbb{T}, G) = i_P(\mathbb{T}, G_1) + i_P(\mathbb{T}, G_2)$.

Lemma 2.10. [4, 13, Lemma 2.1] *Let G be a neighborhood of the origin and $\mathbb{T} : X \to X$ be a continuous mapping satisfying $\mathbb{T}(P) \subset P$. Assume that $\mathbb{T}(G \cap P)$ is relatively compact. Then*

(i) If $\gamma u \neq \mathbb{T}(u)$ for all $(\gamma, u) \in [1, \infty) \times P \cap \partial G$, then $i_P(\mathbb{T}, G) = 1$.
(ii) If there exists $v \in P \backslash \{\mathbf{0}\}$ such that $u \neq \mathbb{T}(u) + \gamma v$ for all $(\gamma, u) \in (0, \infty) \times (P \cap \partial G)$, then $i_P(\mathbb{T}, G) = 0$.

3 Main results

Lemma 3.1. *Let X be a Fréchet space, $P \subset X$ a closed convex set, and let $\mathbb{T} : [0, \infty) \times P \to P$ be a continuous mapping. Suppose that $G \ni \mathbf{0}$ is an open and bounded subset of X and the following conditions are satisfied:*

(i) For each $\gamma \in [0, \infty)$, the set $\mathbb{T}(\gamma, G)$ is relatively compact.
(ii) If there exist $u \in P \backslash \{\mathbf{0}\}$ and $\gamma u = \mathbb{T}(0, u)$, then $\gamma < 1$.
(iii) There exists $\gamma_0 > 0$ such that $i_P(\mathbb{T}(\gamma, \cdot), G) = 0$ for all $\gamma \geq \gamma_0$.

Then there exists $(\gamma, u) \in (0, \infty) \times (P \cap \partial G)$ such that $u = \mathbb{T}(\gamma, u)$.

Proof. Let γ_0 be the number mentioned in condition (ii). Define

$$\alpha = \sup \left\{ \gamma > 0 : i_P(\mathbb{T}(\gamma, \cdot), G) \neq 0 \right\}.$$

First, we show that $\alpha > 0$. Consider

$$\mathbb{H}(\gamma, x) = (1-\gamma)\mathbb{T}(\epsilon, x) + \gamma\mathbb{T}(0, x), \quad \epsilon > 0,\ x \in P \cap \partial G, \gamma \in [0,1].$$

We will prove by contradiction the following claim:

$$\text{There exists } \epsilon > 0 \text{ such that } (\gamma, x) \neq \mathbb{H}(\gamma, x) \text{ for all } (\gamma, x) \in [0,1] \times (P \cap \partial G). \tag{3.1}$$

Indeed, if (3.1) were false, then for each ϵ, one could find $(\gamma_\epsilon, x_\epsilon) \in [0,1] \times P \cap \partial G$ such that

$$x_\epsilon = (1-\gamma_\epsilon)\mathbb{T}(\epsilon, x_\epsilon) + \gamma_\epsilon \mathbb{T}(0, x_\epsilon). \tag{3.2}$$

Since $\mathbb{T}$ is compact, from (3.2) we may assume that $\gamma_\epsilon \to \gamma$ and $x_\epsilon \to x$ as $\epsilon \to 0^+$. Therefore, by the continuity of $\mathbb{T}$, we obtain

$$x = (1-\gamma)\mathbb{T}(0, x) + \gamma\mathbb{T}(0, x) = \mathbb{T}(0, x).$$

This contradicts condition (ii). Hence, claim (3.1) holds. Using property (ii) of Lemma 2.9, we conclude that $i_P(\mathbb{T}(\epsilon, \cdot), G) = i_P(\mathbb{T}(0, \cdot), G)$. Together with Lemma 2.10, this implies $i_P(\mathbb{T}(\epsilon, \cdot), G) = 1$. Consequently, $\alpha \geq \epsilon > 0$.

Next, from the definition of α, we construct a homotopy as follows: for each $\epsilon \in (0, \alpha)$, there exists $\gamma_\epsilon \in (\alpha - \epsilon, \alpha]$ such that $i_P(\mathbb{T}(\gamma_\epsilon, \cdot), G) \neq 0$, and we define

$$H_\epsilon(\gamma, \cdot) = (1-\gamma)\mathbb{T}(\gamma_\epsilon, \cdot) + \gamma\mathbb{T}(\alpha + \epsilon, \cdot).$$

We now prove the assertion of the lemma. Assume to the contrary that $u \neq \mathbb{T}(\gamma, u)$ for all $(\gamma, u) \in (0, \infty) \times (P \cap \partial G)$. By the definition of α and applying property (ii) of Lemma 2.9 to the homotopy $\mathbb{H}_\epsilon(\cdot, \cdot)$, we obtain the contradiction

$$0 \neq i_P(\mathbb{T}(\gamma_\epsilon, \cdot), G) = i_P(\mathbb{T}(\alpha + \epsilon, \cdot), G) = 0.$$

Hence, the lemma is proved. □

We denote by $\mathcal{EV}_+(\Phi)$ the set of all positive eigenvalues of the mapping $\Phi : X \to X$, that is, $\mathcal{EV}_+(\Phi) = \Big\{\lambda \in (0, \infty) : \exists u \in X \backslash \{\mathbf{0}\}, \Phi(u) = \lambda u\Big\}$.

