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Blow-up Solutions to Logarithmic Viscoelastic Equations with Nonlocal Weak Damping, Delay, and Acoustic Boundary Conditions

Qianqian Luo[1], **Jisong Duan**[1,*]

1. School of Mathematical Sciences, Guizhou Normal University, Guiyang 550025, China

[*]Corresponding author

Abstract

We in this paper consider the blow-up behavior of any nontrivial solution to the following initial boundary value problem of the logarithmic viscoelastic equations with delay and nonlocal weak damping terms under acoustic boundary conditions

$$\begin{aligned} u_{tt} - \Delta u + \|u_t\|_2^k u_t + \int_0^t \omega(t-s)\Delta u(s)ds + c_1|u_t(t)|^{m-2}u_t(t) \\ + c_2|u_t(t-\tau)|^{m-2}u_t(t-\tau) = |u|^{p-2}u\ln|u|. \end{aligned}$$

Our results show that the local solution blows up in finite time if $p > m$ and $E(0) < 0$. More accurately, we give the upper bound of the blow-up time by the energy method. Moreover, we also give the lower bound of the blow-up time.

Keywords: Finite time blow-up, Logarithmic viscoelastic equations, Nonlocal weak damping term, Time delay, Acoustic boundary conditions
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1 Introduction

The purpose of this work is to consider the blow-up effects for the following initial boundary value problem:

$$\begin{aligned} & u_{tt} - \Delta u + \|u_t\|_2^k u_t + \int_0^t \omega(t-s)\Delta u(s)ds \\ & \quad + c_1|u_t(t)|^{m-2}u_t(t) + c_2|u_t(t-\tau)|^{m-2}u_t(t-\tau) = |u|^{p-2}u\ln|u| \;\; in\; \Omega \times (0,T), \end{aligned} \tag{1}$$

Contact: Qianqian Luo 13116497060@163.com; Jisong Duan 1814935334@qq.com

$$u = 0 \quad on \quad \Gamma_0 \times (0, T), \tag{2}$$

$$\frac{\partial u}{\partial v} - \int_0^t \omega(t-s)\frac{\partial u(s)}{\partial v} ds = h(x) y_t \quad on \quad \Gamma_1 \times (0, T), \tag{3}$$

$$u_t + f(x) y_t + g(x) y = 0 \quad on \quad \Gamma_1 \times (0, T), \tag{4}$$

$$u(0) = u_0, u_t(0) = u_1 \quad in \; \Omega, \quad y(0) = y_0 \quad on \; \Gamma_1, \tag{5}$$

$$u_t(x, t-\tau) = J_0(x, t-\tau) \; for \; (x, t) \in \; \Omega \; \times (0, \tau), \tag{6}$$

where Ω is a bounded domain with smooth boundary $\partial\Omega$ in $\mathbb{R}^n (n \geq 1), \partial\Omega = \Gamma_0 \cup \Gamma_1, |\Gamma_0| > 0, |\Gamma_1| > 0$ and $\Gamma_0 \cap \Gamma_1 = \phi$. $\tau > 0$ represents the time delay. The coefficients c_1 and c_2 satisfy $c_1 > 0$ and $c_2 \in \mathbb{R}$. The parameters m, p and k satisfy

$$\begin{cases} m \geq 2; \\ 2 < p < \infty, \; if \; n = 1, 2; \\ 2 < p < \frac{2n-2}{n-2}, \; if \; n \geq 3; \\ 0 < k < p - 2. \end{cases}$$

J_0, u_0 and u_1 are the initial data, $\|u_t\|_2^k u_t$ is defined as a nonlocal weak damping term, $w(t) : \mathbb{R}^+ \to \mathbb{R}^+$ is a non-increasing and differentiable function. $h, f, g : \Gamma_1 \to \mathbb{R}^+$ are all essentially bounded functions, where g satisfies

$$g(x) \geq g_0 > 0 \;\; for \; any \; x \in \Gamma_1.$$

The acoustic boundary conditions are (3) and (4). There are a lot of researchers who study wave equations with such boundary conditions (see [1, 2, 3, 4] and references therein).

It is well known that the viscoelastic equations with different damping terms and conditions frequently appeared in many branches of sciences, such as physics and mathematics. In mathematics, the viscoelastic equations have been widely researched. For example, in [5, 6, 7, 8, 9, 10, 11], the authors investigated the global existence; in [5, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17], the authors studied the blow-up effects. Among them, in [6, 7, 11, 13, 14, 17], the authors researched the global existence and blow-up on the viscoelastic equations with delay; in [10, 11, 12, 13], the authors studied the finite time blow-up for the viscoelastic equations with logarithmic nonlinearity. Under the dimension $1 \leq n \leq 3$, Park [13] investigated the following model

$$u_{tt} - \Delta u + \int_0^t \omega(t-s)\Delta u(s) ds + c_1 |u_t(t)|^{m-2} u_t(t) + c_2 |u_t(t-\tau)|^{m-2} u_t(t-\tau) = |u|^{p-2} u \ln |u|,$$

with the bounded domain $\Omega \subset \mathbb{R}^n$, but without the nonlocal damping term $\|u_t\|_2^q u_t$. The author established the upper and lower bounds of the blow-up solutions.

Recently, the nonlocal weak damping effects have been frequently studied in different contexts, such as physics and other applied science. The nonlocal weak damping effects have many forms, such as $M(\|\nabla u\|_2^2) u_t$, $(\|\Delta u\|_2^2 + \|u_t\|_2^2)^q \Delta u_t$ and $\|u_t\|_2^q u_t$. The wave equations with $M(\|\nabla u\|_2^2) u_t$ or $(\|\Delta u\|_2^2 + \|u_t\|_2^2)^q \Delta u_t$ have been studied by many researchers, we refer to [18, 19, 20, 21, 22] and so on. For the wave equations with this kind of nonlocal weak damping $\|u_t\|_2^q u_t$, Zhao et al. [23, 24] studied the asymptotic behavior of following wave equation with nonlocal weak damping, anti-damping and critical nonlinearity

$$u_{tt} - \Delta u + k\|u_t\|_2^q u_t + f(u) = \int_\Omega K(x, y) u_t(y) dy + h(x)$$

under some suitable conditions, which $\|u_t\|_2^q u_t$ and $\int_\Omega K(x, y) u_t(y) dy$ are called nonlocal damping and anti-damping, respectively. For the above wave equations, Zhao et al. in [25] also give the existence of a generalized polynomial attractor. In addition, Zhao et al. [26] investigated the following equation with the nonlocal damping

$$u_{tt} - \Delta u - m(\|\nabla u\|_2^2)\Delta u + k\|u_t\|_2^q u_t + f(u) = h(x).$$

