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Some Fixed Point Theorems of Leggett–Williams Type for Multivalued Mappings and Applications to Second-Order Differential Inclusions

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Abstract

This paper examines the application of fixed point index theory to set-valued mappings in ordered Banach spaces, with a specific focus on Leggett-Williams type fixed point theorems. The study provides novel conditions for the existence of one and multiple fixed points under a range of assumptions. Additionally, it applies these results to prove the existence of positive solutions for the second-order differential inclusion problem $-x''(t) \in f(t, x(t)), t \in [0,1]$ with boundary conditions $x(0) = 0 = x'(1)$. The obtained results contribute to expanding the scope of application of fixed point theory, providing new techniques for solving boundary value problems in different cases.

Keywords: fixed point index, fixed point theorem of Leggett–Williams type, multivalued mapping, second-order differential inclusions
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1 Introduction

The fixed point index is an effective tool for studying the existence of solutions and the structure of solution sets. It has been widely used in nonlinear equations and boundary value problems (see [5, 8]). With the increasingly widespread use of multivalued mappings in mathematics, natural sciences and economics,..., the fixed point index theory for multivalued mappings has been developed and applied to differential inclusions and eigenvalue problems for multivalued mappings [13].

Over the past few decades, fixed point theory has become one of the central tools in nonlinear analysis and applied mathematics. Fixed point theorems play an important role not only in the study of differential and integral equations, but also in many other areas such as boundary value problems, differential inclusions and optimization.

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Among these, Leggett–Williams type fixed point theorems have proved effective in asserting the existence of one or more positive solutions for nonlinear problems (see [10, 11, 12]). However, most previous studies primarily focused on single-valued mappings (see [3, 4]), whereas many mathematical models are naturally described by multivalued mappings and differential inclusions.This motivates extending existing results to more general multivalued settings.

In this paper, we investigate Leggett–Williams type fixed point theorems for multivalued mappings together with their applications, and obtain the following new results:

- We employ the fixed point index for multivalued mappings to establish Leggett–Williams type theorems in ordered spaces.The approach uses two characteristic functionals, one convex and one concave, to express hypotheses without relying on any specific norm. This is a new point compared to [6], which mainly focuses on norm-type Leggett-Williams fixed point theorems for multivalued maps with compact, convex value.
- We construct theorems to prove the existence of one, two and three fixed points for multivalued mappings, combining inward and outward conditions.These results extend Leggett–Williams theorems from single-valued to multivalued maps.
- We apply the theory to prove the existence of positive solutions for a second-order differential inclusion with appropriate boundary conditions, extending the studies of [3, 4] to the multivalued case.

The obtained results not only contribute to expanding the scope of application of fixed point theory but also provide additional tools for studying boundary value problems in the general case, promising significant prospects for future research.

2 Preliminaries

2.1 Fixed point index for multivalued mappings in ordered spaces

Let X be a real Banach space. A set $K \subset X$ is called a *cone* if:

(i) K is a closed subset of X;
(ii) $K + \lambda K \subset K$ for all $\lambda \geq 0$;
(iii) $K \cap (-K) = \{0\}$.

If K is a cone, the order in X generated by K is defined by $x \leq y \iff y - x \in K$. In this case, the pair (X, K) is called an *ordered Banach space*.

Definition 2.1. [7] Let (X, K) be an ordered Banach space, and let A, B be nonempty subsets of X. For $k = 1, 2, 3$, we define relations "$\overset{(k)}{\leq}$" between A and B as follows:

$$A \overset{(1)}{\leq} B \iff (\forall x \in A,\ \exists y \in B :\ x \leq y) \quad \text{(i.e., } A \subset B - K\text{)},$$
$$A \overset{(2)}{\leq} B \iff (\forall y \in B,\ \exists x \in A :\ x \leq y) \quad \text{(i.e., } B \subset A + K\text{)},$$
$$A \overset{(3)}{\leq} B \iff (\forall x \in A,\ \forall y \in B :\ x \leq y) \quad \text{(i.e., } B \subset x + K,\ \forall x \in A\text{)}.$$

These notions coincide with the order induced by K when A and B are singletons.

Definition 2.2. [8] Let X, Y be real Banach spaces, and let $F : D \subset X \to 2^Y \setminus \{\varnothing\}$ be a multivalued mapping.

(i) F is *upper semi-continuous* (u.s.c for short) on D if for every open set $V \subset Y$, the set $F^+(V) = \{x \in D : F(x) \subset V\}$ is open in D.
(ii) F is *compact* if $F(S) = \bigcup\limits_{x \in S} F(x)$ is relatively compact in Y for every bounded set $S \subset D$.

(iii) For $E \subset Y$, we denote $cc(E)$ as the set of nonempty, closed and convex subsets of E.

When $X = Y$, a point $x \in X$ with $x \in F(x)$ is called a *fixed point* of multivalued mapping F; the set of all fixed points of F is denoted by $Fix(F)$.

Proposition 2.3. *[8] Let X, Y be real Banach spaces and $F : D \subset X \to 2^Y \setminus \{\varnothing\}$.*

(i) F is u.s.c. on D iff for every $x_0 \in D$ and every open set V containing $F(x_0)$, there exists $r > 0$ such that $F\big(B(x_0, r) \cap D\big) \subset V$.
(ii) Suppose F is u.s.c. on D, $(x_n) \subset D$, $x_n \to x_0 \in D$, and $y_n \in F(x_n)$ with $y_n \to y_0 \in Y$. If $F(x_0)$ is closed, then $y_0 \in F(x_0)$.
(iii) If F is compact and its graph $G_F = \{(x, y) \in X \times Y : \ x \in D,\ y \in F(x)\}$ is a closed subset of $X \times Y$, then F is u.s.c.

