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The Continuity by Order Derivative for Conformable Parabolic Equations with Exponential Nonlinearity

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Abstract

In this article, we examine the continuity according to the derivative order of conformable parabolic equations. We establish the existence and uniqueness of mild solutions to the problem with source function exponential non-linearity. We prove the existence and uniqueness of a mild solution based on Sobolev embeddings, the Banach space of all continuous functions, and the Banach fixed point theorem. In addition, we will address the nonlinear model's continuity issue and demonstrate the mild solution's convergence to the nonlinear problem when $\gamma' \to \gamma$.

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1 Introduction

In recent decades, fractional derivatives have been successful in describing viscoelastic processes. Some well-known models (such as the Oldroyd-B model) with classical derivatives may not adequately represent phenomena with memory effects in some cases. In such cases, fractional derivatives have been suggested as a replacement for classical derivatives. As a result, many fractional partial differential equation mathematical models have been extensively studied. We recommend [1, 2, 3, 4, 5, 6, 7, 8] for interesting papers on fractional partial differential equations and the references given there. In this paper, we investigate a pseudo-parabolic with the Caputo fractional derivative, It has a wide range of practical applications. The seepage

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of homogeneous fluids from a broken rock, the unidirectional distribution of long waves of nonlinear dispersion (which we can see in [9, 10]), and population aggregation (see [11]) are all examples of this. It was also thoroughly investigated in [12, 13]. Specifically, let $\gamma \in (0,1]$ and Ω be a smooth bounded domain of $\mathbb{R}^\ell$ with $\ell \in \{1,2,3\}$. In this article, we discuss the parabolic equation with a conformable derivative for the initial value problem as follows:

$$\begin{cases} \partial_t^\gamma u(x,t) + Au(x,t) &= F(x,t,u(x,t)), \quad (x,t) \in \Omega \times (0,T], \\ u(x,t) &= 0, \quad (x,t) \in \partial\Omega \times (0,T], \\ u(x,0) &= g(x), \quad x \in \Omega. \end{cases} \tag{1}$$

If $\gamma = 1$, We can easily see that (1) is the normal diffusion model. If $\gamma \in (0,1]$, the operator ∂_t^γ is called the conformable fractional derivative, which is introduced as follows:

$$\partial_t^\gamma u(x,t) = \lim_{h\to 0} \frac{u(t+ht^{1-\gamma}) - u(t)}{h}, \tag{2}$$

for any $u \in B$ where B is a Banach space. The conformable derivative can be viewed as a generalization of the classic limit definition of a function's derivative. and was first introduced in [14], the function g is the initial data which is described later. In problem (1) the function $F : \mathbb{R} \to \mathbb{R}$ is called exponential nonlinearities and satisfies some assumptions as follows:

$$\begin{cases} |F(u) - F(v)| &\le C_F \Big| |u|^{d-1}\exp(u^2) - |v|^{d-1}\exp(v^2) \Big| |u-v|, \\ |F(u)| &\le C_F |u|^d \exp(u^2), \end{cases} \tag{3}$$

where $d > 0$, and satisfies other conditions, which are considered after. According to our understanding, the nonlinear source functions satisfying assumptions like (3) are derived and developed by authors who wrote some interesting works [4, 15, 16, 17, 18, 19, 20]. For the reader's convenience, we want to mention some recent results related to the parabolic problem with exponential nonlinearity. The authors of [4] investigated the local existence and blows up for the parabolic equation with source function

$$F(u) = -u + \lambda[2\alpha_0 u \exp(\alpha_0 u^2)].$$

In [15], the authors looked at a nonlinear parabolic equation with a nonlinearity that was exponential. For small global-in-time solutions, they also calculated decay estimates and asymptotic behavior. Majdoub-Tayachi [19] gave some results for the Cauchy problem of the heat equation on $\mathbb{R}^N$, where $F : \mathbb{R} \to \mathbb{R}$ has an exponential growth at infinity with $F(0) = 0$. In this work, they got the global existence and decay estimates of solutions. We can easily see that if $\gamma = 1$, then the model of problem (1) is the well-known pseudo-parabolic equation. We rely on the above-mentioned works to construct an idea for problem (1) with the assumption condition (3). To the best of the author's knowledge, there are currently no articles that give results for the parabolic equation with exponential nonlinearity. The primary goal of this paper is to investigate the well-posedness of problems (1). The control of exponential nonlinearity is our biggest challenge in this paper. The main reason is that unlike in [21, 22] articles, we can't evaluate directly in L^p space. Therefore, we have to study some more techniques as well as some important inequalities in several papers like [17, 20].

2 Some fundamental fractional calculus outcomes

2.1 Function spaces

In this subsection, we introduce some important function spaces and notations. Noting that $L^2(\Omega)$, $H_0^1(\Omega)$, $H^2(\Omega)$ are understood in the usual sense. The symmetric uniform elliptic operator $A : L^2(\Omega) \to L^2(\Omega)$ is defined by

$$Au(x) = -\sum_{i=1}^{N} \frac{\partial}{\partial x_i}\left(A_{ij}(x)\frac{\partial}{\partial x_j}u(x)\right) + A(x)u(x,t), x \in \overline{\Omega},$$

where $D(A) = H_0^1(\Omega) \cap H^2(\Omega)$, with assumptions that $A(x) \in C(\overline{\Omega}, [0,\infty))$, $A_{ij} \in C^1(\overline{\Omega})$, $A_{ij} = A_{ji}$, $1 \le i, j \le N$, and there exist a positive constant $\widetilde{C} > 0$, for $x \in \overline{\Omega}$, $e = (e_1, e_2, \ldots, e_N) \subset R^N$, such that

$$\widetilde{C}\sum_{i=1}^{N} e_i^2 \le \sum_{1\le i,j\le N} e_i A_{ij}(x) e_j,$$

see e.g. [23, Section 2]. Let us recall that the following spectral problem

$$A\varphi_j(x) = \sigma_j \varphi_j(x) \text{ in } \Omega \quad \text{and} \quad \varphi_j(x) = 0 \text{ on } \partial\Omega, \tag{4}$$

where $\{\varphi_j\}_{j\in\mathbb{Z}^+}$ is a orthonormal basis of $L^2(\Omega)$, admits a family of eigenvalues $\{\sigma_j\}_{j\in\mathbb{Z}^+}$ satisfying

$$0 < \sigma_1 \le \sigma_2 \le \sigma_3 \le \ldots \le \sigma_j \le \ldots \nearrow \infty.$$

Definition 2.1. For $n > 0$, we define the Hilbert scale space $H^n(\Omega)$ in the following way

$$H^n(\Omega) = \left\{ u \in \mathbb{L}^2(\Omega) : \|u\|^2_{H^n(\Omega)} := \sum_{j=1}^{\infty} |\langle u, \varphi_j\rangle|^2 \sigma_j^n < \infty \right\},$$

where $\langle u, \varphi_j\rangle := \int_\Omega u(x)\varphi_j(x)\mathrm{d}x$ is the inner product of $\mathbb{L}^2(\Omega)$.

We can see that $H^0(\Omega) = L^2(\Omega)$. Let us denote that $H^q(\Omega)$ has dual space $H^{-q}(\Omega)$ which is a Hilbert space with respect to the norm

$$\|u\|^2_{H^{-n}(\Omega)} := \sum_{j=1}^{\infty} \sigma_j^{-n} \left|(u, \varphi_j)_{-n,n}\right|^2, \quad u \in H^{-n},$$

where $(u, \varphi_j)_{-n,n}$ is dual product between $H^{-n}(\Omega)$ and $H^n(\Omega)$. From Section 3 of [24] we can see that

$$(u, v)_{-n,n} = (u, v) \text{ if } u \in \mathbb{L}^2(\Omega) \text{ and } v \in H^n.$$

If $n \in (0, \frac{1}{2}) \cup (\frac{1}{2}, 1)$ and $\partial\Omega$ is sufficiently smooth boundary then $H^n(\Omega) = H_0^n(\Omega)$.

Definition 2.2. The Gamma function $\Gamma(a)$is defined by the integral

$$\Gamma(a) = \int_0^\infty e^{-t} t^{a-1} \mathrm{dt}.$$

Definition 2.3. Let $b_1 > 0; b_2 > 0$. The Beta function is denoted and defined as follows:

$$B(b_1, b_2) = \int_0^1 (1-\xi)^{b_1-1} \xi^{b_2-1} \mathrm{d}\xi.$$

Note that

$$B(a,b) = \frac{\Gamma(a)\Gamma(b)}{\Gamma(a+b)} \text{ and } B(1,b) = \frac{1}{b}. \tag{5}$$

Definition 2.4 (Vector-valued Lebesgue spaces)**.** Let $(X, \|\cdot\|_X)$ be a Banach space. For $1 \le p \le \infty$, we denote by $L^p(0,T;X)$ the space of all measurable functions $u : (0,T) \to X$ such that

$$\|u\|_{L^p(0,T;X)} = \begin{cases} \left(\int_0^T \|u(t)\|_X^p \, dt \right)^{1/p}, & 1 \le p < \infty, \\ \operatorname*{ess\,sup}_{t\in(0,T)} \|u(t)\|_X, & p = \infty. \end{cases}$$

Definition 2.5. We denote by $\mathbb{C}((0,T];\mathbb{X})$ the space of all continuous functions which map $(0,T]$ into $\mathbb{X}$. With the convention that $\mathbb{C}((0,T];\mathbb{X}) := \mathbb{C}((0,\infty);\mathbb{X})$ when $T \to \infty$.

For $0 < a < 1$, we define the the following weighted Banach space as below.

$$\mathbb{C}^a([0,T],\mathbb{X}) = \left\{ u \in \mathbb{C}([0,T], H^n(\Omega)) : \sup_{0<t\le T} t^a \|u(\cdot,t)\|_{\mathbb{X}} < \infty \right\},$$

with norm

$$\|u\|_{\mathbb{C}^a((0,T],\mathbb{X})} = \sup_{0<t\le T} t^a \|u(\cdot,t)\|_{\mathbb{X}}.$$

For any $b > 0$ we represent the Hölder continuous space of the exponent b as follows:

$$\mathbb{W}_b([0,T],\mathbb{X}) = \left\{ u \in \mathbb{C}([0,T],\mathbb{X}) : \sup_{0\le s<t\le T} \frac{\|u(\cdot,t) - u(\cdot,s)\|_{\mathbb{X}}}{|t-s|^b} \right\},$$

it has the standard equipment

$$\|u\|_{\mathbb{W}_b([0,T],\mathbb{X})} = \sup_{0\le s<t\le T} \frac{\|u(\cdot,t) - u(\cdot,s)\|_{\mathbb{X}}}{|t-s|^b}.$$

2.2 Representation of the mild solution

We present the formulation of mild solutions to the problem (1) in this subsection. Assume that the solution to problem (1) is unique, and apply the Fourier transform to obtain u in $L^2(\Omega)$ to the form

$$u(x,t) = \sum_{j=1}^{\infty} \langle u(\cdot,t), \varphi_j(\cdot) \rangle \, \varphi_j(x).$$

Therefore, we get

$$\partial_t^\gamma \langle u(\cdot,t), \varphi_j \rangle + \langle \mathscr{A} u(\cdot,t), \varphi_j \rangle = \langle F((\cdot,t,u(\cdot,t)), \varphi_j) \rangle.$$

Thanks to (4), we can deduce

$$\partial_t^\gamma \langle u(\cdot,t), \varphi_j \rangle + \sigma_j \langle u(\cdot,t), \varphi_j \rangle = \langle F((\cdot,t,u(\cdot,t)), \varphi_j) \rangle. \tag{6}$$

The solution of problem (6) is provided by Theorem 5, p. 318, Jaiswal and Bahuguna [25], and Theorem 3.3, p. 1795 [26].

$$\langle u(\cdot,t), \varphi_j \rangle = \exp\Big(-\sigma_j \frac{t^\gamma}{\gamma} \Big) \langle g, \varphi_j \rangle + \int_0^t s^{\gamma-1} \exp\Big(-\sigma_j \frac{t^\gamma - s^\gamma}{\gamma} \Big) \langle F(s,u(s)), \varphi_j \rangle \mathrm{d}s.$$

Therefore, we can get the mild solution formula for problem (1) as follows:

$$u_\gamma(x,t) = \sum_{j=1}^{\infty} \exp\Big(-\sigma_j \frac{t^\gamma}{\gamma}\Big)\langle g, \varphi_j\rangle \varphi_j(x) + \sum_{j=1}^{\infty}\Bigg[\int_0^t s^{\gamma-1}\exp\Big(-\sigma_j\frac{t^\gamma - s^\gamma}{\gamma}\Big)\langle F(s,u(s)),\varphi_j\rangle \mathrm{d}s\Bigg]\varphi_j(x). \tag{7}$$

