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Horadam-Lagrange Interpolation Polynomials: Construction, Recurrence Relations, and Connections to Special Number Sequences

Orhan Dişkaya[1]

1. Department of Mathematics, Mersin University, Mersin 33343, Türkiye

Abstract

This study investigates the construction of polynomials of at most degree n using the first $n+1$ terms of the Horadam sequence through Lagrange interpolation. The paper provides a comprehensive analysis of the recurrence relations and fundamental identities associated with the Horadam-Lagrange interpolation polynomials. Furthermore, it explores the structural properties and special cases of these polynomials, highlighting their connections to well-known sequences such as Fibonacci, Lucas, Pell, Jacobsthal, Mersenne and Fermat sequences.

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1 Introduction

As is well known, Fibonacci numbers have been prominently featured in applied sciences. There have been many studies on Fibonacci numbers and their generalizations over the centuries. Lucas numbers, which share the same recurrence relation but have different initial conditions from Fibonacci numbers, have many relationships with the Fibonacci numbers. Both Fibonacci and Lucas numbers are sequences of second-order recurrence relations. There are many generalized sequences of the same order, some of which include Pell, Jacobsthal, Pell-Lucas, and Jacobsthal-Lucas sequences.

A notable example of such generalizations is the Horadam sequence. One of the most comprehensive generalizations of second-order recurrence sequences is the Horadam sequence. Horadam numbers have attracted significantly interest due to their rich structural properties and diverse applications. Defined by a general second-order recurrence relation with arbitrary initial conditions, Horadam numbers encompass many well-known sequences as special cases,

Contact: Orhan Dişkaya orhandiskaya@mersin.edu.tr

including Fibonacci, Lucas, and Pell sequences. Their flexibility allows for numerous generalizations and extensions in combinatorial analysis, number theory, and applied mathematics. To further explore the properties of Horadam numbers, we consider their recurrence relation and fundamental characteristics as follows:
In [7, 8, 9], Horadam's sequence $\{H_r(d,m;p,q)\}$, or briefly $\{H_r\}$ is defined by the recurrence relation

$$H_0 = d \quad \text{and} \quad H_1 = m, \quad H_r = pH_{r-1} - qH_{r-2}, \quad r \geq 2,$$

where d, m, p, and q are integers with $p > 0$, $q \neq 0$ and $p^2 - 4q \geq 0$. Let $h_1 = \frac{p+\sqrt{p^2-4q}}{2}$ and $h_2 = \frac{p-\sqrt{p^2-4q}}{2}$ be the roots of $h^2 - ph + q = 0$ where p and q are arbitrary integers. Furthermore, the Binet-like formula for the Horadam sequence is

$$\mathcal{H}_n = Uh_1^n + Vh_2^n,$$

where $U = \frac{m-dh_2}{h_1-h_2}$ and $V = \frac{dh_1-m}{h_1-h_2}$. Many well-known sequences arise as special cases of the Horadam sequences when appropriate initial conditions are chosen, as demonstrated below:

- The sequence $\{F_r\}$, is defined as $F_r = H_r(0,1;1,-1)$ and is referred to as the Fibonacci sequence in [7].
- The sequence $\{L_r\}$, is defined as $L_r = H_r(2,1;1,-1)$ and is referred to as the Lucas sequence in [15].
- The sequence $\{P_r\}$, is defined as $P_r = H_r(0,1;2,-1)$ and is referred to as the Pell sequence in [11].
- The sequence $\{J_r\}$, is defined as $J_r = H_r(0,1;1,-2)$ and is referred to as the Jacobsthal sequence in [12].
- The sequence $\{M_r\}$, is defined as $M_r = H_r(0,1;3,2)$ and is referred to as the Mersenne sequence in [21].
- The sequence $\{T_r\}$, is defined as $T_r = H_r(2,3;3,2)$ and is referred to as the Fermat sequence in [13].

