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Finite time blow-up for an inhomogeneous wave equation with Riemann-Liouville fractional integral

Changyong Xiang[1,*], **Jisong Duan**[2] **and Qianqian Luo**[3]

1. School of Mathematical Sciences, Guizhou Normal University, Guiyang, 550025, China
*. Corresponding author

Abstract

This paper is devoted to the study the effect of an inhomogeneity $\omega(x)$ on the blow-up phenomena for the wave equation

$$u_{tt} - \triangle u = I^{\alpha}_{0+}(|u|^p) + \omega(x), \quad (t,x) \in (0,T) \times \mathbb{R}^N,$$

where $N \geq 1$, $p > 1$, $0 \leq \alpha < 1$ and $\omega(x)$ with $\int_{\mathbb{R}^N} \omega(x)dx > 0$. Specifically, by the method of cut-functions, this paper first proves that the any non-trivial solution blows up in finite time under $0 < \alpha < 1$, and later proves that the any non-trivial solution blows up in finite time under $\alpha = 0$, $2 \leq N \leq 4$ and p being the Strauss exponent.

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1 Introduction

In this paper, we are concerned with the blow-up effect of solutions for the equation

$$\begin{cases} u_{tt} - \Delta u = I^{\alpha}_{0+}(|u|^p) + \omega(x), & (t,x) \in (0,T) \times \mathbb{R}^N, \\ u(0,x) = u_0, \quad u_t(0,x) = u_1, & x \in \mathbb{R}^N, \end{cases} \tag{1}$$

where $N \geq 1$, $p > 1$, $0 \leq \alpha < 1$, $\omega(x)$ with $\int_{\mathbb{R}^N} \omega(x)dx > 0$ is inhomogeneous function with respect to x, and all u_0, u_1, $\omega(x) \in C_0(\mathbb{R}^N)$. Here,

$$I^{\alpha}_{0+}(|u|^p) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} u(s,x)ds$$

Contact: Changyong Xiang 2552876289@qq.com; Jisong Duan 18149353334@qq.com; Qianqian Luo 13116497060@163.com

denotes the left Riemann-Liouville fractional integral of order $0 < \alpha < 1$. This equation of type(1) can be used to characterize various phenomena in physics, geophysics and acoustics (see, e.g., Hartle and Hawking[1] and Klibanov[2] and the references therein).

In [3], the authors studied the semilinear wave equation

$$u_{tt} - \Delta u = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} u(s,x) ds, \quad (t,x) \in (0,T) \times \mathbb{R}^N. \tag{2}$$

They proved that the effect of the generalized Strauss exponent $p_0(N,\alpha)$ of the equation, and the concrete results are as following:

$$p_0(N,\alpha) = \begin{cases} \infty & N = 1, \\ \frac{N+1+2\alpha+\sqrt{N^2+(10+4\alpha)N+4\alpha(\alpha+1)-7}}{2(N-1)} & N \geq 2, \end{cases}$$

i.e.

(a) if $1 < p < \infty$ as $N = 1$ or $1 < p < p_0(N,\alpha)$ as $N \geq 2$, u_0 and u_1 be nonnegative, then the solution blows up in finite time;

(b) if $p = p_0(N,\alpha)$ and $N \geq 2$, u_0 and u_1 be nonnegative, then the solution blows up in finite time.

Observe that in the limit case $\alpha \to 0$, we have

$$p_0(N,\alpha) = p_0(N,0) = \frac{N+1+\sqrt{N^2+10N-7}}{2(N-1)} = p(N \geq 2),$$

which is the Strauss exponent [4] for the semilinear wave equation

$$u_{tt} - \Delta u = |u|^p, \quad (t,x) \in (0,T) \times \mathbb{R}^N. \tag{3}$$

This exponent $p_0(N,0)$ plays the role of a bifurcation parameter in distinguishing between existence and non-existence of solutions, but also in determining their asymptotic behavior. Some studies of the above result can be found in [5, 6, 7, 8]. In [9, 10], the authors studied the inhomogeneous semilinear wave equation

$$u_{tt} - \Delta u = |u|^p + \omega(x), \quad (t,x) \in (0,T) \times \mathbb{M}, \tag{4}$$

where $\mathbb{M}$ with $N \geq 3$ is a noncompact complete Riemannian manifold and $\omega(x) > 0$. They proved that $p_* = \frac{N}{N-2}$ is critical for above the equation. Specifically, in [9], the author obtained the following results:

(c) Under $1 < p < p_*$, $\omega(x) > 0$, the problem (4) possesses no global solutions for any u_0, u_1 unless $\omega(x) \equiv 0$.

(d) Under $\mathbb{M}^n = \mathbb{R}^N$ with $N = 3$, $p_* = \frac{N}{N-2}$, and suppose both u_0 and u_1 have compact supports, then (1.4) possesses no global solutions unless $\omega(x) \equiv 0$.