To facilitate the technical processing of the problem, we restate the mild solution formula as follows:

$$u_\gamma(x,t) = P_\gamma(t)g(x) + \int_0^t Q_\gamma(t-s)F(u)(x,s)\mathrm{d}s, \tag{8}$$

where $P_\gamma(t)$ and $Q_\gamma(t)$ are defined by

$$P_\gamma(t)\Psi(x) = \sum_{j=1}^{\infty}\exp\Big(-\sigma_j\frac{t^\gamma}{\gamma}\Big)\langle\Psi,\varphi_j\rangle\varphi_j(x), \quad \Psi \in L^2(\Omega),$$

$$Q_\gamma(t-s)\Psi(x) = \sum_{j=1}^{\infty}\Bigg[s^{\gamma-1}\exp\Big(-\sigma_j\frac{t^\gamma - s^\gamma}{\gamma}\Big)\langle\Psi,\varphi_j\rangle\Bigg]\varphi_j(x), \quad \Psi \in L^2(\Omega).$$

Definition 2.6. For $1 \geq p$, if a function $u \in \mathbb{L}^p(0,T,\mathbb{L}^2(\Omega))$ satisfies equation (7) for every almost where $0 \leq t \leq T$, then it is called a mild solution of Problem (1)

2.3 Some basic theorems related to the problem

Theorem 2.7. *If $0 < \epsilon_1, \epsilon_2 \leq 1, 0 < \epsilon_3$ are positive numbers and $0 < \gamma_0 \leq \gamma \leq 1$ then for all $t \in (0,T]$ we can the following estimate*

$$\Big|\exp\Big(-\sigma_j\frac{t^\gamma}{\gamma}\Big) - \exp(-\sigma_j t)\Big| \leq C(\gamma_0,\epsilon_1,\epsilon_2,T)\sigma_j^{\epsilon_1}t^{\epsilon_1(\gamma-\epsilon_2)}|\overline{C}(1-\gamma,\epsilon_2)|^{\epsilon_1},$$

where $\overline{C}(1-\gamma,\epsilon_2) := (1-\gamma)^{\epsilon_2} + (1-\gamma) + (T^{1-\gamma}-1) \xrightarrow{\gamma\to 1} 0$. Moreover, we also obtain

$$\Big|\exp\Big(-\sigma_j\frac{t^\gamma}{\gamma}\Big) - \exp(-\sigma_j t)\Big| \leq C(\gamma_0,\epsilon_1,\epsilon_2,\epsilon_3,T)\sigma_j^{\epsilon_1-\epsilon_3}t^{\epsilon_1(\gamma-\epsilon_2)-\epsilon_3}|\overline{C}(1-\gamma,\epsilon_2)|^{\epsilon_1}.$$

In addition, with $0 < \gamma_0 \leq \gamma \leq \gamma' \leq 1$, we can deduce that

$$\Big|\exp\Big(-\sigma_j\frac{t^{\gamma'}}{\gamma'}\Big) - \exp(-\sigma_j\frac{t^\gamma}{\gamma})\Big| \leq C(\gamma_0,\epsilon_1,\epsilon_2,\epsilon_3,\sigma_1,T)\overline{C}(\gamma'-\gamma,\epsilon_1,\epsilon_2)t^{\epsilon_1(\gamma-\epsilon_2)-\gamma'\epsilon_3}, \tag{9}$$

where $\overline{C}(\gamma'-\gamma,\epsilon_1,\epsilon_2) := \Big[(\gamma'-\gamma) + (\gamma'-\gamma)^{\epsilon_2} + (T^{\gamma'-\gamma}-1)\Big]^{\epsilon_1} \xrightarrow{\gamma'\to\gamma} 0$.
And with the assumption that $\epsilon_2 \leq \gamma_0 \leq \gamma$ we can get the estimate as follows:

$$\Big|\exp\Big(-\sigma_j\frac{t^{\gamma'}-s^{\gamma'}}{\gamma'}\Big) - \exp(-\sigma_j\frac{t^\gamma - s^\gamma}{\gamma})\Big| \leq C(\gamma_0,\epsilon_1,\epsilon_2,\epsilon_3,\sigma_1,T)\max\Big[\frac{(t^{\gamma'}-s^{\gamma'})^{-\epsilon_3}}{\gamma'^{-\epsilon_3}}, \frac{(t^\gamma - s^\gamma)^{-\epsilon_3}}{\gamma^{-\epsilon_3}}\Big]\overline{C}(\gamma'-\gamma,\epsilon_1,\epsilon_2). \tag{10}$$

Proof of Theorem 2.7. The proof is divided into two parts.

Part 1. Applying the inequality $|\exp(-a)-\exp(-b)| \leq C(\epsilon_1)|a-b|^{\epsilon_1}$, for any $0<\epsilon_1 \leq 1$ we can deduce

$$\Big|\exp\Big(-\sigma_j\frac{t^\gamma}{\gamma}\Big)-\exp(-\sigma_j t)\Big| \leq C(\epsilon_1)\sigma_j^{\epsilon_1}\Big|\frac{t^\gamma}{\gamma}-t\Big|^{\epsilon_1}. \tag{11}$$

We take two cases into consideration in order to eliminate the absolute value sign on the right side of the inequality above.
Case 1. If $0<t\leq 1$, we can easily see that

$$\Big|1-\gamma t^{1-\gamma}\Big| = 1-\gamma t^{1-\gamma} = 1-\gamma+\gamma-\gamma t^{1-\gamma} \leq 1-\gamma+1-\exp\big[-(1-\gamma)\ln(t^{-1})\big].$$

Using the inequality $1-\exp(-a)\leq C(\epsilon_2)a^{\epsilon_2}$ with $0<\epsilon_2\leq 1$, we obtain estimate as follow:

$$\Big|1-\gamma t^{1-\gamma}\Big| = 1-\gamma t^{1-\gamma} \leq 1-\gamma+C(\epsilon_2)(1-\gamma)^{\epsilon_2}t^{-\epsilon_2}. \tag{12}$$

Since $1-\gamma \leq (1-\gamma)t^{-\epsilon_2}$ for any $t\in(0,1]$ and $\epsilon_2>0$. Therefore, we have

$$\frac{t^\gamma}{\gamma}\Big|1-\gamma t^{1-\gamma}\Big| \leq \frac{1}{\gamma}\max[C(\epsilon_2),1]\Big[(1-\gamma)^{\epsilon_2}+(1-\gamma)\Big]t^{\gamma-\epsilon_2}.$$

In this case, we can find that

$$\Big|\frac{t^\gamma}{\gamma}-t\Big| \leq C_1(\gamma_0,\epsilon_2)\Big[(1-\gamma)^{\epsilon_2}+(1-\gamma)\Big]t^{\gamma-\epsilon_2}. \tag{13}$$

Case 2. If $1\leq t\leq T$, we also get

$$\Big|\frac{t^\gamma}{\gamma}-t\Big| = \frac{t^\gamma}{\gamma}|1-\gamma t^{1-\gamma}| = \frac{t^\gamma}{\gamma}|1-\gamma+\gamma-\gamma t^{1-\gamma}|.$$

Thanks to inequality $|a+b|\leq|a|+|b|$, we obtain

$$\begin{aligned}\Big|\frac{t^\gamma}{\gamma}-t\Big| &\leq \frac{t^{\gamma-\epsilon_2}}{\gamma}t^{\epsilon_2}\Big[(1-\gamma)+\gamma(t^{1-\gamma}-1)\Big],\\ &\leq C_2(\gamma_0,\epsilon_2,T)\Big[(1-\gamma)+(T^{1-\gamma}-1)\Big]t^{\gamma-\epsilon_2}.\end{aligned} \tag{14}$$

From (13) and (14), we can deduce

$$\Big|\frac{t^\gamma}{\gamma}-t\Big| \leq C(\gamma_0,\epsilon_2,T)\overline{C}(1-\gamma,\epsilon_2)t^{\gamma-\epsilon_2}. \tag{15}$$

Here $C(\gamma_0,\epsilon_2),T$ is a constant that only depends on γ_0,ϵ_2,T.
Therefore, from (11), we obtain

$$\Big|\exp\Big(-\sigma_j\frac{t^\gamma}{\gamma}\Big)-\exp(-\sigma_j t)\Big| \leq C(\gamma_0,\epsilon_1,\epsilon_2,T)\sigma_j^{\epsilon_1}t^{\epsilon_1(\gamma-\epsilon_2)}|\overline{C}(1-\gamma,\epsilon_2)|^{\epsilon_1},$$

where $C(\gamma_0,\epsilon_1,\epsilon_2,T)$ is a constant that depends on $\gamma_0,\epsilon_1,\epsilon_2,T$.
Part 2. For $0<\varepsilon\leq 1$ and $\varepsilon_3>0$, using the inequality

$$|\exp(-a)-\exp(-b)| \leq C(\epsilon_1,\epsilon_3)\max(a^{-\epsilon_3},b^{-\epsilon_3})|a-b|^{\epsilon_1},$$

we have

$$\begin{aligned}\Big|\exp\Big(-\sigma_j\frac{t^\gamma}{\gamma}\Big)-\exp(-\sigma_j t)\Big| &\le C(\epsilon_1,\epsilon_3)\max(t^{-\gamma\epsilon_3},t^{-\epsilon_3})\sigma_j^{\epsilon_1-\epsilon_3}\Big|\frac{t^\gamma}{\gamma}-t\Big|^{\epsilon_1},\\ &\le C(\epsilon_1,\epsilon_3)\max(t^{\epsilon_3(1-\gamma)}t^{-\epsilon_3},t^{-\epsilon_3})\sigma_j^{\epsilon_1-\epsilon_3}\Big|\frac{t^\gamma}{\gamma}-t\Big|^{\epsilon_1},\\ &\le C(\epsilon_1,\epsilon_3)\max(T^{\epsilon_3(1-\gamma)},1)\sigma_j^{\epsilon_1-\epsilon_3}|C(\gamma_0,\epsilon_2,T)|^{\epsilon_1}|\overline{C}(1-\gamma,\epsilon_2)|^{\epsilon_1}t^{\epsilon_1(\gamma-\epsilon_2)-\epsilon_3},\\ &\le C(\gamma_0,\epsilon_1,\epsilon_2,\epsilon_3,T)\sigma_j^{\epsilon_1-\epsilon_3}|\overline{C}(1-\gamma,\epsilon_2)|^{\epsilon_1}t^{\epsilon_1(\gamma-\epsilon_2)-\epsilon_3},\end{aligned}$$

where $C(\gamma_0,\epsilon_1,\epsilon_2,\epsilon_3,T):=C(\epsilon_1,\epsilon_3)\max(T^{\epsilon_3(1-\gamma)},1)|C(\gamma_0,\epsilon_2,T)|^{\epsilon_1}$ is a constant depends on $\gamma_0,\epsilon_1,\epsilon_2,\epsilon_3,T$.
Part 3. Similar to part 2, apply the inequality

$$|\exp(-a)-\exp(-b)|\le C(\epsilon_1,\epsilon_3)\max(a^{-\epsilon_3},b^{-\epsilon_3})|a-b|^{\epsilon_1},$$

for any $0<\epsilon_1\le 1$ and $0<\gamma_0\le\gamma<\gamma'\le 1$, we obtain

$$\begin{aligned}\Big|\exp\Big(-\sigma_j\frac{t^{\gamma'}}{\gamma'}\Big)-\exp(-\sigma_j\frac{t^\gamma}{\gamma})\Big| &\le C(\epsilon_1,\epsilon_3)\max(\frac{t^{-\gamma'\epsilon_3}}{\gamma'^{-\epsilon_3}},\frac{t^{-\gamma\epsilon_3}}{\gamma^{-\epsilon_3}})\sigma_j^{\epsilon_1-\epsilon_3}\Big|\frac{t^{\gamma'}}{\gamma'}-\frac{t^\gamma}{\gamma}\Big|^{\epsilon_1},\\ &\le C(\epsilon_1,\epsilon_3)\max(\frac{1}{\gamma'^{-\epsilon_3}},\frac{T^{(\gamma'-\gamma)\epsilon_3}}{\gamma^{-\epsilon_3}})\sigma_j^{\epsilon_1-\epsilon_3}t^{-\gamma'\epsilon_3}\Big|\frac{t^{\gamma'}}{\gamma'}-\frac{t^\gamma}{\gamma}\Big|^{\epsilon_1}.\end{aligned}$$