Definition 2.4. [2, 14] Let $\Omega \subset X$ be a bounded open set in the ordered Banach space (X, K) and $A : K \cap \partial\Omega \to cc(K)$ be an u.s.c., compact multivalued mapping such that $x \notin A(x)$ for all $x \in K \cap \partial\Omega$ (i.e. A is non-degenerate on $K \cap \partial\Omega$). Then there exists a single-valued, compact mapping $f : K \cap \partial\Omega \to K$ homotopic to A on $K \cap \partial\Omega$, i.e., there exists an u.s.c., compact multivalued mapping $G : (K \cap \partial\Omega) \times [0, 1] \to cc(K)$ such that

$$x \notin G(x, t)\ , \forall (x, t) \in (K \cap \partial\Omega) \times [0, 1]\ ;\ G(., 1) = A\ ,\ G(., 0) = f.$$

The *fixed point index* $i_K(A, \Omega)$ to cone K of A on Ω is defined by

$$i_K(A, \Omega) = i_K(f, \Omega),$$

where $i_K(f, \Omega)$ is the fixed point index on K for the single-valued mapping f on Ω.

Proposition 2.5. *[2] Let (X, K) be an ordered Banach space and $\Omega \subset X$ a bounded open set. For properties i–iii, we assume mapping A is defined on $K \cap \partial\Omega$, in iv–v, we assume A is defined on $K \cap \overline{\Omega}$, takes values in $cc(K)$ and is an u.s.c., compact.*

i. Normality: If $A(x) \equiv C$, $\forall x \in K \cap \partial\Omega$, where $C \subset K$ is a convex, compact set, then

$$i_K(A, \Omega) = \begin{cases} 1 & \text{when } C \subset K \cap \Omega, \\ 0 & \text{when } C \subset K \setminus \overline{\Omega}. \end{cases}$$

ii. Homotopy Invariance: If $H : (K \cap \partial\Omega) \times [0, 1] \to cc(K)$ is an u.s.c., compact and non-degenerate, then

$$i_K(H(\cdot, 1), \Omega) = i_K(H(\cdot, 0), \Omega).$$

iii. Poincaré Property: Suppose $A_1 : K \cap \partial\Omega \to cc(K)$ is an u.s.c., compact non-degenerate map on $K \cap \partial\Omega$ and satisfies

$$\frac{x - y}{\|x - y\|} \neq -\frac{x - z}{\|x - z\|},\ \ \forall y \in A(x),\ \forall z \in A_1(x),\ \forall x \in K \cap \partial\Omega.$$

Then, $i_K(A, \Omega) = i_K(A_1, \Omega)$.
iv. Additivity: Suppose $\Omega_1, \Omega_2 \subset \Omega$ are disjoint open sets. If $x \notin A(x)$, for all $x \in K \cap \big(\overline{\Omega} \backslash (\Omega_1 \cup \Omega_2)\big)$, then

$$i_K(A, \Omega) = i_K(A, \Omega_1) + i_K(A, \Omega_2).$$

v. If $i_K(A, \Omega) \neq 0$ then A has a fixed point in Ω.

Theorem 2.6. *Let (X, K) be an ordered Banach space, $\Omega \subset X$ a bounded open set and $A : K \cap \partial\Omega \to cc(K)$ be an u.s.c., compact. Suppose there exists a convex, compact set $C \subset K$ such that*

$$t(x-u) \notin A(x) - x\,,\ \forall t \geq 0\,,\ \forall u \in C\,,\ \forall x \in K \cap \partial\Omega$$

Then,

$$i_K(A, \Omega) = \begin{cases} 1 & \text{if } C \subset K \cap \Omega, \\ 0 & \text{if } C \subset K \setminus \overline{\Omega}. \end{cases}$$

In particular, if there exists an element $u \in K$ such that

$$t(x-u) \notin A(x) - x\,,\ \forall t \geq 0\,,\ \forall x \in K \cap \partial\Omega$$

then

$$i_K(A, \Omega) = \begin{cases} 1 & \text{if } u \in \Omega, \\ 0 & \text{if } u \notin \overline{\Omega}. \end{cases}$$

Proof. Consider the constant multivalued map $A_1(x) \equiv C\,, \forall x \in K \cap \partial\Omega$. The condition stated in Theorem 2.6 can be rewritten as:

$$\frac{x-y}{\|x-y\|} \neq -\frac{x-z}{\|x-z\|},\ \forall y \in A(x),\ \forall z \in A_1(x),\ \forall x \in K \cap \partial\Omega$$

Applying Proposition 2.5, we have

$$i_K(A, \Omega) = \begin{cases} 1 \text{ if } C \subset K \cap \Omega\,, \\ 0 \text{ if } C \subset K \setminus \overline{\Omega}\,. \end{cases}$$

□

3 Main Results

Definition 3.1. Let (X, K) be an ordered Banach space, $u, v : K \to \mathbb{R}$ continuous functions. For $a, b \in \mathbb{R}$, we define
$K(u, a) = \{x \in K : u(x) < a\}, K(u, v, a, b) = \{x \in K : u(x) < a \text{ and } v(x) > b\}$.