They established the well-posedness of the above wave equation. More recently, Liu et al. [27] introduced the following equation

$$u_{tt} - \Delta u + \|u_t\|_2^q u_t = |u|^{p-2},$$

with Dirichlet boundary conditions. When $0 < q < p-2$, they obtained the local existence and energy decay by using Fatou-Galerkin method and Nakao's inequality. The authors also gave the finite time blow-up effects under three different initial energy levels.

Motivated by the above problems and research results, we investigate the blow-up results for problem (1)-(6) with delay and nonlocal weak damping. Comparing with the system in [13], the main difficulty of this article is to overcome the barriers generated by the nonlocal weak damping term $\|u_t\|_2^k u_t$. We encountered some difficulties in applying the concavity method to estimate the bounds of blow-up time, where these include: 1. logarithmic nonlinearity and acoustic boundary conditions exist at the same time, which will generate some delicate estimates; 2. How to use some inequalities to control the nonlocal weak damping term.

The remainder of this article is as follows. In Section 2, we give some lemmas, hypotheses and other important conclusions, which play a key role in the study of the blow-up effects. In Section 3, we establish the blow-up solutions and their proofs.

2 Preliminaries

In this section, we give some notations, assumptions, inequalities, constraints and lemmas, which are used in proving the main theorems.

The norms of $L^r(\Omega)$ and $L^r(\Gamma_1)$ are denoted by $\|\cdot\|_r$ and $\|\cdot\|_{\Gamma_1,r}$, respectively. The norm of the space X is denoted by $\|\cdot\|_X$. The notation $(\cdot,\cdot)$ represents the scalar product in the space $L^2(\Omega)$. We denote by $u = u(x,t)$ and $u_t = u_t(x,t)$. Some unimportant positive constants are all denoted by C, which may be different in each condition. We let Λ is a Banach space and satisfying

$$\Lambda = \{\psi \in H_0^1(\Omega) \ : \ \psi = 0 \ on \ \Gamma_0\}.$$

For any $2 \le r \begin{cases} < \infty, & if \ n = 1,2 \\ \le \dfrac{2n}{n-2}, & if \ n \ge 3 \end{cases}$ in $\mathbb{R}^n$, the embedding

$$\Lambda \hookrightarrow L^r(\Omega)$$

holds. Then the embedding equality $\|\psi\|_r \le M_r\|\nabla\psi\|_2$ also holds, where M_r is the embedding constant. The following inequality

$$d^{w_1} - 1 \le d \le (1 + \frac{1}{w_2})(d + w_2) - 1 \tag{7}$$

is also we needs, where $d \ge 0, 1 \ge w_1 > 0, w_2 > 0$.

Similar to the assumption of [13, 28, 29], we introduce the variable

$$\delta(x,\rho,t) = u_t(x, t-\rho\tau) \ \ for \ (x,\rho,t) \in \Omega \times (0,1) \times (0,T)$$

with

$$\tau\delta_t(x,\rho,t) + \delta_\rho(x,\rho,t) = 0 \ for \ (x,\rho,t) \ \epsilon \ \Omega \times (0,T).$$

Then the problem (1)-(6) can be rewritten as

$$\begin{aligned} & u_{tt} - \Delta u + \|u_t\|_2^k u_t + \int_0^t \omega(t-s)\Delta u(s)ds \\ & \quad + c_1|u_t|^{m-2}u_t + c_2|\delta(x,1,t)|^{m-2}\delta(x,1,t) = |u|^{p-2}u\ln|u| \;\; in\; \Omega \times (0,T), \end{aligned} \tag{8}$$

$$\tau\delta_t(x,\rho,t) + \delta_\rho(x,\rho,t) = 0 \; for \; (x,\rho,t) \,\epsilon\, \Omega \times (0,T), \tag{9}$$

$$u = 0 \;\; on\; \Gamma_0 \times (0,T), \tag{10}$$

$$\frac{\partial u}{\partial v} - \int_0^t \omega(t-s)\frac{\partial u(s)}{\partial v}ds = h(x)y_t \;\; on\; \Gamma_1 \times (0,T), \tag{11}$$

$$u_t + f(x)y_t + g(x)y = 0 \;\; on\; \Gamma_1 \times (0,T), \tag{12}$$

$$u(0) = u_0, u_t(0) = u_1 \;\; in\; \Omega,\; y(0) = y_0 \;\; on\; \Gamma_1, \tag{13}$$

$$\delta(x,\rho,0) = J_0(x,-\rho\tau) = \delta_0(x,\rho) \; for \; (x,\rho) \,\epsilon\, \Omega \times (0,1). \tag{14}$$

Our main purpose is to find a blow-up solutions on problem (8)-(14). For this goal, we will employ the following assumptions: **(A_1)** The function $\omega(t)$ verifies

$$\int_0^t \omega(s)ds < \frac{p-2}{p+\frac{1}{p}-2} \tag{15}$$

and

$$0 < 1 - \int_0^\infty \omega(s)ds := \omega_l.$$

(A_2) Let $1 \le n \le 3$.
(A_3) $2 < p < \infty,\; if\; n = 1,2;\; 2 < p < 4,\; if\; n = 3.$
(A_4) $m > max\left(2, \frac{p^2+2p}{p^2-2p+4}, \frac{(p^2+4p)(2p+(p-2)(n-1))-p^2(n(p-2)+4)}{(p^2+4)(2p+(p-2)(n-1))-p^2(n(p-2)+4)}\right)$.
Since $p > 2$ and $1 \le n \le 3$, we obtain

$$p > \frac{p^2+2p}{p^2-2p+4}$$

and

$$p > \frac{(p^2+4p)(2p+(p-2)(n-1))-p^2(n(p-2)+4)}{(p^2+4)(2p+(p-2)(n-1))-p^2(n(p-2)+4)}.$$

We can choose m such that

$$p > m > max\left(2, \frac{p^2+2p}{p^2-2p+4}, \frac{(p^2+4p)(2p+(p-2)(n-1))-p^2(n(p-2)+4)}{(p^2+4)(2p+(p-2)(n-1))-p^2(n(p-2)+4)}\right), \tag{16}$$

where the reason for the choice of coefficient m is to limit the value range of coefficients θ and q generated in the subsequent proof process.
(A_5) $0 < |c_2| < c_1$.