(e) Under $p > p_*$, $\omega(x) > 0$, the problem (4) has global solutions for some $\omega(x)$, u_0 and u_1. In [10], Mohamed Jleli et al proved that the problem (1.4) has no global solutions under $\mathbb{M}^n \neq \mathbb{R}^N$, $p = p_* = \frac{N}{N-2}$. From the results of the study in problems (2) and (4), we can see that the influence of the inhomogeneous term $\omega(x)$ on the wave equation is significant.

In addition, [11, 12, 13, 14, 15, 16] et al. reveal that the inhomogeneous term $\omega(x)$ has a great influence on the behavior of solutions on the evolution equations.

As far as we know, the blow-up phenomena of problem (1) is still unknown. Therefore, we in this article will be devoted to deal with the blow-up effect for the problem (1). In order to make this article self-contained, after accepting local solution existence, the main result of this article is stated as follows:

Theorem 1.1 Let $u_0, u_1 \in C_0(\mathbb{R}^N)$, $\omega(x) \in C_0(\mathbb{R}^N) \cap L^1(\mathbb{R}^N)$ with $\int_{\mathbb{R}^N} \omega(x)dx > 0$. Then

(1) the solution of problem (1.1) blows up in finite time under $N \geq 1, 0 < \alpha < 1$ and $p > 1$;

(2) the solution of problem (1.1) blows up in finite time under $2 \leq N \leq 4$, $\alpha = 0$ and

$$p = \frac{N+1+\sqrt{N^2+10N-7}}{2(N-1)}.$$

Remark 1.2 Theorem 1.1 shows that **(1)** the problem (1) with $\omega(x)$ perturbation admits no global solutions under $N \geq 1, 0 < \alpha < 1$ and $p > 1$, which is different from that of the problem (1) without $\omega(x)$ perturbation in [3] under $1 < p < \infty$ as $N = 1$(or $1 < p < p_0(N, \alpha)$ as $N \geq 2$); **(2)** the problem (1.1) with $\omega(x)$ perturbation admits no global weak solution under $2 \leq N \leq 4$, $\alpha = 0$ and $p = \frac{N+1+\sqrt{N^2+10N-7}}{2(N-1)}$. These results generalize the previous blow-up results.

In the next section, we prove Theorem 1.1.

2 Proof of Theorem 1.1

2.1 Preliminaries

Definition 2.1 ([17]) For any $0 < \alpha < 1$, $u \in C^\infty([0,T], \mathbb{R}^N)$ and $T > 0$,

(1) the left Riemann-Liouville fractional integrals of order α of u is defined as

$$(I_{0^+}^\alpha u)(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} u(s)ds, \qquad 0 \leq t \leq T;$$

(2) the right Riemann-Liouville fractional integrals of order α of u is defined as

$$(I_{T^-}^\alpha u)(t) = \frac{1}{\Gamma(\alpha)} \int_t^T (s-t)^{\alpha-1} u(s)ds, \qquad 0 \leq t \leq T.$$

And since $(I_{0^+}^\alpha u)(t) \to u(t)$ as $\alpha \to 0$ (see [17]), we can define $I^0(u)(t) = u(t)$.

Lemma 2.2 ([17]). Let $0 < \alpha < 1$, $T > 0$, and $f, g \in L_p(0,T)$. Then

$$\int_0^T f(t)(I_{0^+}^\alpha g)(t)dt = \int_0^T g(t)(I_{T^-}^\alpha f)(t)dt.$$

Lemma 2.3 ([17]). If $\alpha > 0$ and $\beta > 0$, then

$$\left(I_{a^+}^\alpha (t-a)^{\beta-1}\right)(x) = \frac{\Gamma(\beta)}{\Gamma(\beta+\alpha)}(x-a)^{\beta+\alpha-1}$$

and

$$\left(I_{b^-}^\alpha (b-t)^{\beta-1}\right)(x) = \frac{\Gamma(\beta)}{\Gamma(\beta+\alpha)}(b-x)^{\beta+\alpha-1}.$$

Definition 2.4 Let $N \geq 1$, $u_0, u_1 \in C_0(\mathbb{R}^N)$ and $\omega(x) \in L^1(\mathbb{R}^N)$. We say $u(x,t) \in L^p\left(L^\infty_{loc}(\mathbb{R}^N), (0,T)\right)$ to be a weak solution of problem (1.1), if the equality

$$\begin{aligned} &\int_0^T \int_{\mathbb{R}^N} (I_{T^-}^\alpha \varphi)|u|^p dxdt + \int_0^T \int_{\mathbb{R}^N} \omega(x)\varphi dxdt + \int_{\mathbb{R}^N} \left(\varphi(0,x)u_1 - \varphi_t(0,x)u_0\right)dx \\ &= \int_0^T \int_{\mathbb{R}^N} u\varphi_{tt} dxdt - \int_0^T \int_{\mathbb{R}^N} u\Delta\varphi dxdt \end{aligned} \tag{5}$$

holds for all $T > 0$ and $\varphi \in C^{2,2}_{t,x}([0,T], \mathbb{R}^N)$ with $supp_x\, \varphi \subset\subset \mathbb{R}^N$, $\varphi(T,\cdot) = 0$ and $\varphi_t(T,\cdot) = 0$.