With the same technique as Part 1, from (12), we have
If $0<t\le 1$ then

$$\Big|1-\frac{\gamma}{\gamma'}t^{\gamma'-\gamma}\Big|\le|1-\frac{\gamma}{\gamma'}|+\frac{\gamma}{\gamma'}C(\epsilon_2)(\gamma'-\gamma)^{\epsilon_2}t^{-\epsilon_2}.$$

Therefore, we can deduce

$$\begin{aligned}\frac{t^\gamma}{\gamma}\Big|1-\frac{\gamma}{\gamma'}t^{\gamma'-\gamma}\Big| &\le\frac{1}{\gamma}\max[\frac{\gamma}{\gamma'}C(\epsilon_2),\frac{1}{\gamma'}]\Big[(\gamma'-\gamma)+(\gamma'-\gamma)^{\epsilon_2}\Big]t^{\gamma-\epsilon_2},\\ &\le\mathfrak{C}_1(\gamma_0,\epsilon_2)\Big[(\gamma'-\gamma)+(\gamma'-\gamma)^{\epsilon_2}\Big]t^{\gamma-\epsilon_2}.\end{aligned}$$

If $1<t\le T$ then we also find that

$$\begin{aligned}\frac{t^\gamma}{\gamma}\Big|1-\frac{\gamma}{\gamma'}t^{\gamma'-\gamma}\Big| &\le\frac{t^{\gamma-\epsilon_2}t^{\epsilon_2}}{\gamma}\Big[(\frac{\gamma'-\gamma}{\gamma'})+\frac{\gamma}{\gamma'}(t^{\gamma'-\gamma}-1)\Big],\\ &\le\mathfrak{C}_2(\gamma_0,\epsilon_2,T)\Big[(\gamma'-\gamma)+(T^{\gamma'-\gamma}-1)\Big]t^{\gamma-\epsilon_2}.\end{aligned}\tag{16}$$

Hence, for all $t\in(0,T]$ we obtain

$$\Big|\frac{t^{\gamma'}}{\gamma'}-\frac{t^\gamma}{\gamma}\Big|\le C(\gamma_0,\epsilon_2,T)\Big[(\gamma'-\gamma)+(\gamma'-\gamma)^{\epsilon_2}+(T^{\gamma'-\gamma}-1)\Big]t^{\gamma-\epsilon_2}.\tag{17}$$

Therefore, suppose that $\epsilon_1\le\epsilon_3$ we can deduce

$$\Big|\exp\Big(-\sigma_j\frac{t^{\gamma'}}{\gamma'}\Big)-\exp(-\sigma_j\frac{t^\gamma}{\gamma})\Big|\le C(\gamma_0,\epsilon_1,\epsilon_2,\epsilon_3,\sigma_1,T)\overline{C}(\gamma'-\gamma,\epsilon_1,\epsilon_2)t^{\epsilon_1(\gamma-\epsilon_2)-\gamma'\epsilon_3},$$

where

$$C(\gamma_0,\epsilon_1,\epsilon_2,\epsilon_3,\sigma_1,T) := C(\epsilon_1,\epsilon_3)\max\Big(\frac{1}{\gamma'^{-\epsilon_3}},\frac{T^{(\gamma'-\gamma)\epsilon_3}}{\gamma^{-\epsilon_3}}\Big)C^{\epsilon_1}(\gamma_0,\epsilon_2,T)\sigma_1^{\epsilon_1-\epsilon_3},$$
$$\overline{C}(\gamma'-\gamma,\epsilon_1,\epsilon_2) := \Big[(\gamma'-\gamma)+(\gamma'-\gamma)^{\epsilon_2}+(T^{\gamma'-\gamma}-1)\Big]^{\epsilon_1}.$$

Part 4. Using the same proof technique as part 3, Using inequality (17), we can deduce

$$\begin{aligned}&\Big|\exp\Big(-\sigma_j\frac{t^{\gamma'}-s^{\gamma'}}{\gamma'}\Big)-\exp(-\sigma_j\frac{t^{\gamma}-s^{\gamma}}{\gamma})\Big|,\\ &\le C(\epsilon_1,\epsilon_3)\max\Big[\frac{(t^{\gamma'}-s^{\gamma'})^{-\epsilon_3}}{\gamma'^{-\epsilon_3}},\frac{(t^{\gamma}-s^{\gamma})^{-\epsilon_3}}{\gamma^{-\epsilon_3}}\Big]\sigma_j^{\epsilon_1-\epsilon_3}\Big|\frac{t^{\gamma'}-s^{\gamma'}}{\gamma'}-\frac{t^{\gamma}-s^{\gamma}}{\gamma}\Big|^{\epsilon_1},\\ &\le C(\epsilon_1,\epsilon_3)\max\Big[\frac{(t^{\gamma'}-s^{\gamma'})^{-\epsilon_3}}{\gamma'^{-\epsilon_3}},\frac{(t^{\gamma}-s^{\gamma})^{-\epsilon_3}}{\gamma^{-\epsilon_3}}\Big]\sigma_j^{\epsilon_1-\epsilon_3}\Big[\Big|\frac{t^{\gamma'}}{\gamma'}-\frac{t^{\gamma}}{\gamma}\Big|+\Big|\frac{s^{\gamma'}}{\gamma'}-\frac{s^{\gamma}}{\gamma}\Big|\Big]^{\epsilon_1}.\end{aligned}$$

Using inequality $(a+b)^\epsilon \le C(\epsilon)(a^\epsilon+b^\epsilon)$, with assumption that we obtain

$$\begin{aligned}&\Big|\exp\Big(-\sigma_j\frac{t^{\gamma'}-s^{\gamma'}}{\gamma'}\Big)-\exp(-\sigma_j\frac{t^{\gamma}-s^{\gamma}}{\gamma})\Big|,\\ &\le C(\epsilon_1,\epsilon_3)\max\Big[\frac{(t^{\gamma'}-s^{\gamma'})^{-\epsilon_3}}{\gamma'^{-\epsilon_3}},\frac{(t^{\gamma}-s^{\gamma})^{-\epsilon_3}}{\gamma^{-\epsilon_3}}\Big]\sigma_j^{\epsilon_1-\epsilon_3},\\ &\qquad\times C(\gamma_0,\epsilon_2,T)C(\epsilon_1)\overline{C}(\gamma'-\gamma,\epsilon_1,\epsilon_2)(t^{\epsilon_1(\gamma-\epsilon_2)}+s^{\epsilon_1(\gamma-\epsilon_2)}),\end{aligned}$$

with condition $\gamma \ge \gamma_0 \ge \epsilon_2$, we can obtain the following estimate

$$\begin{aligned}&\Big|\exp\Big(-\sigma_j\frac{t^{\gamma'}-s^{\gamma'}}{\gamma'}\Big)-\exp(-\sigma_j\frac{t^{\gamma}-s^{\gamma}}{\gamma})\Big|,\\ &\le 2C(\epsilon_1,\epsilon_3)\max\Big[\frac{(t^{\gamma'}-s^{\gamma'})^{-\epsilon_3}}{\gamma'^{-\epsilon_3}},\frac{(t^{\gamma}-s^{\gamma})^{-\epsilon_3}}{\gamma^{-\epsilon_3}}\Big]\sigma_1^{\epsilon_1-\epsilon_3}C(\gamma_0,\epsilon_2,T)C(\epsilon_1)\overline{C}(\gamma'-\gamma,\epsilon_1,\epsilon_2)T^{\epsilon_1(\gamma-\epsilon_2)},\\ &\le C(\gamma_0,\epsilon_1,\epsilon_2,\epsilon_3,\sigma_1,T)\max\Big[\frac{(t^{\gamma'}-s^{\gamma'})^{-\epsilon_3}}{\gamma'^{-\epsilon_3}},\frac{(t^{\gamma}-s^{\gamma})^{-\epsilon_3}}{\gamma^{-\epsilon_3}}\Big]\overline{C}(\gamma'-\gamma,\epsilon_1,\epsilon_2).\end{aligned}$$

□

Theorem 2.8. *Assume that $0<\gamma_0\le\gamma\le 1$ and $t\in(0,T]$. Under these conditions, we have:*
a) If $\epsilon_2>0$ then we always have

$$\int_0^t |s^{\gamma-1}-1|\mathrm{d}s \le C(\gamma_0,\epsilon_2)t^{\gamma-\epsilon_2}\overline{C}(\gamma,\epsilon_2)+2\frac{1-\gamma}{\gamma}.$$

b) For $\beta_2\in(0,1)$ and $\beta_3>0$, we get the following estimate

$$\int_0^t |s^{\gamma-1}-1|(t-s)^{-\beta_2}\mathrm{d}s \le C(\beta_3,T)t^{\gamma-\beta_2-\beta_3}\overline{\mathfrak{B}}(\gamma,\beta_2,\beta_3,T).$$

c) Moreover, for any $0<\beta_2<\gamma_0<1$ and $\beta_3>0$, we also obtain

$$\int_0^t |s^{\gamma-1}-1|s^{-\beta_2}\mathrm{d}s \le C(\gamma_0,\beta_2,\beta_3,T)t^{\gamma-\beta_2-\beta_3}\overline{C}(\gamma,\beta_2,\beta_3,T).$$

Proof. Starting with part a, we observe the integral term below.

$$\int_0^t \left|s^{\gamma-1}-1\right|ds.$$

If $t \in (0,1]$ then $s^{\gamma-1} \geq 1$ for all $s \in [0,t]$. Therefore, using (15) we can easily see that

$$\int_0^t \left|s^{\gamma-1}-1\right|ds = \int_0^t \left(s^{\gamma-1}-1\right)ds = \frac{t^\gamma}{\gamma} - t \leq C(\gamma_0,\epsilon_2,T)\overline{C}(1-\gamma,\epsilon_2)t^{\gamma-\epsilon_2}. \tag{18}$$

Similarly, if $t \geq 1$ then we have

$$\begin{aligned}\int_0^t \left|s^{\gamma-1}-1\right|ds &= \int_0^1 (s^{\gamma-1}-1)ds + \int_1^t \left(1-s^{\gamma-1}\right)ds,\\ &= t - \frac{t^\gamma}{\gamma} + 2\frac{1-\gamma}{\gamma} \leq C(\gamma_0,\epsilon_2,T)\overline{C}(\gamma,\epsilon_2)t^{\gamma-\epsilon_2} + 2\frac{1-\gamma}{\gamma}.\end{aligned} \tag{19}$$

From (18) and (19), we confirm that for all $t \in (0,T]$

$$\int_0^t \left|s^{\gamma-1}-1\right|ds \leq C(\gamma_0,\epsilon_2)t^{\gamma-\epsilon_2}\overline{C}(\gamma,\epsilon_2) + 2\frac{1-\gamma}{\gamma}.$$

Secondly, we consider part b) with the term integral as follows:

$$\int_0^t \left|s^{\gamma-1}-1\right|(t-s)^{-\beta_2}ds.$$

If $t \in (0,1]$ then we can easily that $s^{\gamma-1} > 1$ for all $s \in (0,t)$. With $\beta_2 \in (0,1)$, we deduce

$$\begin{aligned}\int_0^t \left|s^{\gamma-1}-1\right|(t-s)^{-\beta_2}ds &= \int_0^t \left(s^{\gamma-1}-1\right)(t-s)^{-\beta_2}ds,\\ &= t^{-\beta_2}\int_0^t s^{\gamma-1}(1-\frac{s}{t})^{-\beta_2}ds - \int_0^t (t-s)^{-\beta_2}ds.\end{aligned}$$

Using the transformation of variables $r = \frac{s}{t}$ into integral functions of one variable, we get

$$\begin{aligned}\int_0^t \left|s^{\gamma-1}-1\right|(t-s)^{-\beta_2}ds &= t^{\gamma-\beta_2}\int_0^1 r^{\gamma-1}(1-r)^{-\beta_2}dr - \frac{t^{1-\beta_2}}{1-\beta_2},\\ &= t^{\gamma-\beta_2}B(\gamma,1-\beta_2) - \frac{t^{1-\beta_2}}{1-\beta_2},\\ &\leq t^{\gamma-\beta_2}\left|B(\gamma,1-\beta_2) - \frac{1}{1-\beta_2}\right| + \frac{1}{1-\beta_2}\left|t^{\gamma-\beta_2} - t^{1-\beta_2}\right|.\end{aligned} \tag{20}$$