Remark 2.1. (1) Since $p > 2$ and $1 \le n \le 3$, we have

$$min\left(\frac{2n+2}{n}, \frac{4p+2(n-1)(p-2)}{n(p-2)+4}\right) = \frac{4p+2(n-1)(p-2)}{n(p-2)+4}. \tag{17}$$

(2) From assumptions **(A_2)**, **(A_3)** and **(A_4)**, it follows that

$$\frac{p-2}{2p} > \frac{2(p-m)}{p^2(m-1)} \tag{18}$$

and

$$p > \frac{4p+2(n-1)(p-2)}{n(p-2)+4} > \frac{2p^2(m-1)}{p^2(m-1)-4(p-m)} > 2. \tag{19}$$

(3) From **(A$_3$)**, there must exists the constants $\xi_0, \xi_1 > 0$ such that

$$2 < p+\xi_0 \begin{cases} < \infty, & if\ n = 1,2 \\ \leq 6, & if\ n = 3 \end{cases} \ and\ 2 < 2(p-1+\xi_1) \begin{cases} < \infty, & if\ n = 1,2 \\ \leq 6, & if\ n = 3 \end{cases},$$

therefore, we have

$$\Lambda \hookrightarrow L^{p+\xi_0}(\Omega) \quad and \quad \Lambda \hookrightarrow L^{2(p+\xi_1-1)}(\Omega). \tag{20}$$

We need the following lemmas for the blow-up theorem.

Lemma 2.2. *([13]) For any given $\xi > 0$,*

$$|a^\xi \ln a| \leq \frac{1}{e\xi}\ for\ 0 < a < 1\ \ and\ \ 0 \leq a^{-\xi} \ln a \leq \frac{1}{e\xi}\ \ for\ \ a \geq 1.$$

Lemma 2.3. *([28]) Let $\psi \in \Lambda$ and $2 \leq r \begin{cases} < \infty & if\ n = 1,2 \\ \leq 6 & if\ n = 3 \end{cases}$. Then, for any $2 \leq s \leq r$,*

$$\|\psi\|_r^s \leq C(\|\psi\|_r^r + \|\nabla\psi\|_2^2)$$

holds.

Lemma 2.4. *([28]) Let $\psi \in \Lambda$ with $\int_\Omega |\psi|^r \ln |\psi| dx > 0$ and $2 \leq r \begin{cases} < \infty & if\ n = 1,2 \\ \leq 5 & if\ n = 3 \end{cases}$. Then, for any $2 \leq s \leq r$,*

$$\left(\int_\Omega |\psi|^r \ln |\psi| dx\right)^{\frac{s}{r}} \leq C\left(\|\nabla\psi\|_2^2 + \int_\Omega |\psi|^r \ln |\psi| dx\right)$$

holds.

Lemma 2.5. *([28]) Let $r \geq 2$. For $\psi \in L^r(\Omega) \cap \Lambda$ with $\int_\Omega |\psi|^r \ln |\psi| dx > 0$,*

$$\|\psi\|_r^r \leq C\left(\|\nabla\psi\|_2^2 + \int_\Omega |\psi|^r \ln |\psi| dx\right)$$

holds.

Lemma 2.6. *Let $f(t)$ be a continuous function from $(0, +\infty)$ to $(0, +\infty)$. If there exists constants $s > 1$ and $C > 0$ such that*

$$f'(t) \geq C f^{\frac{1}{1-s}}(t),$$

then $f(t)$ blows up at finite time T satisfying

$$0 < T \leq \frac{1}{C(1-s) f(0)^{\frac{s}{1-s}}}.$$

Proof. By integration of this formula $f'(t) \geq C f^{\frac{1}{1-s}}(t)$, we have

$$f^{\frac{1}{1-s}}(t) \geq \frac{1}{f^{\frac{-s}{1-s}}(0) - C\frac{s}{(1-s)} T},$$

hence, $f(t)$ blows up at finite time

$$T \leq \frac{1}{C(1-s)f(0)^{\frac{s}{1-s}}}.$$

Then the proof is completed.

Through the previous works of the research [12, 27, 29, 30], we have the local existence and uniqueness theorem for problem (8)-(14), which can be proved by the Fatou-Galerkin method, for convenience, we omit the proofs here.

Theorem 2.7. *Let* **(A_1)**, **(A_3)** *and* **(A_5)** *hold. Then, for any* $(u_0, u_1, y_0, \delta_0) \in \Lambda \times L^2(\Omega) \times L^2(\Gamma_1) \times L^m(\Omega \times (0,1))$, *the problem (8)-(14) exists a unique local solution* (u, y, δ) *satisfying*

$$u \in \ C(0,T;\Lambda) \ \cap C^1(0,T;L^2(\Omega)),$$

$$\sqrt{h}y \in \ L^\infty(0,T;L^2(\Gamma_1)),$$

and

$$\delta \in L^\infty(0,T;L^m(\Omega \times (0,1))).$$

We now give the energy functional of problem (8)-(14).

By multiplying both sides of (8) by u_t and integrating over Ω, it follows that

$$\begin{aligned} E(t) =& \frac{1}{2}\|u_t\|_2^2 + \frac{1}{2}\left(1-\int_0^t \omega(s)ds\right)\|\nabla u\|_2^2 + \frac{1}{2}(\omega \odot \nabla u) + \frac{1}{p^2}\|u\|_p^p \\ &- \frac{1}{p}\int_\Omega |u|^p \ln|u| dx + \frac{1}{2}\int_{\Gamma_1} hgy^2 d\Gamma + \varepsilon\tau\int_0^1 \|\delta(x,\rho,t)\|_m^m d\rho. \end{aligned} \tag{21}$$

By the definition of $E(t)$ and (8)-(14), we have the following Lemma.