Lemma 2.5([18]) Let $\omega(x) \in L^1(\mathbb{R}^N)$ with $\int_{\mathbb{R}^N} \omega(x)dx > 0$. Then there exists a test function $0 \le \phi \le 1$ such that $\int_{\mathbb{R}^N} \omega(x)\phi dx > 0$.

2.2 Proof of Theorem 1.1

Proof: We divide the proof of Theorem 1.1 into two steps. In step I, we will prove that the problem (1) does not admit any global weak solution under $N \ge 1$, $\alpha \in (0,1)$ and $p > 1$. In step II, we will prove that the problem (1) does not admit any global weak solution under $2 \le N \le 4$, $\alpha = 0$ and $p = \frac{N+1+\sqrt{N^2+10N-7}}{2(N-1)}$.

Step I. For $N \ge 1$, $0 < \alpha < 1$ and $p > 1$. By arguing by contradiction, if not, we can assume that $u(x,t)$ is a global weak solution to the problem (1). Inspired by [16], we choose the test function following

$$\varphi(t,x) = \eta(t)\mu(x),$$

where

$$\eta(t) = T^{-\lambda}(T-t)^{\lambda}, \lambda > \frac{p+\alpha}{p-1}, t \in [0,T], T \in (0,\infty);$$

$$\mu(x) = \phi\left(\frac{|x|^2}{R^2}\right), R \gg 1, x \in \mathbb{R}^N.$$

Let the functions $\mu(x)$ and $\phi(z)$ satisfy

(i) $|\mu'(x)| \le C|\mu(x)|, |\mu''(x)| \le C|\mu(x)|$;

(ii) $\phi(z) \in C^2_0(\mathbb{R}_+)$ is non-increasing function with $\phi(z) \equiv 1$ if $z \in [0,1]$ and $\phi(z) \equiv 0$ if $z \in [2,\infty)$.

Then, it follows from (5) that

$$\begin{aligned}&\int_0^T\int_{\mathbb{R}^N} (I^{\alpha}_{T^-}\varphi)|u|^p dxdt + \int_0^T\int_{\mathbb{R}^N} \omega(x)\varphi dxdt + \int_{\mathbb{R}^N} (\varphi(0,x)u_1 - \varphi_t(0,x)u_0)dx \\ \le &\int_0^T\int_{\mathbb{R}^N} |u||\varphi_{tt}|dxdt + \int_0^T\int_{\mathbb{R}^N} |u||\Delta\varphi|dxdt. \end{aligned} \tag{6}$$

Using the ε-Young inequality in the right-side of (6) with taking $\varepsilon = \frac{p}{2}$, we obtain

$$\begin{aligned}\int_0^T\int_{\mathbb{R}^N} |u||\varphi_{tt}|dxdt \le &\frac{1}{2}\int_0^T\int_{\mathbb{R}^N} |u|^p (I^{\alpha}_{T^-}\varphi)dxdt \\ &+ \frac{p-1}{p}\left(\frac{p}{2}\right)^{-\frac{1}{p-1}}\int_0^T\int_{\mathbb{R}^N} (I^{\alpha}_{T^-}\varphi)^{-\frac{1}{p-1}}|\varphi_{tt}|^{\frac{p}{p-1}}dxdt \end{aligned} \tag{7}$$

and

$$\begin{aligned}\int_0^T\int_{\mathbb{R}^N} |u||\Delta\varphi|dxdt \le &\frac{1}{2}\int_0^T\int_{\mathbb{R}^N} |u|^p (I^{\alpha}_{T^-}\varphi)dxdt \\ &+ \frac{p-1}{p}\left(\frac{p}{2}\right)^{-\frac{1}{p-1}}\int_0^T\int_{\mathbb{R}^N} (I^{\alpha}_{T^-}\varphi)^{-\frac{1}{p-1}}|\Delta\varphi|^{\frac{p}{p-1}}dxdt. \end{aligned} \tag{8}$$

Inserting (7) and (8) into (6) gives

$$\begin{aligned}&\int_0^T\int_{\mathbb{R}^N} \omega\varphi dxdt + \int_{\mathbb{R}^N} (\varphi(0,x)u_1 - \varphi_t(0,x)u_0)dx \\ \le &C(p)\left(\int_0^T\int_{\mathbb{R}^N} (I^{\alpha}_{T^-}\varphi)^{-\frac{1}{p-1}}|\varphi_{tt}|^{\frac{p}{p-1}}dxdt + \int_0^T\int_{\mathbb{R}^N} (I^{\alpha}_{T^-}\varphi)^{-\frac{1}{p-1}}|\Delta\varphi|^{\frac{p}{p-1}}dxdt\right), \end{aligned} \tag{9}$$

where $C(p) = \frac{p-1}{p}\left(\frac{p}{2}\right)^{-\frac{1}{p-1}}$.