Theorem 3.2. *Let (X, K) be an ordered Banach space, $\alpha : K \to \mathbb{R}$ a continuous convex function, $\beta : K \to \mathbb{R}$ a continuous concave function, $A : K \to \mathrm{cc}(K)$ an u.s.c., compact. If there exist real numbers λ, μ such that*

(A1) The set $K(\alpha, \beta, \lambda, \mu)$ is non-empty;
(A2) If $x \in K$ with $\alpha(x) = \lambda$ and $\beta(x) \geq \mu$ then $\alpha(y) < \lambda, \forall y \in A(x)$;
(A3) If $x \in K$ with $\alpha(x) = \lambda$ then $\forall y \in A(x) :\ (\beta(y) < \mu \Rightarrow \alpha(y) < \lambda)$;

and if $\overline{K(\alpha, \lambda)}$ is bounded, then $i_K(A, K(\alpha, \lambda)) = 1$.

Proof. Firstly, we will prove that $x \notin A(x), \forall x \in \partial K(\alpha, \lambda)$.

Assume to the contrary, that there exists $x_0 \in \partial K(\alpha, \lambda)$ such that $x_0 \in A(x_0)$. Since $x_0 \in \partial K(\alpha, \lambda)$, it follows that $\alpha(x_0) = \lambda$.

- If $\beta(x_0) \geq \mu$ then from (A2) it follows that $\alpha(x_0) < \lambda$ (contradiction).
- If $\beta(x_0) < \mu$ then from (A3) it follows that $\alpha(x_0) < \lambda$ (contradiction).

Thus, $x \notin A(x), \forall x \in \partial K(\alpha, \lambda)$.

From the hypothesis, we deduce that $K(\alpha, \lambda)$ is an bounded open set in K. Taking $u \in K(\alpha, \beta, \lambda, \mu)$ then $u \in K(\alpha, \lambda)$.

We will prove that $t(x-u) \notin A(x) - x,\ \forall t \geq 0,\ \forall x \in \partial K(\alpha, \lambda)$. Assume to the contrary, i.e.

$$\exists x_0 \in \partial K(\alpha, \lambda),\ \exists t_0 \geq 0: \ t_0(x_0 - u) \in A(x_0) - x_0.$$

Choosing $y_0 \in A(x_0)$ such that $t_0(x_0 - u) = y_0 - x_0$, we have

$$x_0 = \frac{1}{1+t_0} y_0 + \frac{t_0}{1+t_0} u \quad \text{and} \quad \alpha(x_0) = \lambda.$$

- If $\beta(y_0) \geq \mu$.
  Applying the concavity property of the function $\beta(x)$, we have
  $$\beta(x_0) \geq \frac{1}{1+t_0} \beta(y_0) + \frac{t_0}{1+t_0} \beta(u) > \mu.$$
  Therefore, according to hypothesis (A2), we have $\alpha(y_0) < \lambda$.
  Applying the convexity property of the function $\alpha(x)$, we obtain
  $$\lambda = \alpha(x_0) \leq \frac{1}{1+t_0} \alpha(y_0) + \frac{t_0}{1+t_0} \alpha(u) < \lambda,$$
  which is a contradiction.
- If $\beta(y_0) < \mu$ then by (A3) we have $\alpha(y_0) < \lambda$, thus
  $$\lambda = \alpha(x_0) \leq \frac{1}{1+t_0} \alpha(y_0) + \frac{t_0}{1+t_0} \alpha(u) < \lambda,$$
  we encounter a contradiction. Thus, we have proved that
  $$t(x-u) \notin A(x) - x,\ \forall t \geq 0,\ \forall x \in \partial K(\alpha, \lambda).$$
  Applying Theorem 2.6, we deduce that $i_K(A, K(\alpha, \lambda)) = 1$.

□

Theorem 3.3. *Let (X, K) be an ordered Banach space, $\alpha : K \to \mathbb{R}$ a continuous convex function, $\beta : K \to \mathbb{R}$ a continuous concave function, $A : K \to \mathrm{cc}(K)$ an u.s.c., compact. If there exist real numbers λ, μ such that*

(A4) The set $K(\alpha, \beta, \lambda, \mu)$ is non-empty;
(A5) If $x \in K$ with $\beta(x) = \mu$ and $\alpha(x) \leq \lambda$ then $\beta(y) > \mu, \forall y \in A(x)$;
(A6) If $x \in K$ with $\beta(x) = \mu$ then $\forall y \in A(x) : (\alpha(y) > \lambda \Rightarrow \beta(y) > \mu)$;;

and if $\overline{K(\beta, \mu)}$ is bounded, then $i_K(A, K(\beta, \mu)) = 0$.

Proof. Firstly, we will prove that $x \notin A(x), \forall x \in \partial K(\beta, \mu)$. Assume to the contrary, that there exists $x_0 \in \partial K(\beta, \mu)$ such that $x_0 \in A(x_0)$. Since $x_0 \in \partial K(\beta, \mu)$, it follows that $\beta(x_0) = \mu$.

- If $\alpha(x_0) \leq \lambda$, then from (A5), it follows that $\beta(x_0) > \mu$ (contradiction).
- If $\alpha(x_0) > \lambda$, then from (A6), it follows that $\beta(x_0) > \mu$ (contradiction).

Therefore, $x \notin A(x), \forall x \in \partial K(\beta, \mu)$.

From the hypothesis, we deduce that $K(\beta, \mu)$ is an bounded open set in K. Taking $u \in K(\alpha, \beta, \lambda, \mu)$ then $u \notin \overline{K(\beta, \mu)}$. We will prove that $t(x-u) \notin A(x) - x,\ \forall t \geq 0,\ \forall x \in \partial K(\beta, \mu)$. Assume the contrary, i.e.

$$\exists x_0 \in \partial K(\beta, \mu),\ \exists t_0 \geq 0: \ t_0(x_0 - u) \in A(x_0) - x_0.$$

Choosing $y_0 \in A(x_0)$ such that $t_0(x_0 - u) = y_0 - x_0$, we have

$$x_0 = \frac{1}{1+t_0} y_0 + \frac{t_0}{1+t_0} u \quad \text{and} \quad \beta(x_0) = \mu.$$

- If $\alpha(y_0) \leq \lambda$, applying the convexity property of $\alpha(x)$, we have

$$\alpha(x_0) \leq \frac{1}{1+t_0}\,\alpha(y_0) + \frac{t_0}{1+t_0}\,\alpha(u) < \lambda.$$

Therefore, by hypothesis (A5), we have $\beta(y_0) > \mu$.
Applying the concavity property of the function $\beta(x)$, we obtain

$$\mu = \beta(x_0) \geq \frac{1}{1+t_0}\,\beta(y_0) + \frac{t_0}{1+t_0}\,\beta(u) > \mu,$$

which is a contradiction.