Lemma 2.8. *Assume* **(A_1)**, **(A_3)** *and* **(A_5)** *hold. Let* (u, y, δ) *is any solution for the problem (8)-(14). Then*

$$\frac{d}{dt}E(t) \leq \frac{1}{2}(\omega' \circledcirc \nabla u) - \int_{\Gamma_1} hfy_t^2 d\Gamma - \frac{\omega(t)}{2}\|\nabla u\|_2^2 - \|u_t\|_2^{k+2} - \gamma(\|u_t\|_m^m + \|\delta(x,1,t)\|_m^m) \leq 0, \tag{22}$$

where

$$(\omega \circledcirc \nabla u) = \int_0^t \omega(t-s)\|\nabla u(t) - \nabla u(s)\|_2^2 ds.$$

Proof. By direct calculation, we derive

$$\begin{aligned} \frac{d}{dt}E(t) =& \frac{1}{2}(\omega' \circledcirc \nabla u) - \int_{\Gamma_1} hfy_t^2 d\Gamma - \frac{1}{2}\omega(t)\|\nabla u\|_2^2 - \|u_t\|_2^{k+2} - c_1\|u_t\|_m^m \\ &- c_2\int_\Omega \delta(x,1,t)|\delta(x,1,t)|^{m-2}u_t dx + \frac{\partial}{\partial t}\varepsilon\tau\int_0^1 \|\delta(x,\rho,t)\|_m^m d\rho. \end{aligned} \tag{23}$$

By Young's inequality with ε, it follows that

$$-c_2\int_\Omega \delta(x,1,t)|\delta(x,1,t)|^{m-2}u_t dx \leq \frac{|c_2|(m-1)}{m}\|\delta(x,1,t)\|_m^m + \frac{|c_2|}{m}\|u_t\|_m^m. \tag{24}$$

By (9) and simply calculation, we obtain

$$\frac{\partial}{\partial t}\int_0^1 \|\delta(x,\rho,t)\|_m^m d\rho = -\frac{1}{\tau}\|\delta(x,1,t)\|_m^m + \frac{1}{\tau}\|u_t(t)\|_m^m. \tag{25}$$

Inserting (24) and (25) into (23) gives

$$\begin{aligned}\frac{d}{dt}E(t) \leq& \frac{1}{2}(\omega' \circledcirc \nabla u) - \int_{\Gamma_1} hfy_t^2 d\Gamma - \frac{\omega(t)}{2}\|\nabla u\|_2^2 - \|u_t\|_2^{k+2} \\ &- \left(c_1 - \xi - \frac{|c_2|}{m}\right)\|u_t\|_m^m - \left(-\frac{|c_2|(m-1)}{m} + \xi\right)\|\delta(x,1,t)\|_m^m \\ \leq& \frac{1}{2}(\omega' \circledcirc \nabla u) - \int_{\Gamma_1} hfy_t^2 d\Gamma - \frac{\omega(t)}{2}\|\nabla u\|_2^2 - \|u_t\|_2^{k+2} - \gamma(\|u_t\|_m^m + \|\delta(x,1,t)\|_m^m) \leq 0,\end{aligned} \tag{26}$$

which ends this proof.

3 Blow-up solutions

3.1 The blow-up solutions of problem (8)-(14)

The blow-up solutions of this article are as follows:

Theorem 3.1. *Under the assumptions of Theorem 2.7, if* **(A_2)** *also holds, $u_0 \in H_0^1$ and $u_1 \in L^2$ such that $E(0) < 0$. Then any solution (u, y, δ) for the problem (8)-(14) blows up in finite time T satisfying*

$$T \leq \frac{1-\theta}{C\theta F^{\frac{\theta}{1-\theta}}(0)}.$$

Moreover, the lower bound for T is given by

$$\int_{\varphi(0)}^{\infty} \frac{d\Psi}{2\Psi + \mu_1 \Psi^{(p-1+\xi_1)} + \mu_2} \leq T,$$

where

$$F(0) = \beta^{1-\theta}(0) + \sigma\int_\Omega u_0 u_1 dx - \frac{\sigma}{2}\int_{\Gamma_1} hfy_0^2 d\Gamma - \sigma\int_{\Gamma_1} hy_0 u_0 d\Gamma$$

and

$$\varphi(0) = \|u_1\|_2^2 + \|\nabla u_0\|_2^2 + \int_{\Gamma_1} hgy_0^2 d\Gamma + 2\varepsilon\tau\|\delta_0\|_{L^m(\Omega\times(0,1))}^k,$$

the range of θ will be been given in (31) and both μ_1, μ_2 will be been defined in (63).

3.2 Proof of Theorem 3.1

Proof. This proof is divided into four steps:
Step I. We prove $\frac{1}{p}\int_\Omega |u|^p \ln|u| dx > 0$.
Let

$$\begin{aligned}\beta(t) =& -E(t) \\ =& \frac{1}{p}\int_\Omega |u|^p \ln|u| dx - \frac{1}{2}\|u_t\|_2^2 - \frac{1}{2}(\omega \odot \nabla u) \\ &- \frac{1}{2}\left(1 - \int_0^t \omega(s)ds\right)\|\nabla u\|_2^2 - \frac{1}{p^2}\|u\|_p^p \\ &- \frac{1}{2}\int_{\Gamma_1} hgy^2 d\Gamma - \xi\tau\int_0^1 \|\delta(x,\rho,t)\|_m^m d\rho.\end{aligned} \tag{27}$$

Then, we have

$$\beta'(t) \geq -E'(t) \geq 0. \tag{28}$$

Since $E(0) < 0$, it follows from (21) and (28) that

$$0 < \beta(0) \leq \beta(t) \leq \frac{1}{p}\int_{\Omega} |u|^p \ln |u| dx. \tag{29}$$

Step II. We establish auxiliary function $F(t)$, and prove $F'(t) > 0$ holds.