By the definition of the test function $\varphi(t,x)$ and simple calculation, we easily get

$$\begin{aligned}\int_0^T \int_{\mathbb{R}^N} \omega(x)\varphi dxdt =& \int_0^T T^{-\lambda}(T-t)^{\lambda}dt \int_{\mathbb{R}^N} \omega(x)\mu(x)dx \\ =& C(\lambda)T\int_{\mathbb{R}^N} \omega\mu dx \end{aligned} \tag{10}$$

and

$$\int_{\mathbb{R}^N} (\varphi(0,x)u_1 - \varphi_t(0,x)u_0)dx = \int_{\mathbb{R}^N} (\mu(x)u_1 + \lambda T^{-1}u_0)dx. \tag{11}$$

Plugging (10) and (11) into (9) gives

$$\begin{aligned}&C(\lambda)T\int_{\mathbb{R}^N} \omega\mu dx + \int_{\mathbb{R}^N} (\mu(x)u_1 + \lambda T^{-1}u_0)dx \\ \le& C(p)\Bigg(\underbrace{\int_0^T\int_{\mathbb{R}^N} (I^{\alpha}_{T^-}\varphi)^{-\frac{1}{p-1}}|\varphi_{tt}|^{\frac{p}{p-1}}dxdt}_{I_1} + \underbrace{\int_0^T\int_{\mathbb{R}^N} (I^{\alpha}_{T^-}\varphi)^{-\frac{1}{p-1}}|\Delta\varphi|^{\frac{p}{p-1}}dxdt}_{I_2}\Bigg).\end{aligned} \tag{12}$$

Next, we estimate the integrals I_1 and I_2.

Estimation of I_1. Using the definition of the test function $\varphi(t,x)$ and simple calculation, it derives that

$$\begin{aligned}I_1 =& \int_0^T\int_{\mathbb{R}^N} (I^{\alpha}_{T^-}\eta(t)\mu(x))^{-\frac{1}{p-1}}|\eta''(t)\mu(x)|^{\frac{p}{p-1}}dxdt \\ \le& C(\lambda)T^{-\lambda}\underbrace{\int_0^T \left(I^{\alpha}_{T^-}(T-t)^{\lambda}\right)^{-\frac{1}{p-1}}|T-t|^{\frac{\lambda p-2p}{p-1}}dt}_{I_{11}}\underbrace{\int_{\mathbb{R}^N}|\mu(x)|dx}_{I_{12}}.\end{aligned}$$

Noting that by Lemma 2.3, it derives that

$$\begin{aligned}I_{11} =& \int_0^T \left(\frac{\Gamma(\lambda+1)}{\Gamma(\lambda+1+\alpha)}(T-t)^{\lambda+\alpha}\right)^{-\frac{1}{p-1}}|T-t|^{\frac{\lambda p-2p}{p-1}}dt \\ \le& C(\lambda)\int_0^T |T-t|^{\lambda-\frac{2p+\alpha}{p-1}}dt \\ \le& C(\lambda)T^{\lambda-\frac{2p+\alpha}{p-1}+1}\end{aligned}$$

and by the cut-function $\phi(z)$, it derives that

$$\begin{aligned}I_{12} =& \int_{\mathbb{R}^N}|\phi\left(\frac{|x|^2}{R^2}\right)|dx \\ \le& \int_{0\le|x|\le R}|\phi\left(\frac{|x|^2}{R^2}\right)|dx + \int_{R\le|x|\le\sqrt{2}R}|\phi\left(\frac{|x|^2}{R^2}\right)|dx \\ \le& CR^N.\end{aligned}$$

Hence,

$$I_1 \le C(\lambda)T^{\frac{-2p-\alpha}{p-1}+1}R^N. \tag{13}$$

Estimation of I_2. Similar to the estimation of I_1, we have

$$I_2 = \int_0^T \int_{\mathbb{R}^N} (I_{T^-}^{\alpha}\eta(t)\mu(x))^{-\frac{1}{p-1}} |\eta(t)\Delta\mu(x)|^{\frac{p}{p-1}} dxdt$$
$$\leq T^{-\lambda} \underbrace{\int_0^T (I_{T^-}^{\alpha}(T-t)^{\lambda})^{-\frac{1}{p-1}} |(T-t)|^{\frac{\lambda p}{p-1}} dt}_{I_{21}} \underbrace{\int_{\mathbb{R}^N} |\mu(x)|^{-\frac{1}{p-1}} |\Delta\mu(x)|^{\frac{p}{p-1}} dx}_{I_{22}}.$$