Using Lemma 3.2 as in the paper [27], we can find that

$$\left|t^{\gamma-\beta_2} - t^{1-\beta_2}\right| \leq \max(T^{1+2\beta_3},1)\overline{C}_{\beta_3}t^{\gamma-\beta_2-\beta_3}(1-\gamma)^{\beta_3}. \tag{21}$$

Combine (20)-(21), we obtain the following estimate

$$\int_0^t \left|s^{\gamma-1}-1\right|(t-s)^{-\beta_2}\mathrm{d}s \le C_1(\beta_3,T)t^{\gamma-\beta_2-\beta_3}\Big(\Big|B(\gamma,1-\beta_2)-\frac{1}{1-\beta_2}\Big|+\frac{1}{1-\beta_2}(1-\gamma)^{\beta_3}\Big),$$
$$\le C_1(\beta_3,T)t^{\gamma-\beta_2-\beta_3}\overline{B}_1(\gamma,\beta_2,\beta_3). \quad (22)$$

where

$$\overline{B}_1(\gamma,\beta_2,\beta_3) := \Big|B(\gamma,1-\beta_2)-\frac{1}{1-\beta_2}\Big|+\frac{1}{1-\beta_2}(1-\gamma)^{\beta_3}.$$

Here, from (5) we can easily see that

$$\lim_{\gamma\to 1^-}\overline{B}_1(\gamma,\beta_2,\beta_3) = \lim_{\gamma\to 1^-}\left(\Big|B(\gamma,1-\beta_2)-\frac{1}{1-\beta_2}\Big|+(1-\gamma)^{\beta_3}\right)=0.$$

If $t\ge 1$, using the change of variables $r=\frac{s}{t}$ and the inequality $|ab-1|\le |a(b-1)|+|a-1|$, we get

$$\begin{aligned}
\int_0^t \left|s^{\gamma-1}-1\right|(t-s)^{-\beta_2}\mathrm{d}s &= t^{1-\beta_2}\int_0^1 \left|t^{\gamma-1}r^{\gamma-1}-1\right|(1-r)^{-\beta_2}\mathrm{d}r,\\
&\le t^{1-\beta_2}\left[\int_0^1 r^{\gamma-1}\left|t^{\gamma-1}-1\right|(1-r)^{-\beta_2}\mathrm{d}r+\int_0^1\left|r^{\gamma-1}-1\right|(1-r)^{-\beta_2}\mathrm{d}r\right],\\
&\le t^{1-\beta_2}\left[(1-t^{\gamma-1})\int_0^1 r^{\gamma-1}(1-r)^{-\beta_2}\mathrm{d}r+\int_0^1 (r^{\gamma-1}-1)(1-r)^{-\beta_2}\mathrm{d}r\right],\\
&\le t^{1-\beta_2}\left[(1-t^{\gamma-1})B(\gamma,1-\beta_2)+\int_0^1 r^{\gamma-1}(1-r)^{-\beta_2}\mathrm{d}r-\int_0^1 (1-r)^{-\beta_2}\mathrm{d}r\right],\\
&\le t^{1-\beta_2}\left[(1-t^{\gamma-1})B(\gamma,1-\beta_2)+B(\gamma,1-\beta_2)-\frac{1}{1-\beta_2}\right],\\
&\le t^{\gamma-\beta_2-\beta_3}\left[t^{\beta_3}(t^{1-\gamma}-1)B(\gamma,1-\beta_2)+t^{1-\gamma+\beta_3}\Big|B(\gamma,1-\beta_2)-\frac{1}{1-\beta_2}\Big|\right].
\end{aligned}$$

Therefore, we obtain

$$\int_0^t \left|s^{\gamma-1}-1\right|(t-s)^{-\beta_2}\mathrm{d}s \le C_2(\beta_3,T)t^{\gamma-\beta_2-\beta_3}\overline{B}_2(\gamma,\beta_2,\beta_3,T), \quad (23)$$

where

$$\overline{B}_2(\gamma,\beta_2,\beta_3,T) := (T^{1-\gamma}-1)B(\gamma,1-\beta_2)+\Big|B(\gamma,1-\beta_2)-\frac{1}{1-\beta_2}\Big|. \quad (24)$$

Here, from (5) we also see that

$$\lim_{\gamma\to 1^-}\overline{B}_2(\gamma,\beta_2,\beta_3,T) = \lim_{\gamma\to 1^-}\left((T^{1-\gamma}-1)B(\gamma,1-\beta_2)+\Big|B(\gamma,1-\beta_2)-\frac{1}{1-\beta_2}\Big|\right)=0.$$

From (22) and (23), we get the following estimate

$$\int_0^t \left|s^{\gamma-1}-1\right|(t-s)^{-\beta_2}\mathrm{d}s \leq C(\beta_3,T)t^{\gamma-\beta_2-\beta_3}\overline{\mathfrak{B}}(\gamma,\beta_2,\beta_3,T), \tag{25}$$

where $C(\beta_3,T) := \max(C_1(\beta_3,T), C_2(\beta_3,T))$ and $\overline{\mathfrak{B}}(\gamma,\beta_2,\beta_3,T) := \max(\overline{B}_1(\gamma,\beta_2,\beta_3), \overline{B}_2(\gamma,\beta_2,\beta_3,T))$.

Finally, part b's proof and part c's proof are related. We consider the integral below

$$\int_0^t |s^{\gamma-1}-1|s^{-\beta_2}\mathrm{d}s.$$

If $t \in (0,1]$, with $\beta_2 \in (0,1)$ and using inequality (21) we always get

$$\begin{aligned}\int_0^t |s^{\gamma-1}-1|s^{-\beta_2}\mathrm{d}s = \int_0^t (s^{\gamma-1}-1)s^{-\beta_2}\mathrm{d}s &= \frac{t^{\gamma-\beta_2}}{\gamma-\beta_2} - \frac{t^{1-\beta_2}}{1-\beta_2},\\
&\leq t^{\gamma-\beta_2}\Big|\frac{1}{\gamma-\beta_2} - \frac{1}{1-\beta_2}\Big| + \frac{1}{1-\beta_2}\big|t^{\gamma-\beta_2} - t^{1-\beta_2}\big|,\\
&\leq t^{\gamma-\beta_2}\frac{1-\gamma}{(\gamma-\beta_2)(1-\beta_2)} + \frac{1}{1-\beta_2}\max(T^{1+2\beta_3},1)\overline{C}_{\beta_3}t^{\gamma-\beta_2-\beta_3}(1-\gamma)^{\beta_3},\\
&\leq \overline{C}_1(\gamma_0,\beta_2,\beta_3,T)t^{\gamma-\beta_2-\beta_3}[(1-\gamma)^{\beta_3}+(1-\gamma)]. \end{aligned}\tag{26}$$

If $t > 1$, let $s = rt$ and apply triangle inequality we can also deduce that

$$\begin{aligned}\int_0^t |s^{\gamma-1}-1|s^{-\beta_2}\mathrm{d}s &= \int_0^1 \left|t^{\gamma-1}r^{\gamma-1}-1\right|t^{-\beta_2}r^{-\beta_2}t\mathrm{d}r,\\
&\leq t^{1-\beta_2}\int_0^1 r^{\gamma-1}\Big(1-t^{\gamma-1}\Big)r^{-\beta_2}\mathrm{d}r + t^{1-\beta_2}\int_0^1 r^{-\beta_2}\Big(r^{\gamma-1}-1\Big)\mathrm{d}r.\end{aligned}$$

With condition $\beta_2 < \gamma_0$, we obtain

$$\begin{aligned}\int_0^t |s^{\gamma-1}-1|s^{-\beta_2}\mathrm{d}s &\leq t^{1-\beta_2}(1-t^{\gamma-1})\frac{1}{\gamma-\beta_2} + t^{1-\beta_2}\frac{1-\gamma}{(\gamma-\beta_2)(1-\beta_2)},\\
&\leq \frac{t^{\gamma-\beta_2-\beta_3}}{\gamma-\beta_2}\Big[t^{\beta_3}(t^{1-\gamma}-1) + t^{1-\gamma+\beta_3}\frac{1-\gamma}{1-\beta_2}\Big],\\
&\leq \overline{C}_2(\gamma_0,\beta_2,\beta_3)t^{\gamma-\beta_2-\beta_3}\Big[T^{\beta_3}(T^{1-\gamma}-1) + \frac{T^{1-\gamma+\beta_3}}{1-\beta_2}(1-\gamma)\Big].\end{aligned}\tag{27}$$

From (26)-(27), we get the following estimate

$$\int_0^t |s^{\gamma-1}-1|s^{-\beta_2}\mathrm{d}s \leq C(\gamma_0,\beta_2,\beta_3,T)t^{\gamma-\beta_2-\beta_3}\overline{C}(\gamma,\beta_2,\beta_3),$$

where

$$\overline{C}(\gamma,\beta_2,\beta_3,T) := \Big[(1-\gamma)^{\beta_3}+(1-\gamma)\Big] + \Big[T^{\beta_3}(T^{1-\gamma}-1) + \frac{T^{1-\gamma+\beta_3}}{1-\beta_2}(1-\gamma)\Big]. \tag{28}$$

We can show that $\lim\limits_{\gamma\to 1^-}\overline{C}(\gamma,\beta_2,\beta_3,T) = 0$. □

The following lemmas are introduced in relation to the $P_\gamma(t)$ and $Q_\gamma(t)$ operator which are already defined in Section 2.2 as below.

$$P_\gamma(t)\Psi(x) = \sum_{j=1}^{\infty} \exp\Big(-\sigma_j \frac{t^\gamma}{\gamma}\Big)\langle \Psi, \varphi_j\rangle \varphi_j(x), \quad \Psi \in L^2(\Omega),$$
$$Q_\gamma(t-s)\Psi(x) = \sum_{j=1}^{\infty} \Big[s^{\gamma-1}\exp\Big(-\sigma_j \frac{t^\gamma - s^\gamma}{\gamma}\Big)\langle \Psi, \varphi_j\rangle \Big] \varphi_j(x), \quad \Psi \in L^2(\Omega).$$

Lemma 2.9. *Let $0 < \xi \le 1, n \le 2$ and $0 < \gamma_0 < \gamma < \gamma_1 < 1$. For all $t \ge 0$, we get the following estimate*

$$\|P_\gamma(t)\Psi\|_{H^n(\Omega)} \le C(\xi, \gamma_1, \sigma_1) t^{-\xi\gamma} \|\Psi\|_{H^n(\Omega)}. \tag{29}$$
$$\|Q_\gamma(t-s)\Psi\|_{H^n(\Omega)} \le C(\xi, \gamma_1, \sigma_1) s^{\gamma-1}(t^\gamma - s^\gamma)^{-\xi} \|\Psi\|_{H^n(\Omega)}. \tag{30}$$

Proof. Using inequality $\exp(-z) \le C_\xi z^{-\xi}$ for all $0 < \xi \le 1$ and from formula (8) we can deduce

$$\|P_\gamma(t)\Psi\|^2_{H^n(\Omega)} = \sum_{j=1}^{\infty} \sigma_j^n \Big[\exp\Big(-\sigma_j \frac{t^\gamma}{\gamma}\Big)\Big]^2 \langle \Psi, \varphi_j\rangle^2 \le C_\xi^2 \gamma^{2\xi} t^{-2\xi\gamma} \sum_{j=1}^{\infty} \sigma_j^{n-2\xi} \langle \Psi, \varphi_j\rangle^2,$$
$$\le \frac{C_\xi^2 \gamma_1^{2\xi}}{\sigma_1^{2\xi}} t^{-2\xi\gamma} \|\Psi\|^2_{H^n(\Omega)}.$$

Therefore, we can get

$$\|P_\gamma(t)\Psi\|_{H^n(\Omega)} \le C(\xi, \gamma_1, \sigma_1) t^{-\xi\gamma} \|\Psi\|_{H^n(\Omega)},$$

where $C(\xi, \gamma_1, \sigma_1) := C_\xi \gamma_1^\xi \sigma_1^{-\xi}$. Similarly, we also have

$$\|Q_\gamma(t-s)\Psi\|^2_{H^n(\Omega)} = \sum_{j=1}^{\infty} \sigma_j^n s^{2(\gamma-1)} \Big[\exp\Big(-\sigma_j \frac{t^\gamma - s^\gamma}{\gamma}\Big)\Big]^2 \langle \Psi, \varphi_j\rangle^2,$$
$$\le C_\xi^2 \sum_{j=1}^{\infty} \sigma_j^n s^{2(\gamma-1)} \Big(\sigma_j \frac{t^\gamma - s^\gamma}{\gamma}\Big)^{-2\xi} \langle \Psi, \varphi_j\rangle^2,$$
$$\le \frac{C_\xi^2 \gamma_1^{2\xi}}{\sigma_1^{2\xi}} s^{2(\gamma-1)} (t^\gamma - s^\gamma)^{-2\xi} \|\Psi\|^2_{H^n(\Omega)}.$$