We now define

$$F(t) = \beta^{1-\theta}(t) + \sigma \int_{\Omega} u u_t dx - \frac{\sigma}{2}\int_{\Gamma_1} hfy^2 d\Gamma - \sigma \int_{\Gamma_1} hyud\Gamma, \tag{30}$$

where $\sigma > 0$ and the constant θ satisfies

$$\frac{2(p-m)}{p^2(m-1)} \leq \theta \leq min\left(\frac{p-m}{p(m-1)}, \frac{p-2}{2p}, \frac{m-2}{2m}, \frac{p-k-2}{p(k+2)}\right) \tag{31}$$

for

$$\frac{2p^2(m-1)}{p^2(m-1)-4(p-m)} < q < \frac{4p+2(n-1)(p-2)}{n(p-2)+4}. \tag{32}$$

The ranges of θ and q are reasonable due to (18) and (19).

Combining (8)-(14), (30) with Young's inequality with ε, we have

$$\begin{aligned} F'(t) =&(1-\theta)\beta^{-\theta}(t)\beta'(t) + \sigma\|u_t\|_2^2 + \sigma\int_{\Gamma_1} hgy^2 d\Gamma - \sigma\left(1-\int_0^t \omega(s)ds\right)\|\nabla u\|_2^2 \\ &+ \sigma\int_0^t \omega(t-s)(\nabla u(t), \nabla u(s) - \nabla u(t))ds - \sigma\int_{\Omega}\|u_t\|_2^k u_t u dx + \sigma\int_{\Omega}|u|^p \ln|u|dx \\ &- \sigma c_1\int_{\Omega} u|u_t|^{m-2}u_t dx - \sigma c_2\int_{\Omega} u|\delta(x,1,t)|^{m-2}\delta(x,1,t)dx \\ \geq&(1-\theta)\beta^{-\theta}(t)\beta'(t) + \sigma\|u_t\|_2^2 + \sigma\int_{\Gamma_1} hgy^2 d\Gamma + \sigma\int_{\Omega}|u|^p\ln|u|dx \\ &- \sigma\left(1-\int_0^t \omega(s)ds + \frac{1}{4\eta}\int_0^t \omega(s)ds\right)\|\nabla u\|_2^2 - \sigma\eta(\omega \odot \nabla u) \\ &- \sigma\int_{\Omega}\|u_t\|_2^k u_t u dx - \sigma c_1\int_{\Omega} u|u_t|^{m-2}u_t dx - \sigma c_2\int_{\Omega} u|\delta(x,1,t)|^{m-2}\delta(x,1,t)dx, \end{aligned} \tag{33}$$

where $\eta > 0$.

To get the conclusion of $F'(t) > 0$, we induct the term $\sigma\lambda\int_{\Omega}|u|^p\ln|u|dx$. From (27), we have

$$\begin{aligned} F'(t) \geq&(1-\theta)\beta^{-\theta}(t)\beta'(t) + \sigma\left(1+\frac{p(1-\lambda)}{2}\right)\|u_t\|_2^2 + \sigma\left(1+\frac{p(1-\lambda)}{2}\right)\int_{\Gamma_1} hgy^2 d\Gamma \\ &+ \frac{\sigma(1-\lambda)}{p}\|u\|_p^p + \sigma p(1-\lambda)\beta(t) + \sigma\left(\frac{p(1-\lambda)}{2}-1\right)\int_{\Omega}\|u_t\|_2^k u_t u dx \\ &+ \sigma\lambda\int_{\Omega}|u|^p\ln|u|dx + \sigma\left(\frac{p(1-\lambda)}{2}-\eta\right)(\omega\odot\nabla u) \\ &+ \sigma\left[\left(\frac{p(1-\lambda)}{2}-1\right) - \left(\frac{p(1-\lambda)}{2}-1+\frac{1}{4\eta}\right)\int_0^t \omega(s)ds\right]\|\nabla u\|_2^2 \\ &- \sigma c_1\int_{\Omega} u|u_t|^{m-2}u_t dx - \sigma c_2\int_{\Omega} u|\delta(x,1,t)|^{m-2}\delta(x,1,t)dx, \end{aligned} \tag{34}$$

where $0 < \lambda < \frac{p-2}{p}$.

Let $\gamma_1 = (\alpha\beta^{-\theta}(t))^{-\frac{m-1}{m}}$, where $\alpha > 0$. Then, by (28), Young's inequality with ε and **(A_5)**, we obtain

$$
\begin{aligned}
& c_1 \int_\Omega u|u_t|^{m-2}u_t dx + c_2 \int_\Omega u|\delta(x,1,t)|^{m-2}\delta(x,1,t)dx \\
\le & \frac{c_1(m-1)}{m}\gamma_1^{-\frac{m}{m-1}}\|u_t\|_m^m + \frac{|c_2|(m-1)}{m}\gamma_1^{-\frac{m}{m-1}}\|\delta(x,1,t)\|_m^m + \frac{(c_1+|c_2|)\gamma_1^m}{m}\|u\|_m^m \\
\le & \frac{c_1(m-1)}{m}\gamma_1^{-\frac{m}{m-1}}(\|u_t\|_m^m + \|\delta(x,1,t)\|_m^m) + \frac{(c_1+|c_2|)\gamma_1^m}{m}\|u\|_m^m \\
\le & \frac{c_1(m-1)\alpha}{m\gamma}(\beta(t))^{-\theta}\beta'(t) + \frac{C}{m\alpha^{m-1}}(\beta(t))^{\theta(m-1)}\|u\|_p^m.
\end{aligned} \tag{35}
$$

Theorem 2.7 and **(A_2)** show that $u \in L^{p+1}(\Omega)$. By (29) and Lemma 2.5, we derive

$$
\begin{aligned}
(\beta(t))^{\theta(m-1)}\|u\|_p^m \le & C\left(\int_\Omega |u|^p \ln|u|dx\right)^{\theta(m-1)}\left(\int_\Omega |u|^p\ln|u|dx + \|\nabla u\|_2^2\right)^{\frac{m}{p}} \\
\le & C\left(\int_\Omega |u|^p \ln|u|dx\right)^{\theta(m-1)}\left(\left(\int_\Omega |u|^p\ln|u|dx\right)^{\frac{m}{p}} + \|\nabla u\|_2^{\frac{2m}{p}}\right) \\
\le & C\left(\left(\int_\Omega |u|^p \ln|u|dx\right)^{\frac{\theta p(m-1)+m}{p}} + \left(\int_\Omega |u|^p\ln|u|dx\right)^{\frac{\theta p(m-1)}{p-m}} + \|\nabla u\|_2^2\right).
\end{aligned} \tag{36}
$$