- If $\alpha(y_0) > \lambda$, then by (A6) we have $\beta(y_0) > \mu$, thus we obtain the following contradiction

$$\mu = \beta(x_0) \geq \frac{1}{1+t_0}\,\beta(y_0) + \frac{t_0}{1+t_0}\,\beta(u) > \mu.$$

Thus, we have proved that $t(x-u) \notin A(x) - x,\ \forall t \geq 0,\ \forall x \in \partial K(\beta,\mu)$.
Applying Theorem 2.6, we deduce that $i_K(A, K(\beta,\mu)) = 0$.

□

Definition 3.4. Let (X,K) be an ordered Banach space, $\alpha : K \to \mathbb{R}$ a continuous convex function, $\beta : K \to \mathbb{R}$ a continuous concave function, λ, μ are constants, and $A : K \to \mathrm{cc}(K)$ an u.s.c., compact. We say that

i. A is *LW-inward* with respect to $I(\alpha,\beta,\lambda,\mu)$ if the hypotheses in Theorem 3.2 are satisfied, and

ii. A is *LW-outward* with respect to $O(\alpha,\beta,\lambda,\mu)$ if the hypotheses in Theorem 3.3 are satisfied.

Theorem 3.5. *Let (X,K) be an ordered Banach space, $\alpha,\psi : K \to \mathbb{R}$ continuous convex functions, $\beta,\delta : K \to \mathbb{R}$ continuous concave functions and $A : K \to \mathrm{cc}(K)$ an u.s.c., compact. If there exist real numbers a,b,c,d such that*

(D1) A is LW-inward with respect to $I(\alpha,\beta,a,b)$;
(D2) A is LW-outward with respect to $O(\psi,\delta,c,d)$;

and if

(H1) $\overline{K(\alpha,a)} \subset K(\delta,d)$ then A has a fixed point $x^* \in K(\delta,\alpha,d,a)$;
(H2) $\overline{K(\delta,d)} \subset K(\alpha,a)$ then A has a fixed point $x^* \in K(\alpha,\delta,a,d)$.

Proof. We prove the Theorem in the case where (H1) is satisfied; the case where (H2) is satisfied is proved similarly.

To prove the existence of a fixed point for mapping A in $K(\delta,\alpha,d,a)$, we will demonstrate that

$$i_K(A, K(\delta,\alpha,d,a)) \neq 0.$$

Since A is LW-inward with respect to $I(\alpha,\beta,a,b)$, by Theorem 3.2, we have

$$i_K(A, K(\alpha,a)) = 1,$$

and A is LW-outward with respect to $O(\psi,\delta,c,d)$, by Theorem 3.3, we have

$$i_K(A, K(\delta,d)) = 0.$$

A has no fixed points on $\overline{K(\delta,d)} \setminus \left(K(\alpha,a) \cup K(\delta,\alpha,d,a)\right)$ and the sets $K(\alpha,a)$ and $K(\delta,\alpha,d,a)$ are non-empty, disjoint open subsets of $\overline{K(\delta,d)}$. Therefore, by the additivity property of the fixed point index, we obtain

$$i_K(A,K(\delta,d)) = i_K(A,K(\alpha,a)) + i_K(A,K(\delta,\alpha,d,a)).$$

It follows that,

$$i_K(A,K(\delta,\alpha,d,a)) = -1.$$

Thus, there exists $x^* \in K(\delta,\alpha,d,a)$ such that $x^* \in A(x^*)$. □

The theorems below affirm the existence of multiple Leggett–Williams type fixed points for multivalued mappings in ordered Banach spaces.

Theorem 3.6. *Let (X,K) be an ordered Banach space, $\alpha,\psi,\phi : K \to \mathbb{R}$ continuous convex functions, $\beta,\delta,\theta : K \to \mathbb{R}$ continuous concave functions and $A : K \to \mathrm{cc}(K)$ an u.s.c., compact. If there exist real numbers a_1,a_2,a_3,b_1,b_2,b_3 such that*

(T1) A is LW-outward with respect to $O(\alpha,\beta,a_1,b_1)$;
(T2) A is LW-inward with respect to $I(\psi,\delta,a_2,b_2)$;
(T3) A is LW-outward with respect to $O(\phi,\theta,a_3,b_3)$;

if $\overline{K(\beta,b_1)} \subset K(\psi,a_2)$ and $\overline{K(\psi,a_2)} \subset K(\theta,b_3)$ then A has at least two fixed points x^* and x^{**} such that $x^* \in K(\psi,\beta,a_2,b_1)$; $x^{**} \in K(\theta,\psi,b_3,a_2)$.