Noting that by Lemma 2.3, it derives that

$$\begin{aligned} I_{21} &= \int_0^T \left(\frac{\Gamma(\lambda+1)}{\Gamma(\lambda+1+\alpha)} (T-t)^{\lambda+\alpha} \right)^{-\frac{1}{p-1}} |T-t|^{\frac{\lambda p}{p-1}} dt \\ &\leq C(\lambda) \int_0^T |T-t|^{\lambda - \frac{\alpha}{p-1}} dt \\ &\leq C(\lambda) T^{\lambda - \frac{\alpha}{p-1}+1}. \end{aligned}$$

Combine the definition of the function $\mu(x)$ with the condition **(i)**, we have

$$\begin{aligned} |\Delta\mu(x)| =& \left(\frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2}, \cdots, \frac{\partial}{\partial x_n} \right) \left(\frac{\partial\phi\left(\frac{r^2}{R^2}\right)}{\partial x_1}, \frac{\partial\phi\left(\frac{r^2}{R^2}\right)}{\partial x_2}, \cdots, \frac{\partial\phi\left(\frac{r^2}{R^2}\right)}{\partial x_n} \right) \\ =& \left(\frac{\partial\left(\frac{2x_1}{R^2}\phi'\right)}{\partial x_1}, \frac{\partial\left(\frac{2x_2}{R^2}\phi'\right)}{\partial x_2}, \cdots, \frac{\partial\left(\frac{2x_n}{R^2}\phi'\right)}{\partial x_n} \right) \\ \leq& \frac{2N}{R^2}|\mu'(x)| + \frac{4(x_1^2+x_2^2+\cdots+x_n^2)}{R^4}|\mu''(x)| \\ \leq& \frac{C}{R^2}|\mu(x)|, \end{aligned}$$

where $r = |x| = (x_1^2 + x_2^2 + \cdots + x_n^2)^{\frac{1}{2}}$. Then, we have

$$\begin{aligned} I_{22} \leq& CR^{-\frac{2p}{p-1}} \int_{\mathbb{R}^N} |\mu(x)| dx \\ \leq& CR^{N-\frac{2p}{p-1}}. \end{aligned}$$

Combining the above estimates, we obtain

$$I_2 \leq C(\lambda) T^{\frac{-\alpha}{p-1}+1} R^{N-\frac{2p}{p-1}}. \tag{14}$$

Inserting (13) and (14) into (12) gives

$$\int_{\mathbb{R}^N} \omega\mu dx + C(\lambda)T^{-1} \int_{\mathbb{R}^N} (\mu(x)u_1 + \lambda T^{-1}u_0) dx$$
$$\leq C(p,\lambda)(T^{\frac{-2p-\alpha}{p-1}} R^N + T^{\frac{-\alpha}{p-1}} R^{N-\frac{2p}{p-1}}).$$

Finally, fixing R and taking the limits $T \to \infty$ in the last inequality gives

$$\int_{\mathbb{R}^N} \omega(x)\mu dx \leq 0.$$

By lemma 2.5, which contradicts $\int_{\mathbb{R}^N} \omega\mu dx > 0$.

Step II. For $2 \le N \le 4$, $\alpha = 0$ and $p = \frac{N+1+\sqrt{N^2+10N-7}}{2(N-1)}$. For the proof of this step, we intend to use the proving ideals of step I. Since the test function of step I cannot achieve the study goals of this step, we choose another test function

$$\varphi(t,x) = \vartheta(t)\xi(x)$$

for large enough positive real number R and T, where

$$\vartheta(t) = \varrho\left(\frac{t}{T}\right), t > 0; \qquad \xi(x) = \sigma\left(\frac{\ln(\frac{|x|}{\sqrt{R}})}{\ln(\sqrt{R})}\right), x \in \mathbb{R}^N. \tag{15}$$

Let the functions $\varrho(\tau)$, $\xi(x)$ and $\sigma(s)$ possess the following properties:

(iii) $\varrho \in C^2(\mathbb{R})$ with $\varrho \ge 0$, $\varrho \not\equiv 0$, $supp(\varrho) \subset (0,1)$, $supp(\varrho') \subset (0,1)$;

(iv) $|\xi'(x)| \le C|\xi(x)|$ and $|\xi''(x)| \le C|\xi(x)|$;

(v) σ is a smooth function, which satisfies $0 \le \sigma(s) \le 1$ if $0 < s < 1$, and $\sigma(s) = 0$ if $s \le 0$ and $s \ge 1$.