Since (31) implies $0 < \frac{2(p-m)}{p^2(m-1)} \le \theta \le \frac{p-m}{p(m-1)}$ holds, we have $2 \le \theta p(m-1) + m \le p$ and $2 \le \frac{\theta(m-1)p^2}{p-m} \le p$. By Lemma 2.4, it follows that

$$
\left(\int_\Omega |u|^p \ln|u|dx\right)^{\frac{\theta p(m-1)+m}{p}} \le C\left(\|\nabla u\|_2^2 + \int_\Omega |u|^p\ln|u|dx\right) \tag{37}
$$

and

$$
\left(\int_\Omega |u|^p \ln|u|dx\right)^{\frac{\theta p(m-1)}{p-m}} \le C\left(\|\nabla u\|_2^2 + \int_\Omega |u|^p\ln|u|dx\right). \tag{38}
$$

Inserting (37) and (38) into (36), we obtain

$$
(\beta(t))^{\theta(m-1)}\|u\|_p^m \le C\left(\int_\Omega |u|^p\ln|u|dx + \|\nabla u\|_2^2\right). \tag{39}
$$

Combining (35) with (39), we obtain

$$
\begin{aligned}
& c_1 \int_\Omega u|u_t|^{m-2}u_t dx + c_2 \int_\Omega u|\delta(x,1,t)|^{m-2}\delta(x,1,t)dx \\
\le & \frac{c_1(m-1)\alpha}{m\gamma}(\beta(t))^{-\theta}\beta'(t)) + \frac{C}{m\alpha^{m-1}}\left(\int_\Omega |u|^p\ln|u|dx + \|\nabla u\|_2^2\right).
\end{aligned} \tag{40}
$$

For this term $\int_\Omega \|u_t\|_2^k u_t u dx$, Using Young's inequality with ε and H?lder's inequality, we deduce

$$
\begin{aligned}
|\int_\Omega \|u_t\|_2^k u_t u dx| \le & c_0\|u\|_p\|u_t\|_2^{k+1} = c_0\|u\|_p^{-p\theta_1}\|u\|_p^{1-p\theta_1}\|u_t\|_2^{k+1} \\
\le & c_0\left(\varepsilon\|u\|_p^{(1-p\theta_1)(k+2)} + C(\varepsilon)\|u_t\|_2^{k+2}\right)\|u\|_p^{-p\theta_1}
\end{aligned} \tag{41}
$$

for any $\theta_1 > 0$ and $\varepsilon > 0$. We choose $\theta_1 = \frac{1}{q+2} - \frac{1}{p} > 0$ and let $\theta \in (0, \theta_1)$. From (41), we have

$$\begin{aligned} |\int_\Omega \|u_t\|_2^k u_t u dx| \leq & C\left(\varepsilon \|u\|_p^p + C(\varepsilon)\beta'(t)\right)\beta(t)^{-\alpha_1} \\ \leq & C\left(\varepsilon\beta(0)^{-\theta_1}\|u\|_p^p + C(\varepsilon)\beta(0)^{\theta-\theta_1}\beta(t)^{-\theta}\beta'(t)\right), \end{aligned} \tag{42}$$

where the constant $C(\varepsilon) = (\varepsilon(q+2))^{\frac{-(\frac{q+2}{q+1})}{q+2}}(\frac{q+2}{q+1})^{-1}$, which is generated by Young's inequality with parameter ε.

To eliminate $\left(\frac{p(1-\lambda)}{2} - \eta\right)(\omega \odot \nabla u)$ in (34), we establish the following relation

$$\begin{aligned} & \left(\frac{p(1-\lambda)}{2} - 1\right) - \left(\frac{p(1-\lambda)}{2} - 1 + \frac{1}{4\eta}\right)\int_0^t \omega(s)ds \\ = & \left\{\left(\frac{p(1-\lambda)}{2} - 1\right)\left(1 - \int_0^t \omega(s)ds\right) - \frac{1}{2p(1-\lambda)}\int_0^t \omega(s)ds\right\} \\ \geq & \left\{\left(\frac{p(1-\lambda)}{2} - 1\right)\omega_l - \frac{1}{2p(1-\lambda)}(1-\omega_l)\right\} =: Q_\lambda, \end{aligned}$$

where $\eta = \frac{p(1-\lambda)}{2}$.

Inserting (40), (41) and the definition of Q_λ into (34), we derive

$$\begin{aligned} F'(t) \geq & \left((1-\theta) + C(\varepsilon)\beta(0)^{\theta-\theta_1}\sigma C\left(\frac{p(1-\lambda)-1}{2}\right) - \frac{\sigma c_1\alpha(m-1)}{m\gamma}\right)\beta^{-\theta}(t)\beta'(t) \\ & + \sigma\left(1 + \frac{p(1-\lambda)}{2}\right)\int_{\Gamma_1} hgy^2 d\Gamma + \sigma\left(1 + \frac{p(1-\lambda)}{2}\right)\|u_t\|_2^2 \\ & + \sigma\left(\frac{(1-\lambda)}{p} + C\varepsilon\beta(0)^{-\theta_1}\left(\frac{p(1-\lambda)-1}{2}\right)\right)\|u\|_p^p + \sigma p(1-\lambda)\beta(t) \\ & + \sigma\left(Q_\lambda - \frac{C}{m\theta^{m-1}}\right)\|\nabla u\|_2^2 + \sigma\left(\lambda - \frac{C}{m\theta^{m-1}}\right)\int_\Omega |u|^p \ln|u| dx. \end{aligned} \tag{43}$$