Theorem 3.2. *Let $\Phi \in \mathcal{H}_b(E)$. Suppose that the cone P contains some $v_0 \neq \mathbf{0}$ and that there exists a positive constant α such that $\Phi(v_0) \geq_E \alpha v_0$ (or $\Phi(x) \leq_E \alpha v_0$ for all $x \in P$). Then, the set $\mathcal{EV}_+(\Phi^*) \neq \varnothing$. Moreover,*

$$0 < \alpha \leq \lambda \quad \forall \lambda \in \mathcal{EV}_+(\Phi^*).$$

Proof. Define $P^* = \Big\{f \in E_b^* : f(x) \geq 0 \text{ for all } x \in P\Big\}$. Set $G = \Big\{f \in E_b^* : \big|f(v_0)\big| < 1\Big\}$. Define $\mathbb{T} : [0, \infty) \times P^* \to P^*$ by

$$\mathbb{T}(\gamma, f) = \gamma\Phi^*(f).$$

We now verify the assumptions of Lemma 3.1. Clearly, $G \ni \mathbf{0}$ is open in $\big(E_b^*, \sigma_*\big)$. Next, we show that $\mathbb{T}(G \cap Q)$ is compact in $\big(E_b^*, \sigma_*\big)$, here $Q = P^* \cap B^*$. To this end, $\mathbb{T}(\lambda, G \cap Q)$ is bounded in the space $\big(E_b^*, \|\cdot\|_*\big)$. Indeed, for any $f \in G \cap Q$, we have

$$\Big|f \circ \Phi(x)\Big| \leq \Big|f\big(\Phi(x)\big)\Big| \leq \|\Phi\|_{\mathcal{H}_b(E)} \quad \text{for all } x \in E, \|x\| \leq 1.$$

This gives

$$\|f \circ \Phi\|_* \leq \|\Phi\|_{\mathcal{H}_b(E)} \quad \text{for all } f \in G \cap Q.$$

Consequently, the set $\mathbb{T}(\gamma, G \cap Q) = \gamma\Phi^*(G \cap Q)$ is relatively compact in (E_b^*, σ_*) by Proposition 2.7. Thus, condition (i) is satisfied.

Since $\mathbb{T}(0, f) = \{\mathbf{0}\}$, there cannot exist $\mathbf{0} \neq u \in P^*$ such that $u \in \mathbb{T}(0, u)$; hence condition (ii) is fulfilled.

We now verify condition (iii). Let $\nu^* \in P^* \backslash \{\mathbf{0}\}$. We shall prove that there exists $\mu_0 > 0$ such that for all $\gamma \geq \mu_0$,

$$f \notin \mathbb{T}(\gamma, f) + k\nu^* \quad \text{for all } k \geq 0, f \in Q \cap \partial_{\sigma_*} G. \tag{3.3}$$

Once this claim is proved, applying Lemma 2.9 yields $i_Q(\mathbb{T}(\mu, \cdot), G) = 0$ for all sufficiently large $\mu > \mu_0$. Thus, condition (iii) holds.

If the claim were false, then for every $\mu > 0$ there would exist $\gamma_\mu \geq \mu$, $k_\mu > 0$, and $f_\mu \in P^* \cap \partial_{\sigma_*} G$ such that

$$\begin{aligned} f_\mu &= \mathbb{T}(\gamma_\mu, f_\mu) + k_\mu \nu^*, \text{ i.e.,} \\ f_\mu &= \gamma_\mu \Phi^*(f_\mu) + k_\mu \nu^*, \text{ i.e.,} \\ f_\mu &= \gamma_\mu f_\mu \circ \Phi + k_\mu \nu^*. \end{aligned} \tag{3.4}$$

From (3.4), we obtain $f_\mu(\nu_0) > 0$ and

$$f_\mu(\nu_0) \geq \gamma_\mu f_\mu(\nu_0), \text{ hence } \gamma_\mu \leq 1.$$

This is a contradiction when we choose $\mu > 1$. Therefore, (3.3) is proved. It is also clear that if $\gamma \in \mathcal{EV}_+(\Phi)$ then $\gamma \geq \alpha$. The case corresponding to the other condition can be proved analogously, which completes the proof. □

Conclusion

This work establishes a general framework for constructing the extended adjoint mapping $\Phi^* : E_b^* \to E_b^*$ associated with a possibly nonlinear operator $\Phi : E \to E$. By employing the theory of relative topological degree for condensing mappings, we prove the existence of positive eigenvalues of Φ^* under suitable order and compactness assumptions. In the special case where the range space of these mappings is finite-dimensional—so that the weak and norm topologies coincide—our results recover the classical eigenvalue theory for linear compact operators. The developed framework thus extends the classical concepts to a broader nonlinear setting. Future investigations will focus on applying this approach to convex processes and exploring its implications in control theory.

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