Then, it follows from (5) that

$$\begin{aligned} &\int_0^T \int_{\mathbb{R}^N} \varphi |u|^p dxdt + \int_{\mathbb{R}^N} \omega(x)\varphi dx + \int_{\mathbb{R}^N} (\varphi(0,x)u_1 - \varphi_t(0,x)u_0)dx \\ \le &\int_0^T \int_{\mathbb{R}^N} |u||\varphi_{tt}|dxdt + \int_0^T \int_{\mathbb{R}^N} |u||\Delta\varphi|dxdt. \end{aligned} \tag{16}$$

Using the ε-Young inequality in the right-side of (16) with taking $\varepsilon = \frac{p}{2}$, we obtain

$$\begin{aligned} \int_0^T \int_{\mathbb{R}^N} |u||\varphi_{tt}|dxdt \le &\frac{1}{2}\int_0^T \int_{\mathbb{R}^N} |u|^p \varphi dxdt \\ &+ \frac{p-1}{p}\left(\frac{p}{2}\right)^{-\frac{1}{p-1}} \int_0^T \int_{\mathbb{R}^N} \varphi^{-\frac{1}{p-1}} |\varphi_{tt}|^{\frac{p}{p-1}} dxdt \end{aligned} \tag{17}$$

and

$$\begin{aligned} \int_0^T \int_{\mathbb{R}^N} |u||\Delta\varphi|dxdt \le &\frac{1}{2}\int_0^T \int_{\mathbb{R}^N} |u|^p \varphi dxdt \\ &+ \frac{p-1}{p}\left(\frac{p}{2}\right)^{-\frac{1}{p-1}} \int_0^T \int_{\mathbb{R}^N} \varphi^{-\frac{1}{p-1}} |\Delta\varphi|^{\frac{p}{p-1}} dxdt. \end{aligned} \tag{18}$$

Inserting (17) and (18) into (16) gives

$$\begin{aligned} &\int_0^T \int_{\mathbb{R}^N} \omega\varphi dxdt + \int_{\mathbb{R}^N} (\varphi(0,x)u_1 - \varphi_t(0,x)u_0)dx \\ \le &C(p)\left(\int_0^T \int_{\mathbb{R}^N} \varphi^{-\frac{1}{p-1}} |\varphi_{tt}|^{\frac{p}{p-1}} dxdt + \int_0^T \int_{\mathbb{R}^N} \varphi^{-\frac{1}{p-1}} |\Delta\varphi|^{\frac{p}{p-1}} dxdt\right). \end{aligned} \tag{19}$$

Using the conditions that $supp(\varrho) \subset (0,1)$, $supp(\varrho') \subset (0,1)$ and (15), we easily get

$$\begin{aligned} \int_{\mathbb{R}^N} (\varphi(0,x)u_1 - \varphi_t(0,x)u_0)dx &= \int_{\mathbb{R}^N} \left(\varrho(0)u_1 - \varrho'(0)\frac{1}{T}u_0\right)\xi(x)dx \\ &= 0 \end{aligned} \tag{20}$$

and

$$\int_0^T \int_{\mathbb{R}^N} \omega(x)\varphi dxdt = \int_0^T \varrho\left(\frac{t}{T}\right)dt \int_{\mathbb{R}^N} \omega(x)\xi(x)dx$$
$$= CT \int_{\mathbb{R}^N} \omega\xi dx. \tag{21}$$

Plugging (20) and (21) into (19) gives

$$CT \int_0^T \int_{\mathbb{R}^N} \omega\xi dxdt \leq C(p)\Big(\underbrace{\int_0^T \int_{\mathbb{R}^N} \varphi^{-\frac{1}{p-1}}|\varphi_{tt}|^{\frac{p}{p-1}}dxdt}_{J_1} + \underbrace{\int_0^T \int_{\mathbb{R}^N} \varphi^{-\frac{1}{p-1}}|\Delta\varphi|^{\frac{p}{p-1}}dxdt}_{J_2}\Big), \tag{22}$$

where $C(p) = \frac{p-1}{p}\left(\frac{p}{2}\right)^{-\frac{1}{p-1}}$.

Next, we estimate the integrals J_1 and J_2.

Estimation of J_1. By view of (15) and simple calculation , we have

$$J_1 = \int_0^T \int_{\mathbb{R}^N} (\vartheta(t)\xi(x))^{-\frac{1}{p-1}}|\vartheta''(t)\xi(x)|^{\frac{p}{p-1}}dtdx$$
$$\leq \underbrace{T^{-\frac{2p}{p-1}} \int_0^T \left(\varrho\left(\frac{t}{T}\right)\right)^{-\frac{1}{p-1}}\left|\varrho''\left(\frac{t}{T}\right)\right|^{\frac{p}{p-1}}dt}_{J_{11}} \underbrace{\int_{\mathbb{R}^N} |\xi(x)|dx}_{J_{12}}.$$