Hence, we obtain

$$\|Q_\gamma(t-s)\Psi\|_{H^n(\Omega)} \le C(\xi, \gamma_1, \sigma_1) s^{(\gamma-1)} (t^\gamma - s^\gamma)^{-\xi} \|\Psi\|_{H^n(\Omega)}.$$

□

Lemma 2.10. *Let $0 < \epsilon_1 < \epsilon_3, 0 < \gamma_0 < \gamma < \gamma' < \gamma_1 < 1$ and $\Psi \in H^n(\Omega)$, we always have several estimates as follows:*

$$\|[P_{\gamma'}(t) - P_\gamma(t)]\Psi\|_{H^n(\Omega)} \le C(\gamma_0, \epsilon_1, \epsilon_2, \epsilon_3, \sigma_1, T)\overline{C}(\gamma' - \gamma, \epsilon_1, \epsilon_2) t^{\epsilon_1(\gamma - \epsilon_2) - \gamma'\epsilon_3} \|\Psi\|_{H^n(\Omega)}, \tag{31}$$
$$\|[Q_{\gamma'}(t-s) - Q_\gamma(t-s)]\Psi\|_{H^n(\Omega)} \le \Big(\sqrt{2} s^{\gamma-1} \Big| s^{\gamma'-\gamma} - 1\Big| + \max\Big[\mathscr{C}_1(\gamma' - \gamma), \mathscr{C}_2(\gamma' - \gamma)\Big]\Big) \|\Psi\|_{H^n(\Omega)}. \tag{32}$$

Proof. By using inequality (9) of Theorem (2.7), we show that

$$\begin{aligned}\|[P_{\gamma'}(t)-P_{\gamma}(t)]\Psi\|^2_{H^n(\Omega)} &\leq \sum_{j=1}^{\infty}\sigma_j^n\Big[\exp\Big(\frac{-\sigma_j t^{\gamma'}}{\gamma'}\Big)-\exp\Big(\frac{-\sigma_j t^{\gamma}}{\gamma}\Big)\Big]^2\langle\Psi,\varphi_j\rangle^2,\\ &\leq \sum_{j=1}^{\infty}\sigma_j^n\Big[C(\gamma_0,\epsilon_1,\epsilon_2,\epsilon_3,T)\overline{C}(\gamma'-\gamma,\epsilon_1,\epsilon_2)\sigma_j^{\epsilon_1-\epsilon_3}t^{\epsilon_1(\gamma-\epsilon_2)-\gamma'\epsilon_3}\Big]^2\langle\Psi,\varphi_j\rangle^2,\\ &\leq \Big[C(\gamma_0,\epsilon_1,\epsilon_2,\epsilon_3,T)\overline{C}(\gamma'-\gamma,\epsilon_1,\epsilon_2)\sigma_1^{\epsilon_1-\epsilon_3}t^{\epsilon_1(\gamma-\epsilon_2)-\gamma'\epsilon_3}\Big]^2\|\Psi\|^2_{H^n(\Omega)}.\end{aligned}$$

Therefore, we can get the following estimate

$$\|[P_{\gamma'}(t)-P_{\gamma}(t)]\Psi\|_{H^n(\Omega)} \leq C(\gamma_0,\epsilon_1,\epsilon_2,\epsilon_3,\sigma_1,T)\overline{C}(\gamma'-\gamma,\epsilon_1,\epsilon_2)t^{\epsilon_1(\gamma-\epsilon_2)-\gamma'\epsilon_3}\|\Psi\|_{H^n(\Omega)}.$$

Similarly, using inequality $(a+b)^2 \leq 2(a^2+b^2)$, we can show that

$$\begin{aligned}&\|[Q_{\gamma'}(t)-Q_{\gamma}(t)]\Psi\|^2_{H^n(\Omega)},\\ &\quad= \sum_{j=1}^{\infty}\sigma_j^n\Big[s^{(\gamma'-1)}\exp\Big(-\sigma_j\frac{t^{\gamma'}-s^{\gamma'}}{\gamma'}\Big)-s^{(\gamma-1)}\exp\Big(-\sigma_j\frac{t^{\gamma}-s^{\gamma}}{\gamma}\Big)\Big]^2\langle\Psi,\varphi_j\rangle^2,\\ &\quad\leq 2\sum_{j=1}^{\infty}\sigma_j^n\Big[s^{(\gamma'-1)}\Big(\exp\Big(-\sigma_j\frac{t^{\gamma'}-s^{\gamma'}}{\gamma'}\Big)-\exp\Big(-\sigma_j\frac{t^{\gamma}-s^{\gamma}}{\gamma}\Big)\Big)\Big]^2\langle\Psi,\varphi_j\rangle^2,\\ &\quad+2\sum_{j=1}^{\infty}\sigma_j^n\Big[\exp\Big(-\sigma_j\frac{t^{\gamma}-s^{\gamma}}{\gamma}\Big)\Big(s^{(\gamma'-1)}-s^{(\gamma-1)}\Big)\Big]^2\langle\Psi,\varphi_j\rangle^2.\end{aligned}$$

Thanks to (10) of Theorem (2.7), we also have

$$\begin{aligned}&\|[Q_{\gamma'}(t)-Q_{\gamma}(t)]\Psi\|^2_{H^n(\Omega)}\\ &\quad\leq 2C^2(\gamma_0,\epsilon_1,\epsilon_2,\epsilon_3,\sigma_1,T)\Big[\max\Big[\frac{(t^{\gamma'}-s^{\gamma'})^{-\epsilon_3}}{\gamma'^{-\epsilon_3}},\frac{(t^{\gamma}-s^{\gamma})^{-\epsilon_3}}{\gamma^{-\epsilon_3}}\Big]\Big]^2 s^{2(\gamma'-1)}\overline{C}^2(\gamma'-\gamma,\epsilon_1,\epsilon_2)\|\Psi\|^2_{H^n(\Omega)},\\ &\quad+2s^{2(\gamma-1)}\Big(s^{(\gamma'-\gamma)}-1\Big)^2\|\Psi\|^2_{H^n(\Omega)}.\end{aligned}$$

Therefore, we can get the following estimate

$$\begin{aligned}&\|[Q_{\gamma'}(t)-Q_{\gamma}(t)]\Psi\|_{H^n(\Omega)}\\ &\quad\leq \sqrt{2}s^{(\gamma-1)}\Big|s^{(\gamma'-\gamma)}-1\Big|\|\Psi\|_{H^n(\Omega)}+\max\Big[\mathscr{C}_1(\gamma'-\gamma),\mathscr{C}_2(\gamma'-\gamma)\Big]\|\Psi\|_{H^n(\Omega)},\end{aligned}$$

where

$$\mathscr{C}_1(\gamma'-\gamma) := \sqrt{2}C(\gamma_0,\epsilon_1,\epsilon_2,\epsilon_3,\sigma_1,T)\frac{(t^{\gamma'}-s^{\gamma'})^{-\epsilon_3}}{\gamma'^{-\epsilon_3}}s^{(\gamma'-1)}\overline{C}(\gamma'-\gamma,\epsilon_1,\epsilon_2); \tag{33}$$

$$\mathscr{C}_2(\gamma'-\gamma) := \sqrt{2}C(\gamma_0,\epsilon_1,\epsilon_2,\epsilon_3,\sigma_1,T)\frac{(t^{\gamma}-s^{\gamma})^{-\epsilon_3}}{\gamma^{-\epsilon_3}}s^{(\gamma'-1)}\overline{C}(\gamma'-\gamma,\epsilon_1,\epsilon_2). \tag{34}$$

□

3 The existence and uniqueness of mild solutions

Theorem 3.1. *Assume F satisfies the assumption (3) function with the d constraint: $0 < \xi < \min(1, \frac{1}{d})$, $0 \leq \gamma_0 \leq \gamma \leq \gamma_1 \leq 1$ and $g \in H^n(\Omega)$. Then, there exists a unique mild solution to the problem (1) that belongs to the space $\mathbb{C}^{\xi\gamma}((0,T],H^n(\Omega))$.*

Proof. We consider the following space

$$\mathbb{X}_A := \Big\{ u \in \mathbb{C}((0,T], H^n(\Omega)) \quad \text{such that} \quad u(\cdot,0) = g \text{ and } \|u\|_{\mathbb{X}_A} < A \Big\},$$

where the $\mathbb{X}$ norm is defined as follows: $(\ell/2 < m < n)$

$$\|u\|_{\mathbb{X}_A} := \sup_{t\in(0,T]} \Big(\|u\|_{H^m(\Omega)} + t^{\xi\gamma}\|u\|_{H^n(\Omega)} \Big),$$

and A satisfies several specific assumptions, which are defined after.

Considering mapping $\mathscr{M} : \mathbb{X}_A \to \mathbb{X}_A$ is defined as follows:

$$\mathscr{M}u(x,t) := P_\gamma(t)g(x) + \int_0^t Q_\gamma(t-s)F(u)(x,s)\mathrm{d}s. \tag{35}$$

To use the Banach fixed point theorem, we must first prove that $\mathscr{M}$ is well-defined on $\mathbb{X}_A$. Now, we divided the proof into two parts.

Part 1. $\mathscr{M}u$ is bounded by A for any $u \in \mathbb{C}^{\xi\gamma}((0,T], H^n(\Omega))$. Using Lemma (2.9), from (35), we get the following estimate

$$\begin{aligned}\|\mathscr{M}(u(t))\|_{H^n(\Omega)} &\le \|P_\alpha(t)g\|_{H^n(\Omega)} + \Big\| \int_0^t Q_\alpha(t-s)F(u)(s)\mathrm{d}s \Big\|_{H^n(\Omega)}, \\ &\le C(\xi,\gamma_1,\sigma_1)t^{-\xi\gamma}\|g\|_{H^n(\Omega)} + \int_0^t C(\xi,\gamma_1,\sigma_1)s^{\gamma-1}(t^\gamma - s^\gamma)^{-\xi}\|\mathrm{F(u)(s)}\|_{H^n(\Omega)}\mathrm{d}s.\end{aligned}$$

Multiply both sides of the above inequality by $t^{\xi\gamma}$, we get

$$t^{\xi\gamma}\|\mathscr{M}(u(t))\|_{H^n(\Omega)} \le C(\xi,\gamma_1,\sigma_1)\|g\|_{H^n(\Omega)} + C(\xi,\gamma_1,\sigma_1)t^{\xi\gamma}\int_0^t s^{\gamma-1}(t^\gamma - s^\gamma)^{-\xi}\|\mathrm{F(u)(s)}\|_{H^n(\Omega)}\mathrm{d}s.$$

Thanks to inequality (3), we can show that

$$\begin{aligned}&t^{\xi\gamma}\|\mathscr{M}(u(t))\|_{H^n(\Omega)} \\ &\le C(\xi,\gamma_1,\sigma_1)\|g\|_{H^n(\Omega)} + C(\xi,\gamma_1,\sigma_1)C_F t^{\xi\gamma}\int_0^t s^{\gamma-1-d\xi\gamma}(t^\gamma - s^\gamma)^{-\xi}[s^{\xi\gamma}\|u(s)\|_{H^n(\Omega)}]^d \mathrm{e}^{\|u(s)\|^2_{H^n(\Omega)}}\mathrm{d}s, \\ &\le C(\xi,\gamma_1,\sigma_1)\|g\|_{H^n(\Omega)} + C(\xi,\gamma_1,\sigma_1)C_F C^d_{m,n}A^d\exp(C^2_{m,n}A^2)t^{\xi\gamma}\int_0^t s^{\gamma-1-d\xi\gamma}(t^\gamma - s^\gamma)^{-\xi}\mathrm{d}s,\end{aligned}$$

where $C_{m,n}$ depends on m,n. Using transformation $s = tr^{\frac{1}{\gamma}}$, with condition $\xi < \min(1, \frac{1}{d})$ we can deduce

$$\begin{aligned}&t^{\xi\gamma}\|\mathscr{M}(u(t))\|_{H^n(\Omega)} \\ &\le C(\xi,\gamma_1,\sigma_1)\|g\|_{H^n(\Omega)} + C(\xi,\gamma_1,\sigma_1)C_F C^d_{m,n}A^d\exp(C^2_{m,n}A^2)\frac{t^{\gamma-d\xi\gamma}}{\gamma}\int_0^1 r^{-d\xi}(1-r)^{-\xi}\mathrm{d}r, \\ &\le C(\xi,\gamma_1,\sigma_1)\|g\|_{H^n(\Omega)} + C(\xi,\gamma_1,\sigma_1)C_F\frac{T^{\gamma-d\xi\gamma}}{\gamma}C^d_{m,n}A^d\exp(C^2_{m,n}A^2)B(1-\xi,1-d\xi).\end{aligned}$$

We can choose T and A such that $C(\xi,\gamma_1,\sigma_1)\|g\|_{H^n(\Omega)} = \frac{\Phi A}{2}$ and

$$T^{\gamma-d\xi\gamma} \le \frac{\gamma_1\Phi}{2C(\xi,\gamma_1,\sigma_1)C_F C^d_{m,n}A^{d-1}\exp(C^2_{m,n}A^2)B(1-\xi,1-d\xi)}, \tag{36}$$

with $\Phi \in (0,1)$. For any $t \in [0,T]$, we obtain

$$\sup_{t\in(0,T]} t^{\xi\gamma}\|\mathscr{M}(u(t))\|_{H^n(\Omega)} \le \Phi A.$$

Part 2. Proving that $\mathscr{M}(u(t))$ is continuous on $(0,T]$ is not a difficult task, so we omit the proof here.