Numerous studies in the literature have explored special number sequences and their various properties [1, 8, 10, 14, 16, 17, 18, 19, 20, 22]. In this context, interpolation techniques provide a powerful tool for analyzing and extending these sequences. In the following, we introduce the concept of Lagrange interpolation, which we will later relate to the Horadam sequence.

The Lagrange interpolating polynomial is essentially a rephrased version of the Newton polynomial that eliminates the need to calculate divided differences. Lagrange interpolation is beneficial because it is effective for data points that are unevenly spaced along the independent variable. In the realm of numerical analysis, interpolation refers to the method of identifying the most suitable function based on certain given points. The most basic form of interpolation uses a polynomial. This implies that for a set of specified points, there is a polynomial that intersects all these points. This polynomial closely approximates the underlying function. One technique for polynomial interpolation is the Lagrange interpolation method [5].

Let P_n be the set of all real-valued polynomials of degree at most n defined over the set $\mathbb{R}$ of real numbers, given that n is a nonnegative integer. The basic interpolation problem is as follows: identify a polynomial $p_0 \in P_0$ such that $p_0(x_0) = y_0$, given x_0 and y_0 in $\mathbb{R}$. This can be solved by using the formula $p_0(x) \equiv y_0$. The primary purpose of this work is to examine the subsequent, more general problem [2].

Let $n \geq 1$, and assume that x_i for $i = 0,1,\ldots,n$ are distinct real numbers (i.e., $x_i \neq x_j$ for $i \neq j$), and y_i for $i = 0,1,\ldots,n$ are real numbers. We want to find $p_n \in \mathcal{P}_n$ such that $p_n(x_i) = y_i$ for $i = 0,1,\ldots,n$.

There exist polynomials $L_k \in \mathcal{P}_n$ for $k = 0, 1, \ldots, n$, such that

$$L_k(x_i) = \begin{cases} 1 & \text{if } i = k, \\ 0 & \text{if } i \neq k, \end{cases}$$

for all $i, k = 0, 1, \ldots, n$. Furthermore,

$$p_n(x) = \sum_{k=0}^{n} L_k(x) y_k$$

satisfies the interpolation conditions mentioned above. In other words, $p_n \in \mathcal{P}_n$ and $p_n(x_i) = y_i$ for $i = 0, 1, \ldots, n$. For each fixed k, $0 \leq k \leq n$, L_k is required to have n zeros at x_i for $i = 0, 1, \ldots, n$ and $i \neq k$. Thus, $L_k(x)$ is of the form

$$L_k(x) = \prod_{\substack{i=0 \\ i \neq k}}^{n} \frac{x - x_i}{x_k - x_i}.$$

Once these basis polynomials are constructed, the Lagrange interpolation polynomial can be expressed as follows:

$$p_n(x) = \sum_{k=0}^{n} L_k(x) y_k = \sum_{k=0}^{n} \left(\prod_{\substack{i=0 \\ i \neq k}}^{n} \frac{x - x_i}{x_k - x_i} \right) y_k.$$

Based on these statements, Mufid et al. demonstrated that a polynomial of at most degree n can be constructed from the first $n + 1$ terms of the Fibonacci sequence using Lagrange interpolation. They further established that this Fibonacci Lagrange Interpolation Polynomial (FLIP) can be derived both recursively and implicitly [3].

Similarly, Hopkins and Tangbookduangjit investigate a class of interpolating rational polynomials that generate Fibonacci numbers at specific integer values. By design, a polynomial of degree m produces Fibonacci numbers for $m + 1$ input values. Their analysis reveals that several additional outputs of these polynomials also exhibit close relationships with Fibonacci numbers. Furthermore, they employ an alternative formulation of these polynomials to derive various Fibonacci-related identities [4].

In this study, we first investigate the formation of polynomials of degree at most n using the first $n + 1$ terms of the sequences $\{H_n\}$ via Lagrange interpolation. Then, we present a detailed examination of the recurrence relations and various identities associated with the Horadam Lagrange interpolation polynomials.