By taking $p = \frac{N+1+\sqrt{N^2+10N-7}}{2(N-1)}$ putting it into J_{11}, and by calculation, it can be obtained

$$J_{11} \leq CT^{-\frac{N+5+\sqrt{N^2+10N-7}}{4}+1}.$$

Applying the cut-off function $\sigma(s)$, we obtain

$$J_{12} = \int_{\mathbb{R}^N} \left|\sigma\left(\frac{\ln(\frac{|x|}{\sqrt{R}})}{\ln(\sqrt{R})}\right)\right|dx$$
$$\leq \int_{\sqrt{R}\leq|x|\leq R} \left|\sigma\left(\frac{\ln(\frac{|x|}{\sqrt{R}})}{\ln(\sqrt{R})}\right)\right|dx$$
$$\leq \frac{\pi^{\frac{N}{2}}}{\Gamma(1+\frac{N}{2})}(R^N - R^{\frac{N}{2}})$$
$$\leq CR^N.$$

Combining the discussions above, we get

$$J_1 \leq CT^{-\frac{N+5+\sqrt{N^2+10N-7}}{4}+1}R^N. \tag{23}$$

Estimation of J_2. By view of (15) and simple calculation , we have

$$J_2 = \int_0^T \int_{\mathbb{R}^N} (\vartheta(t)\xi(x))^{-\frac{1}{p-1}}|\vartheta(t)\Delta\xi(x)|^{\frac{p}{p-1}}dtdx$$
$$\leq \int_0^T \left|\varrho\left(\frac{t}{T}\right)\right|dt \int_{\mathbb{R}^N} |\xi(x)|^{-\frac{1}{p-1}}|\Delta\xi(x)|^{\frac{p}{p-1}}dx$$
$$\leq CT \underbrace{\int_{\mathbb{R}^N} |\xi(x)|^{-\frac{1}{p-1}}|\Delta\xi(x)|^{\frac{p}{p-1}}dx}_{J_{21}}.$$

Combining (15) with Condition **(iv)**, similar to the calculation of $|\Delta\mu|$ in step I, we have

$$|\Delta\xi(x)| \leq \frac{C}{r^2 \ln R}|\xi(x)|,$$

where $r = |x| = (x_1^2 + x_2^2 + \cdots + x_n^2)^{\frac{1}{2}}$.
Taking $p = \frac{N+1+\sqrt{N^2+10N-7}}{2(N-1)}$, we have

$$J_{21} \leq C(\ln R)^{-\frac{N+5+\sqrt{N^2+10N-7}}{8}} \int_{\mathbb{R}^N} \frac{|\xi(x)|}{|x|^{\frac{N+5+\sqrt{N^2+10N-7}}{4}}} dx.$$

Noting that $N - \frac{N+5+\sqrt{N^2+10N-7}}{4} = \frac{3N-5-\sqrt{N^2+10N-7}}{4} \leq 0$ for $2 \leq N \leq 4$. And since R is big enough, taking $R \geq 1$ in the cut-off function σ is reasonable, too. Hence

$$\begin{aligned}
\int_{\mathbb{R}^N} \frac{|\xi(x)|}{|x|^{\frac{N+5+\sqrt{N^2+10N-7}}{4}}} dx \leq & \int_{\sqrt{R}\leq|x|\leq R} \frac{|\sigma\left(\frac{\ln(\frac{|x|}{\sqrt{R}})}{\ln(\sqrt{R})}\right)|}{|x|^{\frac{N+5+\sqrt{N^2+10N-7}}{4}}} dx \\
\leq & \int_{\sqrt{R}\leq|x|\leq R} \frac{|\sigma\left(\frac{\ln(\frac{|x|}{\sqrt{R}})}{\ln(\sqrt{R})}\right)|}{|x|^N} dx \\
\leq & C\left((\ln R)^N - (\ln\sqrt{R})^N\right) \\
\leq & C(\ln R)^N.
\end{aligned}$$

Further, we have

$$J_{21} \leq C(\ln R)^{\frac{3-N-\sqrt{N^2+10N-7}}{8}}.$$

Combining the above calculations, we obtain

$$J_2 \leq CT(\ln R)^{\frac{3-N-\sqrt{N^2+10N-7}}{8}}. \tag{24}$$

Inserting (23) and (24) into (22) gives

$$\int_{\mathbb{R}^N} \omega\xi dx \leq C\left(T^{-\frac{N+5+\sqrt{N^2+10N-7}}{4}} R^N + (\ln R)^{\frac{3-N-\sqrt{N^2+10N-7}}{8}}\right). \tag{25}$$

Taking $T = R^j$, $j > 0$ in (25), we have

$$\int_{\mathbb{R}^N} \omega\xi dx \leq C\left(CR^{-\frac{5+\sqrt{N^2+10N-7}}{4}j+(1-\frac{1}{4}j)N} + C(\ln R)^{\frac{3-N-\sqrt{N^2+10N-7}}{8}}\right). \tag{26}$$

Since $2 \leq N \leq 4$, $\frac{3-N-\sqrt{N^2+10N-7}}{8} < 0$. Taking $j > 4$ in (26), we have $1 - \frac{1}{4}j < 0$. Passing to the limit as $R \to \infty$ in (26) gives

$$\int_{\mathbb{R}^N} \omega\xi dx \leq 0.$$

By lemma 2.5, which contradicts $\int_{\mathbb{R}^N} \omega\xi dx > 0$. $\square$

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Competing Interests

The authors declare that they have no competing interests.