Proof. Similar to the proof of Theorem 3.5, we have

$$i_K(A,K(\beta,b_1)) = 0, \qquad i_K(A,K(\psi,a_2)) = 1, \qquad i_K(A,K(\theta,b_3)) = 0.$$

Applying the additivity property of the fixed point index, we obtain

$$i_K(A,K(\psi,\beta,a_2,b_1)) = 1, \qquad i_K(A,K(\theta,\psi,b_3,a_2)) = -1.$$

Therefore, there exist at least two fixed points x^* and x^{**} such that $x^* \in K(\psi,\beta,a_2,b_1)$; $x^{**} \in K(\theta,\psi,b_3,a_2)$. □

Theorem 3.7. *Let (X,K) be an ordered Banach space, $\alpha,\psi,\phi : K \to \mathbb{R}$ continuous convex functions, $\beta,\delta,\theta : K \to \mathbb{R}$ continuous concave functions and $A : K \to \mathrm{cc}(K)$ an u.s.c., compact. If there exist real numbers a_1,a_2,a_3,b_1,b_2,b_3 such that*

(T1) A is LW-inward with respect to $I(\alpha,\beta,a_1,b_1)$;
(T2) A is LW-outward with respect to $O(\psi,\delta,a_2,b_2)$;
(T3) A is LW-inward with respect to $I(\phi,\theta,a_3,b_3)$;

if $\overline{K(\alpha,a_1)} \subset K(\delta,b_2)$ and $\overline{K(\delta,b_2)} \subset K(\phi,a_3)$ then A has at least three fixed points x^*, x^{**} and x^{***} such that $x^* \in K(\alpha,a_1)$; $x^{**} \in K(\delta,\alpha,b_2,a_1)$; $x^{***} \in K(\phi,\delta,a_3,b_2)$.

Proof. Similar to the proof of Theorem 3.5, we have

$$i_K(A,K(\alpha,a_1)) = 1, \qquad i_K(A,K(\delta,b_2)) = 0, \qquad i_K(A,K(\phi,a_3)) = 1.$$

Applying the additivity property of the fixed point index, we obtain

$$i_K(A,K(\delta,\alpha,b_2,a_1)) = -1, \qquad i_K(A,K(\phi,\delta,a_3,b_2)) = 1.$$

Therefore, there exist at least three fixed points x^*, x^{**} and x^{***} such that $x^* \in K(\alpha,a_1)$; $x^{**} \in K(\delta,\alpha,b_2,a_1)$; $x^{***} \in K(\phi,\delta,a_3,b_2)$. □

4 Applications to second-order differential inclusions

Consider the problem

$$\begin{cases} -x''(t) \in f(t, x(t)), & 0 \le t \le 1, \\ x(0) = 0 = x'(1), \end{cases} \tag{4.1}$$

with the following assumptions

$f : [0,1] \times \mathbb{R}^+ \to cc(\mathbb{R}^+)$ is a compact multivalued map and an upper Carathéodory map in the sense that:

+ a1) $\forall x \in \mathbb{R}^+$, the function $t \mapsto f(t,x)$ is measurable, i.e. $\forall y \in \mathbb{R}$, the function $D(t) = \inf\{|y - z|, z \in f(t,x)\}$ is measurable.

+ a2) The function $x \mapsto f(t,x)$ is an u.s.c. a.e. in $[0,1]$.

+ a3) $\forall r > 0$, $\exists \varphi_r \in L^1([0,1])$ such that

$$\sup_{0 \le x \le r} f(t,x) \le \varphi_r(t) \quad \text{a.e. in } [0,1].$$

Problem (4.1) can be rewritten as follows: find a function $x \in C([0,1])$ such that there exists $y \in L^1([0,1])$ with $y(t) \in f(t, x(t))$ a.e. and

$$x(t) = \int_0^1 G(t,s) y(s)\, ds, \tag{4.2}$$

where

$$G(t,s) = \begin{cases} s, & 0 \le s \le t \le 1, \\ t, & 0 \le t \le s \le 1. \end{cases}$$

In $C([0,1])$, we consider the norm $\|x\| = \max_{0 \le t \le 1} |x(t)|$ and define the cone $K \subset C([0,1])$ by

$$K = \{x \in C([0,1]) : x \text{ is concave, nonnegative, nondecreasing and } x(0) = 0\}.$$

For $x \in K$ and $\mu \in (0,1)$, define the concave function β_μ on K by

$$\beta_\mu(x) = \min_{t \in [\mu,1]} x(t) = x(\mu),$$

and the convex function α on K by

$$\alpha(x) = \max_{t \in [0,1]} x(t) = x(1).$$

Thus, if $x \in K$ and $\mu \in (0,1)$ then by the concavity of x, we have $x(\mu) \ge \mu x(1)$, since

$$\frac{x(\mu) - x(0)}{\mu - 0} \ge \frac{x(1) - x(0)}{1 - 0}.$$

Therefore, for all $x \in \mathbb{R}$, we have

$$\mu \alpha(x) \le \beta_\mu(x). \tag{4.3}$$

For each $x \in K$, we denote

$$F_x = \{y \in L^1([0,1]) : y(t) \in f(t, x(t)) \text{ a.e. in } [0,1]\},$$

by assumptions a1)–a3) and the results in [2, 9], then $F_x \neq \varnothing$.

We define the mapping $T : K \to 2^K \setminus \{\varnothing\}$ by

$$T(x) = \left\{ h \in K : \exists y \in F_x,\ h(t) = \int_0^1 G(t,s) y(s)\, ds,\ \forall t \in [0,1] \right\}.$$

Lemma 4.1. *[4, 5]*

i) $0 \leq G(t,s) \leq 1,\ \forall (t,s) \in [0,1] \times [0,1]$,

ii) *For each fixed* $s \in [0,1]$, *the function* $G(t,s)$ *is non-decreasing with respect to* t,

iii) $\min\limits_{s\in[0,1]} \frac{G(y,s)}{G(w,s)} \geq \frac{y}{w}$,

iv) $|G(t_1,s) - G(t_2,s)| \leq |t_1 - t_2|,\ \forall t_1, t_2 \in [0,1],\ \forall s \in [0,1]$.