2 The Lagrange interpolation of the Horadam sequences

Before we commence the interpolations of the sequence $\{H_n\}$, we will plot these sequences on the $xy-$coordinate system. Let's denote the Horadam point as $h_n = (n, H_n)$, representing the points associated with the $n-$th terms of the sequence $\{H_n\}$.

Definition 1. Let $\{H_k\}_{k \geq 0}$ be a sequence of the Horadam numbers. The polynomial $H_n(x)$, called the Horadam-Lagrange interpolation polynomial, is defined via Lagrange interpolation using the points $(x_k, y_k) = (k, H_k)$ as follows:

$$\mathbb{H}_n(x) = \sum_{k=0}^{n} H_k \prod_{\substack{i=0 \\ i \neq k}}^{n} \frac{x - i}{k - i}. \tag{1}$$

Here, the product (1) can be simplified as:

$$\prod_{\substack{i=0\\i\neq k}}^{n}(k-i)=(-1)^{n-k}(n-k)!\,k!=(-1)^{n-k}\binom{n}{k}n!.$$

Using this simplification, the polynomial $\mathbb{H}_n(x)$ can be equivalently expressed as:

$$\mathbb{H}_n(x)=\frac{1}{n!}\sum_{k=0}^{n}\left((-1)^{n-k}\binom{n}{k}\prod_{\substack{i=0\\i\neq k}}^{n}(x-i)\right)H_k.$$

the polynomial $\mathbb{H}_n(x)$ provides an explicit interpolation formula for the given sequence H_k based on equally spaced nodes.

Several special cases of the polynomials $\mathbb{H}_n(x)$ are as follows:

- If $F_r = H_r(1,1;1,-1)$, then Eq. (1) reduces to the Fibonacci-Lagrange interpolation polynomial $\mathbb{F}_n(x)$ which is defined via Lagrange interpolation using the points $(x_k, y_k) = (k, F_k)$ in [3].
- If $L_r = H_r(2,1;1,-1)$, then Eq. (1) reduces to the Lucas-Lagrange interpolation polynomial $\mathbb{L}_n(x)$ which is defined via Lagrange interpolation using the points $(x_k, y_k) = (k, L_k)$.
- If $P_r = H_r(1,2;2,-1)$, then Eq. (1) reduces to the Pell-Lagrange interpolation polynomial $\mathbb{P}_n(x)$ which is defined via Lagrange interpolation using the points $(x_k, y_k) = (k, P_k)$.
- If $J_r = H_r(1,1;1,-2)$, then Eq. (1) reduces to the Jacobsthal-Lagrange interpolation polynomial $\mathbb{J}_n(x)$ which is defined via Lagrange interpolation using the points $(x_k, y_k) = (k, J_k)$ in [6].
- If $M_r = H_r(1,3;3,2)$, then Eq. (1) reduces to the Mersenne-Lagrange interpolation polynomial $\mathbb{M}_n(x)$ which is defined via Lagrange interpolation using the points $(x_k, y_k) = (k, M_k)$.
- If $T_r = H_r(2,3;3,2)$, then Eq. (1) reduces to the Fermat-Lagrange interpolation polynomial $\mathbb{T}_n(x)$ which is defined via Lagrange interpolation using the points $(x_k, y_k) = (k, T_k)$.

The graphical representation of the first five special Horadam numbers and the graphs of the polynomial values H for $n = 1,2,3,4$ which pass through these points following a specific rule under special conditions are shown below.