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Authors's Contributions

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References

[1] J. B. Hartle and S. W. Hawking, *Wave function of the universe*, Phys. Rev. D **28** (1983), no. 12, 2960–2975, DOI 10.1103/PhysRevD.28.2960. MR0723174.

[2] M. V. Klibanov, *Global convexity in a three-dimensional inverse acoustic problem*, SIAM J. Math. Anal. **28** (1997), no. 6, 1371–1388, DOI 10.1137/S0036141095292582. MR1456081.

[3] W. Chen and A. Palmieri, *Blow-up result for a semilinear wave equation with a nonlinear memory term*, in *Mathematics without Boundaries*, Springer International Publishing, 2021, 77–97. DOI 10.1007/978-3-030-XXXXX-X_5.

[4] W. A. Strauss, *Nonlinear Wave Equations*, CBMS Regional Conf. Ser. in Math., vol. 73, Amer. Math. Soc., Providence, RI, 1989.

[5] H. Jiao and Z. Zhou, *An elementary proof of the blow-up for semilinear wave equations in high space dimensions*, J. Differ. Equ. **189** (2003), no. 2, 355–365, DOI 10.1016/S0022-0396(03)00191-2.

[6] B. T. Yordanov and Q. S. Zhang, *Finite time blow up for critical wave equations in high dimensions*, J. Funct. Anal. **231** (2006), no. 2, 361–374, DOI 10.1016/j.jfa.2005.06.013.

[7] Y. Zhou, *Blow up of solutions to semilinear wave equations with critical exponent in high dimensions*, Chin. Ann. Math. Ser. B **28** (2007), no. 2, 205–212, DOI 10.1007/s11401-007-0224-8.

[8] Y. Zhou and W. Han, *Life-span of solutions to critical semilinear wave equations*, Commun. Partial Differ. Equ. **39** (2014), no. 3, 439–451, DOI 10.1080/03605302.2013.841124.

[9] Q. S. Zhang, *A new critical behavior for nonlinear wave equations*, J. Comput. Anal. Appl. **2** (2000), no. 4, 277–292, MR1816012.

[10] M. Jleli, B. Samet, and C. Vetro, *A blow-up result for a nonlinear wave equation on manifolds: the critical case*, Appl. Anal. **102** (2023), no. 5, 1463–1472, DOI 10.1080/00036811.2021.2006237.

[11] A. G. Kartsatos and V. V. Kurta, *On blow-up results for solutions of inhomogeneous evolution equations and inequalities*, J. Math. Anal. Appl. **290** (2004), 76–85, DOI 10.1016/j.jmaa.2003.09.022.

[12] M. Jleli, B. Samet, and C. Vetro, *Large time behavior for inhomogeneous damped wave equations with nonlinear memory*, Symmetry (Basel) **12** (2020), no. 10, Article 1609, DOI 10.3390/sym12101609.

[13] A. Alqahtani, M. Jleli, and B. Samet, *Finite-time blow-up for inhomogeneous parabolic equations with nonlinear memory*, Complex Var. Elliptic Equ. **66** (2021), no. 1, 84–93, DOI 10.1080/17476933.2020.1745013.

[14] L. Kirianova, *Wave equation with fractional derivative and stationary inhomogeneities*, Math. Model. Syst. Anal. **28** (2023), 1–10, DOI 10.1051/mmsa/2023001.

[15] M. Jleli and B. Samet, *New critical behaviors for semilinear wave equations and systems with linear damping terms*, Math. Methods Appl. Sci. **43** (2020), 1–11, DOI 10.1002/mma.6400.

[16] M. Borikhanov and B. T. Torebek, *Nonexistence of global solutions for an inhomogeneous pseudo-parabolic equation*, Appl. Math. Lett. **134** (2022), 108366, DOI 10.1016/j.aml.2022.108366.

[17] A. A. Kilbas, H. M. Srivastava, and J. J. Trujillo, *Theory and Applications of Fractional Differential Equations*, North-Holland Mathematics Studies, Elsevier, Amsterdam, 2006.

[18] S. I. Pokhozhaev, *Nonexistence of global solutions of nonlinear evolution equations*, Differ. Equ. **49** (2013), 599–606, DOI 10.1134/S0012266113050102.