Part 3. To finish this theorem, we must demonstrate that $\mathscr{M}$ is also a contraction mapping on $\mathbb{C}((0,T],H^n(\Omega))$. Indeed, for $u,v \in B(0,\mathscr{R}) \subset \mathbb{C}((0,T],H^n(\Omega))$, from Lemma (2.9) we can deduce that

$$\begin{aligned}\|\mathscr{M}(u(t)) - \mathscr{M}(v(t))\|_{H^n(\Omega)} &= \Big\| \int_0^t Q_\gamma(t-s)\big[F(u)(s) - F(v)(s)\big]\mathrm{d}s \Big\|_{H^n(\Omega)}, \\ &\le C(\xi,\gamma_1,\sigma_1)\int_0^t s^{\gamma-1}(t^\gamma - s^\gamma)^{-\xi}\|F(u)(s) - F(v)(s)\|_{H^n(\Omega)}\mathrm{d}s.\end{aligned}$$

Using condition (3), we get the following estimate

$$\begin{aligned}&t^{\xi\gamma}\|\mathscr{M}(u(t)) - \mathscr{M}(v(t))\|_{H^n(\Omega)} \\ &\le C(\xi,\gamma_1,\sigma_1)\mathbb{C}_F t^{\xi\gamma}\int_0^t s^{\gamma-1}(t^\gamma - s^\gamma)^{-\xi}\|u(t)\|^{d-1}_{H^n(\Omega)}\mathrm{e}^{\|u(t)\|^2_{H^n(\Omega)}}\|u(t)-v(t)\|_{H^n(\Omega)}\mathrm{d}s, \\ &+ C(\xi,\gamma_1,\sigma_1)\mathbb{C}_F t^{\xi\gamma}\int_0^t s^{\gamma-1}(t^\gamma - s^\gamma)^{-\xi}\|v(t)\|^{d-1}_{H^n(\Omega)}\mathrm{e}^{\|v(t)\|^2_{H^n(\Omega)}}\|u(t)-v(t)\|_{H^n(\Omega)}\mathrm{d}s, \\ &\le 2C(\xi,\gamma_1,\sigma_1)\mathbb{C}_F C^{d-1}_{m,n}A^{d-1}\exp(C^2_{m,n}A^2)t^{\xi\gamma}\int_0^t s^{\gamma-1-\xi\gamma}(t^\gamma - s^\gamma)^{-\xi}\|[s^{\xi\gamma}\|u(t)-v(t)\|_{H^n(\Omega)}]\mathrm{d}s.\end{aligned} \tag{37}$$

By the same technology as **Part 1**, we can deduce

$$\begin{aligned}&t^{\xi\gamma}\|\mathscr{M}(u(t)) - \mathscr{M}(v(t))\|_{H^n(\Omega)} \\ &\le 2C(\xi,\gamma_1,\sigma_1)\mathbb{C}_F C^{d-1}_{m,n}A^{d-1}\exp(C^2_{m,n}A^2)\|u(t)-v(t)\|_{\mathbb{C}^{\xi\gamma}([0,T],H^n(\Omega))}\frac{t^{\gamma-d\xi\gamma}}{\gamma}\int_0^t r^{-d\xi}(1-r)^{-\xi}\mathrm{d}r, \\ &\le 2C(\xi,\gamma_1,\sigma_1)\mathbb{C}_F C^{d-1}_{m,n}A^{d-1}\exp(C^2_{m,n}A^2)\|u(t)-v(t)\|_{\mathbb{C}^{\xi\gamma}([0,T],H^n(\Omega))}\frac{T^{\gamma-d\xi\gamma}}{\gamma}B(1-\xi,1-d\xi).\end{aligned}$$

Using condition (36) and for any $t \in 0,T]$, we can get the following estimate

$$\|\mathscr{M}(u(t)) - \mathscr{M}(v(t))\|_{\mathbb{C}^{\xi\gamma}((0,T],H^n(\Omega))} \le \Phi\|u-v\|_{\mathbb{C}^{\xi\gamma}((0,T],H^n(\Omega))}.$$

This completes the proof of the theorem. □

4 The continuous dependence of mild solutions on the fractional order of a nonlinear fractional pseudo-parabolic equation with exponential nonlinearity.

Theorem 4.1. *Let $\gamma,\gamma',\gamma_0,\gamma_1$ such that $0 < \gamma_0 \le \gamma \le \gamma' \le \gamma_1 < 1$ and $\sigma_0 \in (0,1)$. For $g \in H^n(\Omega)$, assume there exist positive numbers $\epsilon_1,\epsilon_2,\epsilon_3$ and ξ satisfying $0 < \epsilon_1 \le \epsilon_3 \le \frac{\gamma_0}{2\gamma_1}$ and*

$0 < \xi \le 1 - \frac{1}{2\gamma_0}$, suppose that $u_{\gamma'}$, $u_\gamma \in \mathbb{C}^{\xi\gamma}((0,T], H^n(\Omega))$ are two solutions to Problem (1) with respect to the fractional orders γ and γ' respectively, we have the following estimate

$$\begin{aligned}&\|u_{\gamma'}(t) - u_\gamma(t)\|_{\mathbb{C}^\beta((0,T],H^n(\Omega))}\\&\quad\le \mathscr{K}^{\epsilon_1,\epsilon_2,\epsilon_3,\xi}_{\sigma_1,\beta_2,\beta_3,d}(\gamma_0,\gamma_1,T,C_{m,n},A,C_F,\|g\|_{H^n(\Omega)}) \times \overline{\mathscr{C}}(\gamma'-\gamma),\\&\qquad\times \exp\left(\mathscr{V}^{\epsilon_1,\epsilon_2,\epsilon_3,\xi}_{\sigma_1,\beta_2,\beta_3,d}(\gamma_0,\gamma_1,T,C_{m,n},A,C_F)t^{\frac{1}{2}}\right),\end{aligned}$$

where $\beta := \gamma'\epsilon_3 - \gamma\epsilon_1 + \epsilon_1\epsilon_2$.

$$\begin{aligned}&\left[\mathscr{K}^{\epsilon_1,\epsilon_2,\epsilon_3,\xi}_{\sigma_1,\beta_2,\beta_3,d}(\gamma_0,\gamma_1,T,C_{m,n},A,C_F,\|g\|_{H^n(\Omega)})\right],\\&\quad:= \sqrt{3\Big[C(\gamma_0,\epsilon_1,\epsilon_2,\epsilon_3,\sigma_1,T)\|g\|_{H^n(\Omega)}\Big]^2 + 3\Big[\mathscr{W}^{\epsilon_1,\epsilon_2,\epsilon_3,\xi}_{\sigma_1,\beta_2,\beta_3,d}(\gamma_0,\gamma_1,T,C_{m,n},A,C_F)\Big]^2}.\end{aligned}$$

$$\begin{aligned}&\mathscr{V}^{\epsilon_1,\epsilon_2,\epsilon_3,\xi}_{\sigma_1,\beta_2,\beta_3,d}(\gamma_0,\gamma_1,T,C_{m,n},A,C_F)\\&\quad:= 2C(\xi,\gamma_1,\sigma_1)C_F C^{d-1}_{m,n}A^{d-1}e^{C^2_{m,n}A^2}T^{\gamma-\frac{1}{2}-\xi\gamma}\sqrt{B(1-2\xi,\gamma-2\gamma'\epsilon_3+2\gamma\epsilon_1-2\epsilon_1\epsilon_2)}.\end{aligned}$$

Proof of Theorem 4.1. The continuity of solutions to the problem (1) with respect to fractional orders, which is examined in this section, Now, we use the solution formula (8) with $0 < \gamma_0 \le \gamma < \gamma' \le \gamma_1 \le 1$ to deduce that

$$u_\gamma(x,t) = P_\gamma(t)g(x) + \int_0^t Q_\gamma(t-s)F(u)(x,s)\mathrm{d}s,$$

and

$$u_{\gamma'}(x,t) = P_{\gamma'}(t)g(x) + \int_0^t Q_{\gamma'}(t-s)F(u)(x,s)\mathrm{d}s.$$

Subtracting the two sides of the above two equations, respectively, we have

$$u_{\gamma'}(x,t) - u_\gamma(x,t) = [P_{\gamma'}(t) - P_\gamma(t)]g(x) + \int_0^t [Q_{\gamma'}(t-s) - Q_\gamma(t-s)]F(u)(x,s)\mathrm{d}s.$$

Thanks to the inequality $(a+b+c)^2 \le 3(a^2+b^2+c^2)$, we can obtain the following estimate

$$\begin{aligned}\|u_{\gamma'}(t) - u_\gamma(t)\|^2_{H^n(\Omega)} &\le 3\Big\|[P_{\gamma'}(t) - P_\gamma(t)]g\Big\|^2_{H^n(\Omega)},\\&\quad+3\Big\|\int_0^t [Q_{\gamma'}(t-s) - Q_\gamma(t-s)]F(u_\gamma)(s)\mathrm{d}s\Big\|_{H^n(\Omega)},\\&\quad+3\Big\|\int_0^t Q_{\gamma'}(t-s)[F(u_{\gamma'})(s) - F(u_\gamma)(s)]\mathrm{d}s\Big\|_{H^n(\Omega)}.\end{aligned}\tag{38}$$

Using the first inequality in Lemma (2.10), we can get

$$\|[P_{\gamma'}(t) - P_\gamma(t)]g\|_{H^n(\Omega)} \le C(\gamma_0,\epsilon_1,\epsilon_2,\epsilon_3,\sigma_1,T)\overline{C}(\gamma'-\gamma,\epsilon_1,\epsilon_2)t^{\epsilon_1(\gamma-\epsilon_2)-\gamma'\epsilon_3}\|g\|_{H^n(\Omega)}.$$

With $0 < \epsilon_1 < \epsilon_3$ and $\gamma' > \gamma$ therefore $\gamma'\epsilon_3 - \gamma\epsilon_1 + \epsilon_1\epsilon_2 > 0$ we can deduce that

$$t^{\gamma'\epsilon_3-\gamma\epsilon_1+\epsilon_1\epsilon_2}\|[P_{\gamma'}(t) - P_\gamma(t)]g\|_{H^n(\Omega)} \le C(\gamma_0,\epsilon_1,\epsilon_2,\epsilon_3,\sigma_1,T)\overline{C}(\gamma'-\gamma,\epsilon_1,\epsilon_2)\|g\|_{H^n(\Omega)}.\tag{39}$$