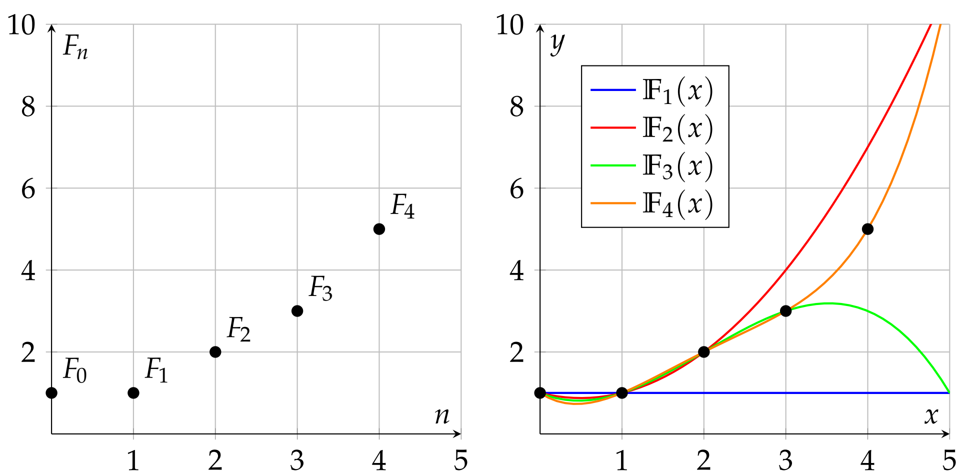

Figure 1: Graphs of the Lagrange interpolation polynomials $\mathbb{F}_n(x)$ for Fibonacci numbers based on the data points (n, F_n) for $n = 0,1,\ldots,4$.

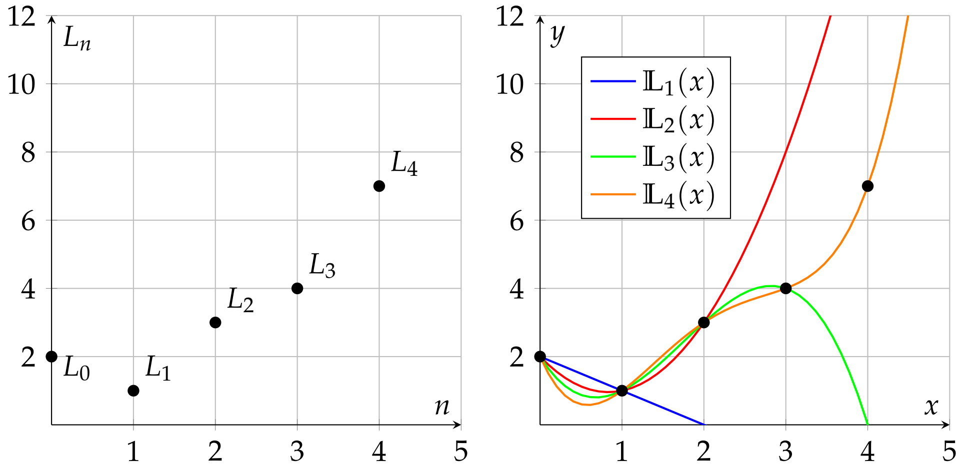

Figure 2: Graphs of the Lagrange interpolation polynomials $\mathbb{L}_n(x)$ for Lucas numbers based on the data points (n, L_n) for $n = 0, 1, \ldots, 4$.