Lemma 4.2. *[1] Suppose* $I \subset \mathbb{R}$ *is a bounded interval,* $f : I \times \mathbb{R} \to 2^{\mathbb{R}} \setminus \{\varnothing\}$ *is an upper Carathéodory mapping and has compact convex values such that* $F_x = \{y \in L^1(I) : y(t) \in f(t, x(t)) \text{ a.e. in } I\} \neq \emptyset,\ \forall x \in C(I)$. *Thus, if* $L : L^1(I) \to C(I)$ *is a continuous linear map then the multivalued map* $L \circ F_x,\ x \mapsto L(F_x)$ *has a closed graph in* $C(I) \times C(I)$.

Lemma 4.3. *T is an u.s.c., compact multivalued map with convex and closed values.*

Proof. 1) $T(K) \subset K$.
For each $x \in K,\ h \in T(x)$ there exists $y \in F_x$ such that

$$h(t) = \int_0^1 G(t,s)y(s)ds = \int_0^t sy(s)ds + \int_t^1 ty(s)ds\,,\ \forall t \in [0,1]$$

Since $G(t,s) \geq 0,\ y(s) \geq 0$ it implies $h(t) \geq 0$ and

$$h'(t) = ty(t) + \int_0^1 y(s)\,ds - ty(t) = \int_0^1 y(s)\,ds \geq 0,$$

$$h''(t) = -y(t) \leq 0.$$

Therefore, h is a non-negative, non-decreasing, concave function on $[0,1]$ and $h(0) = 0$. Thus, $h \in K$ and therefore $T(K) = \bigcup\limits_{x\in K} T(x) \subset K$.

2) T has a convex value.
Suppose $x \in K, h_1, h_2 \in T(x), \lambda \in (0,1)$, there exist $y_1, y_2 \in F_x$ such that

$$h_k(t) = \int_0^1 G(t,s)y_k(s)\,ds, \quad \forall t \in [0,1],\ k = 1,2.$$

Then

$$\lambda h_1(t) + (1-\lambda)h_2(t) = \int_0^1 G(t,s)\,[\lambda y_1(s) + (1-\lambda)y_2(s)]\,ds, \quad \forall t \in [0,1].$$

Since F_x is convex, $\lambda y_1 + (1-\lambda)y_2 \in F_x$. This implies $\lambda h_1 + (1-\lambda)h_2 \in T(x)$.

3) T has a closed value.
Suppose $x \in K$ and $\{u_n\} \subset T(x),\ u_n \to u$ and $\{y_n\} \subset F_x$ such that

$$u_n(t) = \int_0^1 G(t,s)y_n(s)\,ds, \quad \forall t \in [0,1]. \tag{4.4}$$

For $r = \|x\| + 1$ and by assumption a3), there exists $\varphi_r \in L^1([0,1])$ such that

$$\sup_{0\leq x\leq r} f(t,x) \leq \varphi_r(t) \quad \text{a.e. in } [0,1].$$

Because of this, $\{y_n\} \subset F_x$ so $y_n(t) \in f(t,x(t))$ and

$$0 \leq y_n(t) \leq \varphi_r(t), \quad \forall n \in \mathbb{N}.$$

Applying the corollary of the Dunford–Pettis theorem, we deduce the existence of a subsequence $\{y_{n_k}\} \subset \{y_n\}$ and $y \in L^1([0,1])$ such that $y_{n_k} \rightharpoonup y$. Since F_x is convex and closed, it is weakly closed, therefore $y \in F_x$.
Taking the limit in (4.4), we obtain

$$u(t) = \int_0^1 G(t,s)y(s)\,ds, \quad \forall t \in [0,1].$$

Thus, $u \in T(x)$.
4) T is a compact operator.
We will prove that for each $r > 0$, the set $T(\overline{K_r})$ is relatively compact.
According to hypothesis a3), there exists $\varphi_r \in L^1([0,1])$ such that

$$\sup_{0 \le x \le r} f(t,x) \le \varphi_r(t) \quad \text{a.e. in } [0,1].$$

- $T(\overline{K_r})$ is uniformly bounded.
  For each $x \in \overline{K_r}$ and $h \in T(x)$, there exists $y \in F_x$ such that

$$h(t) = \int_0^1 G(t,s)y(s)\,ds \le \int_0^1 \varphi_r(s)\,ds, \quad \forall t \in [0,1].$$

- $T(\overline{K_r})$ is equicontinuous.
  Assume $t_1, t_2 \in [0,1]$ and $t_1 < t_2$. We have

$$h(t_1) - h(t_2) = \int_0^1 \big(G(t_1,s) - G(t_2,s)\big)y(s)\,ds, \quad \forall t \in [0,1].$$

  Therefore,

$$|h(t_1) - h(t_2)| \le \int_0^1 |t_1 - t_2|\varphi_r(s)\,ds.$$

  It implies $\lim_{t_1 \to t_2} |h(t_1) - h(t_2)| = 0$ (Lemma 4.1).