Similarly, using the second evaluation in Lemma (2.10), we obtain

$$
\begin{aligned}
&t^{\gamma'\epsilon_3-\gamma\epsilon_1+\epsilon_1\epsilon_2}\left\| \int_0^t [Q_{\gamma'}(t-s)-Q_\gamma(t-s)]F(u_\gamma)(s)\mathrm{d}s\right\|_{H^n(\Omega)},\\
&\leq t^{\gamma'\epsilon_3-\gamma\epsilon_1+\epsilon_1\epsilon_2}\int\limits_0^t \left(\sqrt{2}s^{\gamma-1}\left|s^{\gamma'-\gamma}-1\right| + \max\left[\mathscr{C}_1(\gamma'-\gamma),\mathscr{C}_2(\gamma'-\gamma)\right]\right)\|F(u_\gamma)\|_{H^n(\Omega)}\mathrm{d}s,\\
&\leq C_F t^{\gamma'\epsilon_3-\gamma\epsilon_1+\epsilon_1\epsilon_2}\int\limits_0^t \sqrt{2}s^{\gamma-1}\left|s^{\gamma'-\gamma}-1\right|\|u_\gamma(s)\|^d_{H^n(\Omega)}\mathrm{e}^{\|u_\gamma(s)\|^2_{H^n(\Omega)}}\mathrm{d}s,\\
&\quad + C_F t^{\gamma'\epsilon_3-\gamma\epsilon_1+\epsilon_1\epsilon_2}\int\limits_0^t \max\left[\mathscr{C}_1(\gamma'-\gamma),\mathscr{C}_2(\gamma'-\gamma)\right]\|u_\gamma(s)\|^d_{H^n(\Omega)}\mathrm{e}^{\|u_\gamma(s)\|^2_{H^n(\Omega)}}\mathrm{d}s. \qquad (40)
\end{aligned}
$$

By similarly estimating as Part 1 in Theorem (3.1), we can get the following estimate

$$
\begin{aligned}
\mathcal{I}_1 &:= C_F t^{\gamma'\epsilon_3-\gamma\epsilon_1+\epsilon_1\epsilon_2}\int\limits_0^t \sqrt{2}\left|s^{\gamma'-1}-s^{\gamma-1}\right|s^{-d\gamma\xi}\|s^{-\gamma\xi}u_\gamma(s)\|^d_{H^n(\Omega)}\mathrm{e}^{\|u_\gamma(s)\|^2_{H^n(\Omega)}}\mathrm{d}s,\\
&\leq \sqrt{2}C^d_{n,m}A^d\exp(C^2_{m,n}A^2)C_F t^{\gamma'\epsilon_3-\gamma\epsilon_1+\epsilon_1\epsilon_2}\int\limits_0^t \left|s^{1+\gamma-\gamma'-1}-1\right|s^{\gamma'-1-d\gamma\xi}\mathrm{d}s. \qquad (41)
\end{aligned}
$$

Using part c) of Theorem (2.8), with condition $0<\beta_3\leq \gamma-d\gamma\xi$ we can deduce

$$
\begin{aligned}
\mathcal{I}_1 &\leq \sqrt{2}C^d_{n,m}A^d\mathrm{e}^{C^2_{m,n}A^2}C_F t^{\gamma'\epsilon_3-\gamma\epsilon_1+\epsilon_1\epsilon_2}C(\gamma_0,\beta_2,\beta_3,T)t^{\gamma-d\gamma\xi-\beta_3}\overline{C}(\gamma'-\gamma,\beta_3,T),\\
&\leq \sqrt{2}C^d_{n,m}A^d\mathrm{e}^{C^2_{m,n}A^2}C_F C(\gamma_0,\beta_2,\beta_3,T)T^{\gamma'\epsilon_3-\gamma\epsilon_1+\epsilon_1\epsilon_2+\gamma-d\gamma\xi-\beta_3}\overline{C}(\gamma'-\gamma,\beta_3,T), \qquad (42)
\end{aligned}
$$

where $\beta_2 = 1-\gamma'+d\gamma\xi$

$$
\overline{C}(\gamma'-\gamma,\beta_3,T) := \left[(\gamma'-\gamma)^{\beta_3}+(\gamma'-\gamma)\right] + \left[T^{\beta_3}(T^{\gamma'-\gamma}-1)+\frac{T^{\gamma'-\gamma+\beta_3}}{\gamma'-d\gamma\xi}(\gamma'-\gamma)\right].
$$

We can see that $\lim\limits_{\gamma'\to\gamma}\overline{C}(\gamma'-\gamma,\beta_3,T)=0$. From (33), we can get the following estimate

$$
\begin{aligned}
\mathcal{I}_2 &\leq C_F t^{\gamma'\epsilon_3-\gamma\epsilon_1+\epsilon_1\epsilon_2}\int\limits_0^t \mathscr{C}_1(\gamma'-\gamma)\|u_\gamma(s)\|^d_{H^n(\Omega)}\mathrm{e}^{\|u_\gamma(s)\|^2_{H^n(\Omega)}}\mathrm{d}s,\\
&\leq \sqrt{2}C^d_{n,m}A^d\mathrm{e}^{C^2_{m,n}A^2}C_F t^{\gamma'\epsilon_3-\gamma\epsilon_1+\epsilon_1\epsilon_2}\int\limits_0^t \mathscr{C}_1(\gamma'-\gamma)s^{-d\gamma\xi}\mathrm{d}s,\\
&\leq \sqrt{2}C^d_{n,m}A^d\mathrm{e}^{C^2_{m,n}A^2}C_F t^{\gamma'\epsilon_3-\gamma\epsilon_1+\epsilon_1\epsilon_2},\\
&\qquad\times\int\limits_0^t \sqrt{2}C(\gamma_0,\epsilon_1,\epsilon_2,\epsilon_3,\sigma_1,T)\frac{(t^{\gamma'}-s^{\gamma'})^{-\epsilon_3}}{\gamma'^{-\epsilon_3}}s^{(\gamma'-1)}\overline{C}(\gamma'-\gamma,\epsilon_1,\epsilon_2)s^{-d\gamma\xi}\mathrm{d}s,\\
&\leq 2C^d_{n,m}A^d\mathrm{e}^{C^2_{m,n}A^2}C_F C(\gamma_0,\epsilon_1,\epsilon_2,\epsilon_3,\sigma_1,T)\overline{C}(\gamma'-\gamma,\epsilon_1,\epsilon_2),\\
&\qquad\times t^{\gamma'\epsilon_3-\gamma\epsilon_1+\epsilon_1\epsilon_2}\int\limits_0^t \frac{(t^{\gamma'}-s^{\gamma'})^{-\epsilon_3}}{\gamma'^{-\epsilon_3}}s^{\gamma'-1-d\gamma\xi}\mathrm{d}s.
\end{aligned}
$$

Similar to the technique of proving **part 1** of Theorem (3.1), we have the evaluation as follows:

$$
\begin{aligned}
\mathcal{I}_2 \le\ & 2C^d_{n,m}A^d e^{C^2_{m,n}A^2} C_F C(\gamma_0,\epsilon_1,\epsilon_2,\epsilon_3,\sigma_1,T)\overline{C}(\gamma'-\gamma,\epsilon_1,\epsilon_2),\\
&\times (\gamma')^{1+\epsilon_3} t^{\gamma'\epsilon_3-\gamma\epsilon_1+\epsilon_1\epsilon_2} t^{\gamma'-\gamma'\epsilon_3-d\gamma\xi}\int_0^1 (1-r)^{-\epsilon_3} r^{\frac{-d\gamma\xi}{\gamma'}}\mathrm{d}r,\\
\le\ & 2C^d_{n,m}A^d e^{C^2_{m,n}A^2} C_F C(\gamma_0,\epsilon_1,\epsilon_2,\epsilon_3,\sigma_1,T)\overline{C}(\gamma'-\gamma,\epsilon_1,\epsilon_2),\\
&\times (\gamma')^{1+\epsilon_3} T^{\gamma'-\gamma\epsilon_1+\epsilon_1\epsilon_2-d\gamma\xi}\mathrm{B}\Big(1-\epsilon_3,\frac{\gamma'-d\gamma\xi}{\gamma'}\Big).
\end{aligned} \tag{43}
$$

Using method estimate as $\mathcal{I}_2$. From (34), we also have the following evaluation

$$
\begin{aligned}
\mathcal{I}_3 \le\ & C_F t^{\gamma'\epsilon_3-\gamma\epsilon_1+\epsilon_1\epsilon_2}\int_0^t \mathscr{C}_2(\gamma'-\gamma)\|u_\gamma(s)\|^d_{H^n(\Omega)} e^{\|u_\gamma(s)\|^2_{H^n(\Omega)}}\mathrm{d}s,\\
\le\ & \sqrt{2}C^d_{n,m}A^d e^{C^2_{m,n}A^2} C_F t^{\gamma'\epsilon_3-\gamma\epsilon_1+\epsilon_1\epsilon_2}\int_0^t \mathscr{C}_2(\gamma'-\gamma)s^{-d\gamma\xi}\mathrm{d}s,\\
\le\ & \sqrt{2}C^d_{n,m}A^d e^{C^2_{m,n}A^2} C_F t^{\gamma'\epsilon_3-\gamma\epsilon_1+\epsilon_1\epsilon_2},\\
&\times \int_0^t \sqrt{2}C(\gamma_0,\epsilon_1,\epsilon_2,\epsilon_3,\sigma_1,T)\frac{(t^\gamma-s^\gamma)^{-\epsilon_3}}{\gamma^{-\epsilon_3}} s^{\gamma'-1}\overline{C}(\gamma'-\gamma,\epsilon_1,\epsilon_2)s^{-d\gamma\xi}\mathrm{d}s,\\
\le\ & 2C^d_{n,m}A^d e^{C^2_{m,n}A^2} C_F C(\gamma_0,\epsilon_1,\epsilon_2,\epsilon_3,\sigma_1,T)\overline{C}(\gamma'-\gamma,\epsilon_1,\epsilon_2),\\
&\times t^{\gamma'\epsilon_3-\gamma\epsilon_1+\epsilon_1\epsilon_2}\int_0^t \frac{(t^\gamma-s^\gamma)^{-\epsilon_3}}{\gamma^{-\epsilon_3}} s^{\gamma'-1-d\gamma\xi}\mathrm{d}s,\\
\le\ & 2C^d_{n,m}A^d e^{C^2_{m,n}A^2} C_F C(\gamma_0,\epsilon_1,\epsilon_2,\epsilon_3,\sigma_1,T)\overline{C}(\gamma'-\gamma,\epsilon_1,\epsilon_2),\\
&\times \gamma^{1+\epsilon_3} t^{\gamma'\epsilon_3-\gamma\epsilon_1+\epsilon_1\epsilon_2} t^{\gamma'-\gamma\epsilon_3-d\gamma\xi}\int_0^1 (1-r)^{-\epsilon_3} r^{\frac{\gamma'-\gamma-d\gamma\xi}{\gamma}}\mathrm{d}r,\\
\le\ & 2C^d_{n,m}A^d e^{C^2_{m,n}A^2} C_F C(\gamma_0,\epsilon_1,\epsilon_2,\epsilon_3,\sigma_1,T)\overline{C}(\gamma'-\gamma,\epsilon_1,\epsilon_2),\\
&\times \gamma^{1+\epsilon_3} T^{\gamma'+\gamma'\epsilon_3-\gamma\epsilon_3-\gamma\epsilon_1+\epsilon_1\epsilon_2-d\gamma\xi}\mathrm{B}\Big(1-\epsilon_3,\frac{\gamma'-d\gamma\xi}{\gamma}\Big).
\end{aligned} \tag{44}
$$

From (40), (41), (43), and (44), we can obtain

$$
\begin{aligned}
& t^{\gamma'\epsilon_3-\gamma\epsilon_1+\epsilon_1\epsilon_2}\Big\| \int_0^t [Q_{\gamma'}(t-s)-Q_\gamma(t-s)]F(u_{\alpha'})(s)\mathrm{d}s\Big\|_{H^n(\Omega)},\\
&\quad \le \sqrt{2}C^d_{n,m}A^d e^{C^2_{m,n}A^2} C_F C(\gamma_0,\beta_2,\beta_3,T)T^{\gamma'\epsilon_3-\gamma\epsilon_1+\epsilon_1\epsilon_2+\gamma-d\gamma\xi-\beta_3}\overline{C}(\gamma'-\gamma,\beta_3,T),\\
&\quad + 2C^d_{n,m}A^d e^{C^2_{m,n}A^2} C_F C(\gamma_0,\epsilon_1,\epsilon_2,\epsilon_3,\sigma_1,T)\overline{C}(\gamma'-\gamma,\epsilon_1,\epsilon_2),\\
&\qquad \times (\gamma')^{1+\epsilon_3} T^{\gamma'-\gamma\epsilon_1+\epsilon_1\epsilon_2-d\gamma\xi}\mathrm{B}\Big(1-\epsilon_3,\frac{\gamma'-d\gamma\xi}{\gamma'}\Big),\\
&\quad + 2C^d_{n,m}A^d e^{C^2_{m,n}A^2} C_F C(\gamma_0,\epsilon_1,\epsilon_2,\epsilon_3,\sigma_1,T)\overline{C}(\gamma'-\gamma,\epsilon_1,\epsilon_2),\\
&\qquad \times \gamma^{1+\epsilon_3} T^{\gamma'+\gamma'\epsilon_3-\gamma\epsilon_3-\gamma\epsilon_1+\epsilon_1\epsilon_2-d\gamma\xi}\mathrm{B}\Big(1-\epsilon_3,\frac{\gamma'-d\gamma\xi}{\gamma}\Big).
\end{aligned}
$$