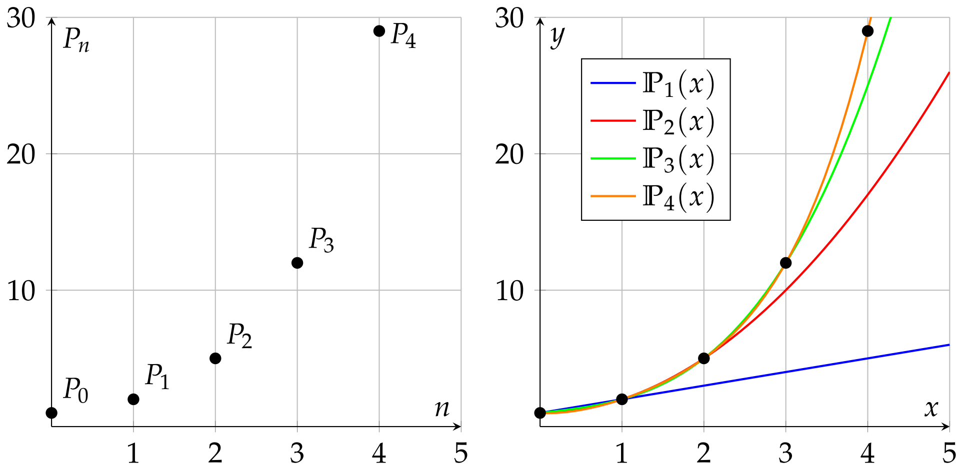

Figure 3: Graphs of the Lagrange interpolation polynomials $\mathbb{P}_n(x)$ for Pell numbers based on the data points (n, P_n) for $n = 0, 1, \ldots, 4$.

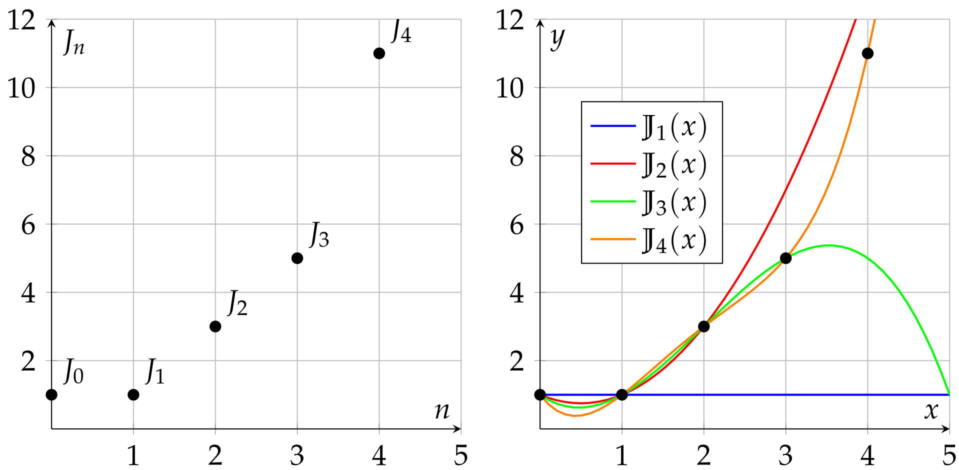

Figure 4: Graphs of the Lagrange interpolation polynomials $\mathbb{J}_n(x)$ for Jacobsthal numbers based on the data points (n, J_n) for $n = 0, 1, \ldots, 4$.

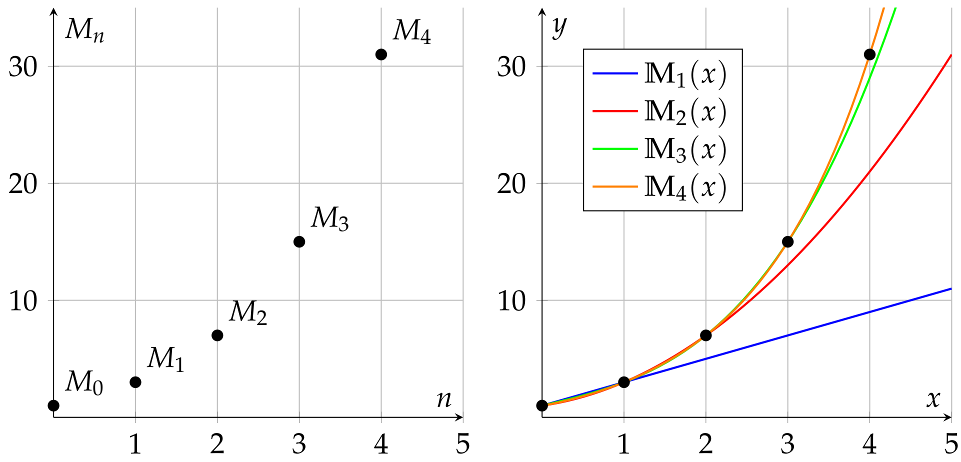

Figure 5: Graphs of the Lagrange interpolation polynomials $\mathbb{M}_n(x)$ for Mersenne numbers based on the data points (n, M_n) for $n = 0, 1, \ldots, 4$.