By the Ascoli–Arzelà Theorem, we deduce that T is a compact operator.
5) T has a closed graph.
Assume $x_n, x \in K$ and $x_n \to x$, $h_n \in T(x_n)$, $\forall n \in \mathbb{N}^*$ and $h_n \to h$. We shall prove $h \in T(x)$.
Since $L : L^1([0,1]) \to C([0,1])$, $w \mapsto \int_0^1 G(t,s)w(s)\,ds$ is a continuous linear operator, which implies that the multivalued operator $L \circ F_x : C([0,1]) \to cc\,(C([0,1]))$ has a closed graph in $C([0,1]) \times C([0,1])$ (Lemma 4.2).
Since $h_n \in L \circ F_{x_n}$ and $h_n \to h$, then $h \in L \circ F_x$, or $h \in T(x)$.
Because T is a compact operator and has a closed graph, T is an upper semi-continuous operator. □

Theorem 4.4. *Assume there exist real numbers $\tau, \mu \in (0,1)$; $b, d > 0$ with $b < d\mu$ and a multivalued operator f satisfying the following conditions:*

(1) $f(s,x) \overset{(1)}{\le} 3bs$, $\forall s \in [0,\tau]$, $\forall x \in [bs, b\tau]$,

(2) $f(s,x) \overset{(1)}{\le} 2b\tau$, $\forall s \in [\tau,1]$, $\forall x \in [b\tau, b]$,

(3) $f(s,x) \overset{(2)}{\ge} Mx$, $\forall s \in [0,1]$, $\forall x \in [d\mu, d]$, $M > \frac{1}{\mu(1-\mu)}$.

Then, problem (4.1) has at least one positive solution $x^* \in K(\beta_\mu, \alpha, c, b)$.

Proof. Let $a = b\tau,\ c = d\mu$. Then, we have $a < b < c < d$. We will prove that T is LW-inward with respect to $I(\alpha, \beta_\tau, b, a)$ and LW-outward with respect to $O(\alpha, \beta_\mu, d, c)$.
Step 1. T is LW-inward with respect to $I(\alpha, \beta_\tau, b, a)$.
1a. Proving that the set $K(\alpha, \beta_\tau, b, a) = \{x \in K : \alpha(x) < b \text{ and } \beta_\tau(x) > a\} \neq \varnothing$.
For each $L \in \left(\frac{2b}{2-\tau}, 2b\right)$, the function x_L defined by

$$x_L(t) = \int_0^1 LG(t,s)\,ds = \frac{Lt(2-t)}{2}, \quad \forall t \in [0,1].$$

Since $\alpha(x_L) = \max\limits_{t\in[0,1]} x_L(t) = x_L(1) = \frac{L}{2} < b$ and

$$\beta_\tau(x_L) = \min_{t\in[\tau,1]} x_L(t) = x_L(\tau) = \frac{L\tau(2-\tau)}{2} > b\tau = a,$$

then $x_L \in K(\alpha, \beta_\tau, b, a)$ or $K(\alpha, \beta_\tau, b, a) \neq \varnothing$.
If $x \in K(\alpha, b)$ then $\|x\| = \max\limits_{t\in[0,1]} x(t) = \alpha(x) \leq b$. This implies that $\overline{K(\alpha, b)}$ is bounded.

1b. Taking $x \in K$ with $\alpha(x) = b$ and $\beta_\tau(x) \geq a$. By the concavity of x and for each $s \in [0, \tau]$, we have

$$x(s) \geq \frac{x(\tau)}{\tau} s \geq bs,$$

and for all $s \in [\tau, 1]$, we have

$$b\tau \leq x(s) \leq b.$$

For each $h \in T(x)$ there exists $y \in F_x$ such that

$$h(t) = \int_0^1 G(t,s)y(s)\,ds, \quad \forall t \in [0,1].$$

Therefore

$$\alpha(h) = \max_{t\in[0,1]} h(t) = h(1) = \int_0^1 G(1,s)y(s)\,ds = \int_0^\tau sy(s)\,ds + \int_\tau^1 sy(s)\,ds.$$

Since $y(t) \in f(t, x(t))$, then $y(t) \leq 3bt,\ \forall t \in [0, \tau]$ (by (1)) and $y(t) \leq 2b\tau,\ \forall t \in [\tau, 1]$ (by (2)). Consequently,

$$\alpha(h) = \int_0^\tau sy(s)\,ds + \int_\tau^1 sy(s)\,ds \leq \int_0^\tau s(3bs)\,ds + \int_\tau^1 s(2b\tau)\,ds \leq b\tau^3 + b\tau(1-\tau^2) = b\tau < b.$$

1c. Taking $x \in K$ with $\alpha(x) = b$. For each $h \in T(x)$ there exists $y \in F_x$ such that

$$h(t) = \int_0^1 G(t,s)y(s)\,ds, \quad \forall t \in [0,1].$$

Assuming that $\beta_\tau(h) < a$, then $\alpha(h) < b$. Indeed,

$$\alpha(h) = \int_0^1 G(1,s)y(s)\,ds \leq \frac{1}{\tau}\int_0^1 G(\tau,s)y(s)\,ds = \frac{1}{\tau}h(\tau) = \frac{1}{\tau}\beta_\tau(h) < \frac{a}{\tau} = b. \left(\frac{G(\tau,s)}{G(1,s)} \geq \tau\right)$$

Step 2. T is LW-outward with respect to $O(\alpha, \beta_\mu, d, c)$.
2a. Proving that the set $K(\alpha, \beta_\mu, d, c) = \{x \in K : \alpha(x) < d \text{ and } \beta_\mu(x) > c\} \neq \varnothing$.