We now impose restrictions on the coefficients of the right-hand terms in (43). Let

$$Q = \frac{\omega_l(p-2)}{2} - \frac{1-\omega_l}{2p}.$$

Then

$$Q > 0,\ Q_\lambda < Q,\ \lim_{\lambda\to 0^+} Q_\lambda = Q.$$

Since $0 < \lambda < \frac{p-2}{p}, \omega_l > 0, m \geq 2, 0 < k < p-2, Q_\lambda > 0$ and $\left(\frac{p(1-\lambda)}{2} - 1\right) > \left(\frac{p(1-\frac{p-2}{p})}{2} - 1\right) = 0$. By the arbitrariness of $\alpha > 0$, we see that for sufficiently large α, one knows

$$Q_\lambda - \frac{C}{m\theta^{m-1}} > 0 \textit{ and } \lambda - \frac{C}{m\theta^{m-1}} > 0.$$

Choosing $\sigma > 0$ appropriately small to gain

$$\left((1-\theta) + C(\varepsilon)\beta(0)^{\theta-\theta_1}\sigma C\left(\frac{p(1-\lambda)-1}{2}\right) - \frac{\sigma c_1\alpha(m-1)}{m\gamma}\right) > 0$$

and

$$F(0) = \beta^{1-\theta}(0) + \sigma\int_\Omega u_0 u_1 dx - \frac{\sigma}{2}\int_{\Gamma_1} hfy_0^2 d\Gamma - \sigma\int_{\Gamma_1} hy_0 u_0 d\Gamma > 0.$$

Therefore

$$F'(t) \geq C\left(\|u_t\|_2^2 + \|u\|_p^p + \beta(t) + \|\nabla u\|_2^2 + \int_{\Gamma_1} hgy^2 d\Gamma + \int_\Omega |u|^p \ln|u| dx\right) > 0. \tag{44}$$

Step III. We establish the upper bound for the finite time blow-up solution.
From (30), (31) and (34), we have

$$F^{\frac{1}{1-\theta}}(t) \leq C\left(\beta(t) + \left(\int_\Omega |uu_t| dx\right)^{\frac{1}{1-\theta}} + \left(\int_{\Gamma_1} |hyu| d\Gamma\right)^{\frac{1}{1-\theta}}\right). \tag{45}$$

Since (31) implies $2 \leq \frac{2}{1-2\alpha} \leq p$, by Young's inequality with ε and Lemma 2.3, we deduce

$$\begin{aligned}\left(\int_\Omega |uu_t| dx\right)^{\frac{1}{1-\theta}} &\leq \|u\|_2^{\frac{1}{1-\theta}} \|u_t\|_2^{\frac{1}{1-\theta}} \leq C\|u\|_p^{\frac{1}{1-\theta}} \|u_t\|_2^{\frac{1}{1-\theta}} \\ &\leq C(\|u_t\|_2^2 + \|u\|_p^{\frac{2}{1-2\theta}}) \leq C((\|u_t\|_2^2 + \|u\|_p^p + \|\nabla u\|_2^2).\end{aligned} \tag{46}$$

Once again using Young's inequality with ε, we obtain

$$\begin{aligned}\left(\int_{\Gamma_1} h(x) yu d\Gamma\right)^{\frac{1}{1-\theta}} &= \left(\int_{\Gamma_1} \frac{hgyu}{g} d\Gamma\right)^{\frac{1}{1-\theta}} \\ &\leq \left(\frac{\|h\|_{\Gamma_1,\infty}\|g\|_{\Gamma_1,\infty}}{g_0^2}\right)^{\frac{1}{2(1-\theta)}} \left(\int_{\Gamma_1} hgy^2 d\Gamma\right)^{\frac{1}{2(1-\theta)}} \left(\int_{\Gamma_1} u^2 d\Gamma\right)^{\frac{1}{2(1-\theta)}} \\ &\leq C\left(\int_{\Gamma_1} hgy^2 d\Gamma\right)^{\frac{1}{2(1-\theta)}} \|u\|_{\Gamma_1,q}^{\frac{1}{1-\theta}} \\ &\leq C\left(\int_{\Gamma_1} hgy^2 d\Gamma + \|u\|_{\Gamma_1,q}^{\frac{2}{1-2\theta}}\right).\end{aligned} \tag{47}$$

Since (31) implies $\frac{2}{q(1-2\theta)} \leq 1$, by applying (7) with $w_1 = \frac{2}{q(1-2\theta)}$ and $w_2 = \beta(0)$, we obtain

$$\|u\|_{\Gamma_1,q}^{\frac{2}{1-2\theta}} = \left(\int_{\Gamma_1} u^q d\Gamma\right)^{\frac{2}{q(1-2\theta)}} \leq \frac{\beta(0)+1}{\beta(0)}\left(\int_{\Gamma_1} u^q d\Gamma + \beta(0)\right) \leq C(\|u\|_{\Gamma_1,q}^q + \beta(t)). \tag{48}$$

Combining (45)-(48) with (29), we derive

$$F^{\frac{1}{1-\theta}}(t) \leq C\left(\|u_t\|_2^2 + \|u\|_p^p + \beta(t) + \|\nabla u\|_2^2 + \int_{\Gamma_1} hgy^2 d\Gamma + \|u\|_{\Gamma_1,q}^q\right). \tag{49}$$

We now estimate $\|u\|_{\Gamma_1,q}^q$. From (17), (19) and (32), we have

$$2 < q < \frac{2n+2}{n},$$

which implies

$$0 < \frac{n}{2} - \frac{n-1}{q} < \frac{2}{q} < 1. \tag{50}$$

We let $\frac{n}{2} - \frac{n-1}{q} \leq \iota < \frac{2}{q}$, it is effective due to (50). Hence the embedding $H^\iota(\Omega) \hookrightarrow L^q(\Gamma_1)$ holds. By the interpolation theorem, we have

$$\|u\|_{\Gamma_1,q}^k \leq C\|u\|_{H^\iota(\Omega)}^k \leq C\|u\|_2^{(1-\iota)q} \|\nabla u\|_2^{\iota q} \leq C\|u\|_p^{(1-\iota)q} \|\nabla u\|_2^{\iota q}. \tag{51}$$

By Young's inequality with ε, it follows that

$$\|u\|_{\Gamma_1,q}^q \leq C(\|\nabla u\|_2^2 + (\|u\|_p^p)^{\frac{2k(1-\iota)}{p(2-\iota q)}}). \tag{52}$$