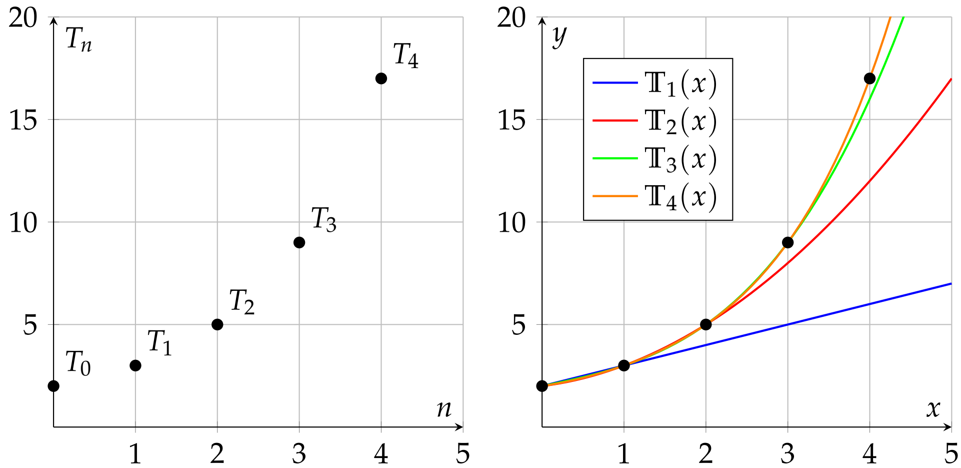

Figure 6: Graphs of the Lagrange interpolation polynomials $\mathbb{T}_n(x)$ for Fermat numbers based on the data points (n, T_n) for $n = 0, 1, \ldots, 4$.

We shall establish a leading coefficient theorem for $\mathbb{H}_n(x)$, before obtaining more formulas for it.

Theorem 2. *The leading coefficients of the Horadam-Lagrange interpolation polynomial $\mathbb{H}_n(x)$ are*

$$\frac{(-q)^n H_{-n}}{n!}. \tag{2}$$

Proof. The leading coefficient of $\mathbb{H}_n(x)$ is equal to $\frac{1}{n!}\sum_{k=0}^{n}(-p)^{n-k}H_k\binom{n}{k}$. Thus, we just have to demonstrate that

$$\begin{aligned}
\sum_{k=0}^{n}(-p)^{n-k}H_k\binom{n}{k} &= \sum_{k=0}^{n}(-p)^{n-k}\left(Uh_1^k+Vh_2^k\right)\binom{n}{k}\\
&= U\sum_{k=0}^{n}(-p)^{n-k}h_1^k\binom{n}{k}+V\sum_{k=0}^{n}(-p)^{n-k}h_2^k\binom{n}{k}\\
&= U(h_1-p)^n+V(h_2-p)^n\\
\text{From}\quad h_1+h_2=p,\quad &= U(-h_2)^n+V(-h_1)^n\\
\text{From}\quad h_1h_2=q,\quad &= (-q)^n(Uh_1^{-n}+Vh_2^{-n})\\
&= (-q)^nH_{-n}.
\end{aligned}$$