For each $J \in \left(\frac{2d}{2-\mu}, 2d\right)$, the function x_J defined by

$$x_J(t) = \int_0^1 JG(t,s)\,ds = \frac{Jt(2-t)}{2}, \quad \forall t \in [0,1].$$

Since $\alpha(x_J) = \max\limits_{t\in[0,1]} x_J(t) = x_J(1) = \frac{J}{2} < d$ and

$$\beta_\mu(x_J) = \min_{t\in[\mu,1]} x_J(t) = x_J(\mu) = \frac{J\mu(2-\mu)}{2} > d\mu = c,$$

then $x_J \in K(\alpha, \beta_\mu, d, c)$ or $K(\alpha, \beta_\mu, d, c) \neq \varnothing$. Moreover, $K(\beta_\mu, c)$ is a bounded set because if $x \in K(\beta_\mu, c)$ then by (4.3), we have

$$\mu\alpha(x) \leq \beta_\mu(x) \leq c,$$

and therefore

$$\|x\| = \max_{t\in[0,1]} x(t) = \alpha(x) \leq \frac{\beta_\mu(x)}{\mu} \leq \frac{c}{\mu} = d.$$

2b. Taking $x \in K$ with $\beta_\mu(x) = c$ and $\alpha(x) < d$. For each $h \in T(x)$ there exists $y \in F_x$ such that

$$h(t) = \int_0^1 G(t,s)y(s)\,ds, \quad \forall t \in [0,1].$$

If $s \in [\mu, 1]$ then $d\mu = c \leq x(s) \leq d$.
Consequently,

$$\beta_\mu(h) = h(\mu) = \int_0^1 G(\mu,s)y(s)ds \geq \int_\mu^1 G(\mu,s)y(s)ds = \int_\mu^1 \mu y(s)ds.$$

Since $y(t) \in f(t, x(t))$ then $y(t) \geq Mx(t)$, $\forall t \in [0,1]$ (by (3)).
Hence,

$$\beta_\mu(h) \geq \int_\mu^1 \mu y(s)ds \geq \int_\mu^1 \mu Mx(s)ds \geq M\mu \int_\mu^1 cds = Mc\mu(1-\mu) > c.$$

2c. Taking $x \in K$ with $\beta_\mu(x) = c$, for each $h \in T(x)$ there exists $y \in F_x$ such that

$$h(t) = \int_0^1 G(t,s)y(s)ds, \quad \forall t \in [0,1].$$

Assuming that $\alpha(h) > d$, we have

$$\beta_\mu(h) = h(\mu) = \int_0^1 G(\mu,s)y(s)ds = \int_\mu^1 \mu y(s)ds,$$

$$\mu\alpha(h) = \mu h(1) = \mu \int_0^1 G(1,s)y(s)ds,$$

$$\beta_\mu(h) \geq \mu\alpha(h) > \mu d = c.$$

Thus, T is LW-outward with respect to $O(\alpha, \beta_\mu, d, c)$.

For each $N \in (2b, 2c)$, the function x_N defined by

$$x_N(t) = \int_0^1 NG(t,s)ds = \frac{Nt(2-t)}{2}, \quad \forall t \in [0,1].$$

Since $\alpha(x_N) = \max_{t\in[0,1]} x_N(t) = x_N(1) = \frac{N}{2} > b$ and

$$\beta_\mu(x_N) = \min_{t\in[\mu,1]} x_N(t) = x_N(\mu) = \frac{N\mu(2-\mu)}{2} < \frac{2c\mu(2-\mu)}{2} < c,$$

then $x_N \in K(\beta_\mu, \alpha, c, b)$ or $K(\beta_\mu, \alpha, c, b) \neq \emptyset$. If $x \in \overline{K(\alpha, b)}$ then $\beta_\mu(x) \leq \alpha(x) < b < c$. Consequently,

$$\overline{K(\alpha, b)} \subset K(\beta_\mu, c)$$

Applying Theorem 3.5, we deduce that the operator T has at least one fixed point $x^* \in K\left(\beta_\mu, \alpha, c, b\right)$, which is a positive solution to problem (4.1). □

Example 1. *We choose the constants $\tau = \frac{1}{3}$, $\mu = \frac{1}{2}$, $b = 1$, $d = 12$. Then, $b < d\mu$. Choosing $M = 5 > \frac{1}{\mu(1-\mu)} = 4$ and multivalued function $f(t,x)$ defined by*

$$f(t,x) = \begin{cases} [tx, 3t], & 0 \leq t \leq \frac{1}{3},\ 0 \leq x \leq 1, \\ \left[\frac{1}{3}x, \frac{2}{3}\right], & \frac{1}{3} < t \leq 1,\ 0 \leq x \leq 1, \\ \left[\frac{(6-x)}{5}tx + x^2 - x,\ \frac{(6-x)}{5}3t + x^2 - x + \frac{x-1}{5}\right], & 0 \leq t \leq \frac{1}{3},\ 1 < x < 6, \\ \left[\frac{(6-x)}{15}x + x^2 - x,\ \frac{2(6-x)}{15} + x^2 - x + \frac{x-1}{5}\right], & \frac{1}{3} < t \leq 1,\ 1 < x < 6, \\ [5x, 5x+1], & 0 \leq t \leq 1,\ 6 \leq x. \end{cases}$$

Then, $f(t,x)$ satisfies the conditions $a1) - a3)$ and $(1) - (3)$ of Theorem 4.4. Thus, according to Theorem 4.4, problem (4.1) has at least one positive solution x^* with $x^*(0.5) < 6$ and $x^*(1) > 1$.

5 Conclusion

In this study, we obtained results concerning the existence of one and multiple fixed points for Leggett-Williams type theorems for certain classes of multivalued operators acting in ordered spaces. These results are applied to prove the existence of positive solutions for a class of second-order differential inclusions with appropriate boundary conditions.

In the fixed point theory of multivalued mappings and its applications, many promising research directions yielding interesting results still exist. Next, we will investigate the existence of expansion-compression fixed points of Leggett-Williams type for the sum of two mappings and their applications to certain classes of boundary value problems.

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The authors declare that they have no competing interests.

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