From (32), we have

$$\frac{n}{2} - \frac{n-1}{q} \leq \frac{2(p-q)}{q(p-2)}.$$

Hence, we can take ι with $\frac{n}{2} - \frac{n-1}{q} \leq \iota \leq \frac{2(p-q)}{q(p-2)} < \frac{2}{q}$, which implies

$$0 < \frac{2k(1-\iota)}{p(2-\iota q)} \leq 1.$$

By applying (7) with $\omega_1 = \frac{2q(1-\iota)}{p(2-\iota q)}$ and $\omega_2 = \beta(0)$, we have

$$(\|u\|_p^p)^{\frac{2q(1-\iota)}{p(2-\iota q)}} \leq \left(\frac{\beta(0)+1}{\beta(0)}\right)(\|u\|_p^p + \beta(0)) \leq C(\|u\|_p^p + \beta(t)). \tag{53}$$

Combining (49), (52) with (53), we obtain

$$F^{\frac{1}{1-\theta}}(t) \leq C\left(\|u_t\|_2^2 + \|u\|_p^p + \beta(t) + \|\nabla u\|_2^2 + \int_{\Gamma_1} hgy^2 d\Gamma\right). \tag{54}$$

From (44), (54) and Lemma 2.6, we conclude that

$$F'(t) \geq CF^{\frac{1}{1-\theta}}(t),\ t \geq 0, \tag{55}$$

which implies

$$T \leq \frac{1-\theta}{C\theta F^{\frac{\theta}{1-\theta}}(0)}. \tag{56}$$

Step IV. We establish the lower bound for the finite time blow-up solution.

Let $(u, y, \delta; u_0)$ is any finite time blow-up solution of problem (8)-(14). If $E(u_0) < 0$, we conclude from (27), (30) and (55) that

$$\lim_{t \to T^-} \int_\Omega |u|^p \ln |u| dx = +\infty. \tag{57}$$

By (20), (29), Lemma 2.2 and the Sobolev embedding inequality, it follows that

$$\begin{aligned}
0 < \int_\Omega |u|^p \ln |u| dx &= \int_{\Omega_-} |u|^p \ln |u| dx + \int_{\Omega_+} |u|^p \ln |u| dx \\
&< \frac{1}{e\xi_0} \int_{\Omega_+} |u|^{p+\xi_0} dx \leq \frac{1}{e\xi_0} \|u\|_{p+\xi_0}^{p+\xi_0} \leq \frac{M_{p+\xi_0}^{p+\xi_0}}{e\xi_0} \|\nabla u\|_2^{p+\xi_0},
\end{aligned} \tag{58}$$

where

$$\Omega_+ = \{x | x \in \Omega, |u| \geq 1\},\ \ \Omega_- = \{x | x \in \Omega, |u| < 1\}.$$

Hence,

$$\lim_{t \to T^-} \|\nabla u\|_2 = +\infty. \tag{59}$$

Let

$$\varphi(t) = \|u_t\|_2^2 + \left(1 - \int_0^t \omega(s) ds\right) \|\nabla u\|_2^2 + (\omega \odot \nabla u) + \int_{\Gamma_1} hgz^2 d\Gamma + 2\xi\tau \int_0^1 \|\delta(x, \rho, t)\|_m^m d\rho. \tag{60}$$

Then, (59) and (60) imply

$$\lim_{t\to T^-} L(t) \geq \lim_{t\to T^-} w_l\|\nabla u\|_2^2 = +\infty. \tag{61}$$

Combining (20), (21),(22) and (60), together with Young's inequality with ε and Lemma 2.2, it follows that

$$\begin{aligned}
\varphi'(t) &= 2E'(t) + \frac{d}{dt}\Big(\frac{1}{p}\int_\Omega |u|^p \ln|u|dx - \frac{1}{p^2}\|u\|_p^p\Big)\\
&\leq 2\int_\Omega u_t|u|^{p-2}u\ln|u|dx\\
&\leq 2\|u_t\|_2^2 + \int_\Omega ||u|^{p-1}\ln|u||^2dx\\
&= 2\|u_t\|_2^2 + \int_{\Omega_+} ||u|^{p-1}\ln|u||^2dx + \int_{\Omega_-} ||u|^{p-1}\ln|u||^2dx\\
&\leq 2\|u_t\|_2^2 + \frac{1}{e^2\xi_1^2}\int_{\Omega_+} |u|^{2(p-1+\xi_1)}dx + \frac{|\Omega|}{e^2(p-1)^2}\\
&\leq 2\|u_t\|_2^2 + \frac{M_{2(p-1+\xi_1)}^{2(p-1+\xi_1)}}{e^2\xi_1^2}\|\nabla u\|_2^{2(p-1+\xi_1)} + \frac{|\Omega|}{e^2(p-1)^2}\\
&\leq 2\varphi(t) + \mu_1\varphi^{(p-1+\xi_1)}(t) + \mu_2,
\end{aligned} \tag{62}$$

where

$$\mu_1 = \frac{M_{2(p-1+\xi_1)}^{2(p-1+\xi_1)}}{e^2\xi_1^2 w_l^{(p-1+\xi_1)}} \quad \textit{and} \quad \mu_2 = \frac{|\Omega|}{e^2(p-1)^2}. \tag{63}$$

From (62), we obtain

$$\frac{\varphi'(t)}{2\varphi(t) + \mu_1\varphi^{(p-1+\xi_1)}(t) + \mu_2} \leq 1. \tag{64}$$

Integrating (64) on $[0, T)$ gives

$$\int_0^T \frac{\varphi'(t)dt}{2\varphi(t) + \mu_1\varphi^{(p-1+\xi_1)}(t) + \mu_2} \leq \int_0^T dt.$$

Let $\Psi = \varphi(t)$. Then the above formula implies

$$\int_{\varphi(0)}^{+\infty} \frac{d\Psi}{2\Psi + \mu_1\Psi^{(p-1+\xi_1)} + \mu_2} \leq T.$$

Thus, we complete the proof of Theorem 3.1.

4 Declarations

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The authors declare that they have no competing interests.

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Not applicable.

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