□

- If $F_n = H_n(1,1;1,-1)$, then according to Eq. (2), the leading coefficients of the Fibonacci-Lagrange interpolation polynomial $\mathbb{F}_n(x)$ are $\frac{F_{-n}}{n!}$ in [3].
- If $L_r = H_r(2,1;1,-1)$, then according to Eq. (2), the leading coefficients of the Lucas-Lagrange interpolation polynomial $\mathbb{L}_n(x)$ are $\frac{L_{-n}}{n!}$.
- If $P_r = H_r(1,2;2,-1)$, then according to Eq. (2), the leading coefficients of the Pell-Lagrange interpolation polynomial $\mathbb{P}_n(x)$ are $\frac{P_{-n}}{n!}$.
- If $J_r = H_r(1,1;1,-2)$, then according to Eq. (2), the leading coefficients of the Jacobsthal-Lagrange interpolation polynomial $\mathbb{J}_n(x)$ are $\frac{2^nJ_{-n}}{n!}$ in [6].
- If $M_r = H_r(1,3;3,2)$, then according to Eq. (2), the leading coefficients of the Mersenne-Lagrange interpolation polynomial $\mathbb{M}_n(x)$ are $\frac{(-2)^nM_{-n}}{n!}$.
- If $T_r = H_r(2,3;3,2)$, then according to Eq. (2), the leading coefficients of the Fermat-Lagrange interpolation polynomial $\mathbb{T}_n(x)$ are $\frac{(-2)^nT_{-n}}{n!}$.

Theorem 3. *The polynomials $\mathbb{H}_n(x)$ have degree n and admits alternative representations:*

$$\mathbb{H}_n(x)=\sum_{k=0}^{n}H_k\binom{x}{k}\binom{n-x}{n-k}.$$

Proof. Using Eq. (1), we have

$$\begin{aligned}
\mathbb{H}_n(x) &= \sum_{k=0}^{n}H_k\prod_{\substack{i=0\\ i\neq k}}^{n}\frac{x-i}{k-i}\\
&= \sum_{k=0}^{n}H_k\left(\prod_{\substack{i=0\\ i\neq k}}^{n}\frac{1}{k-i}\right)\left(\prod_{\substack{i=0\\ i\neq k}}^{n}(x-i)\right)\\
&= \sum_{k=0}^{n}H_k\cdot(-1)^{n-k}\frac{1}{k!(n-k)!}\cdot k!\binom{x}{k}(n-k)!\binom{x-k-1}{n-k}\\
&= \sum_{k=0}^{n}H_k\binom{x}{k}\binom{n-x}{n-k}.
\end{aligned}$$

where the last step applies the identity

$$\binom{r}{s} = (-1)^s \binom{s-r-1}{s}.$$

valid for all integers r and non-negative integers s [23]. □

3 Recurrence relations of the $\mathbb{H}_n(x)$

In this part, we will derive the additional formulas for $\mathbb{H}_n(x)$. It's remarkable that $\mathbb{H}_n(x)$ can be constructed recurrence relations, much like the sequences $\{H_n\}$.

Consider the polynomials $\mathbb{H}_{n+1}(x)$ and $\mathbb{H}_n(x)$. For each $i \in \{0,1,2,\ldots,n\}$, we have $\mathbb{H}_{n+1}(i) = \mathbb{H}_n(i)$. In other words, the polynomials $\mathbb{H}_{n+1}(x)$ and $\mathbb{H}_n(x)$ intersect at $n+1$ points. Consequently, we can express the relationship as follows:

$$\mathbb{H}_{n+1}(x) - \mathbb{H}_n(x) = a \cdot x(x-1)\cdots(x-n),$$

where a denotes the leading coefficients (2) of $\mathbb{H}_{n+1}(x)$. As a results, when $Q_n(x) = x(x-1)\cdots(x-n)$,

$$\mathbb{H}_{n+1}(x) = \mathbb{H}_n(x) + \frac{(-q)^{n+1}H_{-n-1}}{(n+1)!}Q_n(x)$$

is a recurrence relation.

By successively applying the recurrence relation for $\mathbb{H}_n(x)$, $\mathbb{H}_{n-1}(x)$, ..., $\mathbb{H}_2(x)$, we derive the following implicit formula:

$$\mathbb{H}_{n+1}(x) = \mathbb{H}_1(x) + \sum_{i=1}^{n} \frac{(-q)^{i+1}H_{-i-1}}{(i+1)!}Q_i(x)$$

which simplifies to

$$\mathbb{H}_n(x) = (m-d)x + d + \sum_{i=1}^{n} \frac{(-q)^{i+1}H_{-i-1}}{(i+1)!}Q_i(x). \tag{3}$$

where the parameters d and m denote the initial terms of the Horadam sequence.

- If $F_r = H_r(1,1;1,-1)$, then according to Eq. (3), the recurrence relation of the Fibonacci-Lagrange interpolation polynomial $\mathbb{F}_n(x)$ are given by

$$\mathbb{F}_n(x) = 1 + \sum_{i=1}^{n} \frac{F_{-i-1}}{(i+1)!}Q_i(x).$$

  in [3].
- If $L_r = H_r(2,1;1,-1)$, then according to Eq. (3), the recurrence relation of the Lucas-Lagrange interpolation polynomial $\mathbb{L}_n(x)$ are given by

$$\mathbb{L}_n(x) = -x + 2 + \sum_{i=1}^{n} \frac{L_{-i-1}}{(i+1)!}Q_i(x).$$

- If $P_r = H_r(1,2;2,-1)$, then according to Eq. (3), the recurrence relation of the Pell-Lagrange interpolation polynomial $\mathbb{P}_n(x)$ are given by

$$\mathbb{P}_n(x) = x + 1 + \sum_{i=1}^{n} \frac{P_{-i-1}}{(i+1)!}Q_i(x).$$

- If $J_r = H_r(1,1;1,-2)$, then according to Eq. (3), the recurrence relation of the Jacobsthal-Lagrange interpolation polynomial $\mathbb{J}_n(x)$ are given by

$$\mathbb{J}_n(x) = 1 + \sum_{i=1}^{n} \frac{2^{i+1} J_{-i-1}}{(i+1)!} Q_i(x).$$

in [6].

- If $M_r = H_r(1,3;3,2)$, then according to Eq. (3), the recurrence relation of the Mersenne-Lagrange interpolation polynomial $\mathbb{M}_n(x)$ are given by

$$\mathbb{M}_n(x) = 2x + 1 + \sum_{i=1}^{n} \frac{(-2)^{i+1} M_{-i-1}}{(i+1)!} Q_i(x).$$

- If $T_r = H_r(2,3;3,2)$, then according to Eq. (3), the recurrence relation of the Fermat-Lagrange interpolation polynomial $\mathbb{T}_n(x)$ are given by

$$\mathbb{T}_n(x) = x + 2 + \sum_{i=1}^{n} \frac{(-2)^{i+1} T_{-i-1}}{(i+1)!} Q_i(x).$$

4 Conclusion

In this study, we examine the construction of polynomials of at most degree n using the first $n+1$ terms of the Horadam sequence by means Lagrange interpolation. We provide a detailed analysis of the recurrence relations and fundamental identities associated with the Horadam-Lagrange interpolation polynomials. Furthermore, we explore their structural properties and special cases, demonstrating their connections to well-known sequences such as Fibonacci, Lucas, Pell, Jacobsthal, Mersenne, and Fermat sequences.

Lagrange interpolation plays a crucial role in various disciplines, including numerical analysis, approximation theory, coding theory, and signal processing, where interpolation polynomials are employed for function approximation, error correction, and data reconstruction. Given the generality and flexibility of the Horadam sequence, the potential applications of Horadam-Lagrange interpolation polynomials in these fields merit further investigation. Future research may focus on their utility in computational mathematics, combinatorial structures, and cryptographic algorithms, as well as their effectiveness in modeling and approximation problems within applied sciences.

5 Declarations

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Competing Interests

The authors declare that they have no competing interests.

Ethical Approval

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Authors's Contributions

All authors contributed equally. All the authors read and approved the final manuscript.

Availability Data and Materials

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