To simplify the calculation, we rewrite the above inequality in condensed form as follows:

$$t^{\gamma'\epsilon_3-\gamma\epsilon_1+\epsilon_1\epsilon_2}\Big\|\int_0^t[Q_{\gamma'}(t-s)-Q_\gamma(t-s)]F(u_{\alpha'})(s)\mathrm{d}s\Big\|_{H^n(\Omega)}, \\ \le \mathscr{W}^{\epsilon_1,\epsilon_2,\epsilon_3,\xi}_{\sigma_1,\beta_2,\beta_3,d}(\gamma_0,\gamma_1,T,C_{m,n},A,C_F)\times\overline{\mathscr{C}}(\gamma'-\gamma), \tag{45}$$

where $\mathscr{W}^{\epsilon_1,\epsilon_2,\epsilon_3,\xi}_{\sigma_1,\beta_2,\beta_3,d}(\gamma_0,\gamma_1,T,C_{m,n},A,C_F)$ is a positive number that depends only on the constants written in the formula and $\overline{\mathscr{C}}(\gamma'-\gamma):=\overline{C}(\gamma'-\gamma,\epsilon_1,\epsilon_2)+\overline{C}(\gamma'-\gamma,\beta_3,T)\xrightarrow{\gamma'\to\gamma}0$. Finally, to complete the proof of the theorem. From Theorem (2.9), we also have the following evaluation:

$$t^{\gamma'\epsilon_3-\gamma\epsilon_1+\epsilon_1\epsilon_2}\Big\|\int_0^t Q_{\gamma'}(t-s)[F(u_{\gamma'})(s)-F(u_\gamma)(s)]\mathrm{d}s\Big\|_{H^n(\Omega)}, \\ \le C(\xi,\gamma_1,\sigma_1)\int_0^t s^{\gamma'-1}(t^\gamma-s^\gamma)^{-\xi}\|F(u_{\gamma'})(s)-F(u_\gamma)(s)\|_{H^n(\Omega)}\mathrm{d}s.$$

Using the condition (3), we can see that

$$\Big(t^{\gamma'\epsilon_3-\gamma\epsilon_1+\epsilon_1\epsilon_2}\Big\|\int_0^t Q_{\gamma'}(t-s)[F(u_{\gamma'})(s)-F(u_\gamma)(s)]\mathrm{d}s\Big\|_{H^n(\Omega)}\Big)^2, \\ \le\Big(C(\xi,\gamma_1,\sigma_1)C_Ft^{\gamma'\epsilon_3-\gamma\epsilon_1+\epsilon_1\epsilon_2}\Big)^2, \\ \times\left(\int_0^t s^{\gamma-1}(t^\gamma-s^\gamma)^{-\xi}\Big(\|u_{\gamma'}\|^{d-1}_{H^n(\Omega)}\mathrm{e}^{\|u_{\gamma'}\|^2_{H^n(\Omega)}}+\|u_\gamma\|^{d-1}_{H^n(\Omega)}\mathrm{e}^{\|u_\gamma\|^2_{H^n(\Omega)}}\Big)\|\mathrm{u_{fl'}}-u_\gamma\|_{H^n(\Omega)}\mathrm{d}s\right)^2, \\ \le\Big(2C(\xi,\gamma_1,\sigma_1)C_FC^{d-1}_{m,n}A^{d-1}\mathrm{e}^{C^2_{m,n}A^2}t^{\gamma'\epsilon_3-\gamma\epsilon_1+\epsilon_1\epsilon_2}\Big)^2, \\ \times\left(\int_0^t s^{\gamma-1-\gamma'\epsilon_3+\gamma\epsilon_1-\epsilon_1\epsilon_2}(t^\gamma-s^\gamma)^{-\xi}\Big(s^{\gamma'\epsilon_3-\gamma\epsilon_1+\epsilon_1\epsilon_2}\|\mathrm{u_{fl'}}-u_\gamma\|_{H^n(\Omega)}\Big)\mathrm{d}s\right)^2.$$

Using Hölder's inequality, we can deduce

$$\Big(t^{\gamma'\epsilon_3-\gamma\epsilon_1+\epsilon_1\epsilon_2}\Big\|\int_0^t Q_{\gamma'}(t-s)[F(u_{\gamma'})(s)-F(u_\gamma)(s)]\mathrm{d}s\Big\|_{H^n(\Omega)}\Big)^2, \\ \le\Big(2C(\xi,\gamma_1,\sigma_1)C_FC^{d-1}_{m,n}A^{d-1}\mathrm{e}^{C^2_{m,n}A^2}t^{\gamma'\epsilon_3-\gamma\epsilon_1+\epsilon_1\epsilon_2}\Big)^2, \\ \times\int_0^t s^{2(\gamma-1-\gamma'\epsilon_3+\gamma\epsilon_1-\epsilon_1\epsilon_2)}(t^\gamma-s^\gamma)^{-2\xi}\mathrm{d}s\int_0^t\Big(s^{\gamma'\epsilon_3-\gamma\epsilon_1+\epsilon_1\epsilon_2}\|\mathrm{u_{fl'}}-u_\gamma\|_{H^n(\Omega)}\Big)^2\mathrm{d}s, \\ \le\Big(2C(\xi,\gamma_1,\sigma_1)C_FC^{d-1}_{m,n}A^{d-1}\mathrm{e}^{C^2_{m,n}A^2}t^{\gamma-\frac{1}{2}-\xi\gamma}\Big)^2, \\ \times\int_0^t r^{\gamma-1-2\gamma'\epsilon_3+2\gamma\epsilon_1-2\epsilon_1\epsilon_2}(1-r)^{-2\xi}\mathrm{d}s\int_0^t\Big(s^{\gamma'\epsilon_3-\gamma\epsilon_1+\epsilon_1\epsilon_2}\|\mathrm{u_{fl'}}-u_\gamma\|_{H^n(\Omega)}\Big)^2\mathrm{d}s.$$

Assumption that $0 < \epsilon_1 < \epsilon_3 < \frac{\gamma_0}{2\gamma_1}$ and $0 < \xi < 1 - \frac{1}{2\gamma_0}$, we can get the following estimate

$$\begin{aligned}
&\left(t^{\gamma'\epsilon_3-\gamma\epsilon_1+\epsilon_1\epsilon_2}\left\| \int_0^t Q_{\gamma'}(t-s)[F(u_{\gamma'})(s)-F(u_\gamma)(s)]\mathrm{d}s\right\|_{H^n(\Omega)}\right)^2,\\
&\leq \left(2C(\xi,\gamma_1,\sigma_1)C_F C_{m,n}^{d-1}A^{d-1}\mathrm{e}^{C_{m,n}^2A^2}T^{\gamma-\frac{1}{2}-\xi\gamma}\right)^2,\\
&\qquad \times \int_0^1 r^{\gamma-1-2\gamma'\epsilon_3+2\gamma\epsilon_1-2\epsilon_1\epsilon_2}(1-r)^{-2\xi}\mathrm{d}s \int_0^t \left(s^{\gamma'\epsilon_3-\gamma\epsilon_1+\epsilon_1\epsilon_2}\|\mathrm{u}_{\mathrm{fl}'}-u_\gamma\|_{H^n(\Omega)}\right)^2\mathrm{d}s,\\
&\leq \left(2C(\xi,\gamma_1,\sigma_1)C_F C_{m,n}^{d-1}A^{d-1}\mathrm{e}^{C_{m,n}^2A^2}T^{\gamma-\frac{1}{2}-\xi\gamma}\right)^2,\\
&\qquad \times \mathrm{B}(1-2\xi,\gamma-2\gamma'\epsilon_3+2\gamma\epsilon_1-2\epsilon_1\epsilon_2)\int_0^t \left(s^{\gamma'\epsilon_3-\gamma\epsilon_1+\epsilon_1\epsilon_2}\|\mathrm{u}_{\mathrm{fl}'}-u_\gamma\|_{H^n(\Omega)}\right)^2\mathrm{d}s.
\end{aligned}$$

We can rewrite the above assessment as follows:

$$\begin{aligned}
&\left(t^{\gamma'\epsilon_3-\gamma\epsilon_1+\epsilon_1\epsilon_2}\left\| \int_0^t Q_{\gamma'}(t-s)[F(u_{\gamma'})(s)-F(u_\gamma)(s)]\mathrm{d}s\right\|_{H^n(\Omega)}\right)^2,\\
&\qquad \leq \left[\mathscr{V}^{\epsilon_1,\epsilon_2,\epsilon_3,\xi}_{\sigma_1,\beta_2,\beta_3,d}(\gamma_0,\gamma_1,T,C_{m,n},A,C_F)\right]^2 \times \int_0^t \left(s^{\gamma'\epsilon_3-\gamma\epsilon_1+\epsilon_1\epsilon_2}\|\mathrm{u}_{\mathrm{fl}'}-u_\gamma\|_{H^n(\Omega)}\right)^2\mathrm{d}s,
\end{aligned} \tag{46}$$

where

$$\begin{aligned}
&\mathscr{V}^{\epsilon_1,\epsilon_2,\epsilon_3,\xi}_{\sigma_1,\beta_2,\beta_3,d}(\gamma_0,\gamma_1,T,C_{m,n},A,C_F),\\
&:= 2C(\xi,\gamma_1,\sigma_1)C_F C_{m,n}^{d-1}A^{d-1}\mathrm{e}^{C_{m,n}^2A^2}T^{\gamma-\frac{1}{2}-\xi\gamma}\sqrt{\mathrm{B}(1-2\xi,\gamma-2\gamma'\epsilon_3+2\gamma\epsilon_1-2\epsilon_1\epsilon_2)}.
\end{aligned}$$

From (38), (39), (45) and (46), we obtain

$$\begin{aligned}
&\left[t^{\gamma'\epsilon_3-\gamma\epsilon_1+\epsilon_1\epsilon_2}\|u_{\gamma'}(t)-u_\gamma(t)\|_{H^n(\Omega)}\right]^2,\\
&\qquad \leq 3\left[C(\gamma_0,\epsilon_1,\epsilon_2,\epsilon_3,\sigma_1,T)\|g\|_{H^n(\Omega)}\times \overline{C}(\gamma'-\gamma,\epsilon_1,\epsilon_2)\right]^2,\\
&\qquad + 3\left[\mathscr{W}^{\epsilon_1,\epsilon_2,\epsilon_3,\xi}_{\sigma_1,\beta_2,\beta_3,d}(\gamma_0,\gamma_1,T,C_{m,n},A,C_F)\times \overline{\mathscr{C}}(\gamma'-\gamma)\right]^2,\\
&\qquad + 3\left[\mathscr{V}^{\epsilon_1,\epsilon_2,\epsilon_3,\xi}_{\sigma_1,\beta_2,\beta_3,d}(\gamma_0,\gamma_1,T,C_{m,n},A,C_F)\right]^2 \times \int_0^t \left(s^{\gamma'\epsilon_3-\gamma\epsilon_1+\epsilon_1\epsilon_2}\|\mathrm{u}_{\mathrm{fl}'}-u_\gamma\|_{H^n(\Omega)}\right)^2\mathrm{d}s.
\end{aligned}$$

More simply, we can present the above inequality in the following form:

$$\begin{aligned}
&\left[t^{\gamma'\epsilon_3-\gamma\epsilon_1+\epsilon_1\epsilon_2}\|u_{\gamma'}(t)-u_\gamma(t)\|_{H^n(\Omega)}\right]^2,\\
&\qquad \leq \left[\mathscr{K}^{\epsilon_1,\epsilon_2,\epsilon_3,\xi}_{\sigma_1,\beta_2,\beta_3,d}(\gamma_0,\gamma_1,T,C_{m,n},A,C_F,\|g\|_{H^n(\Omega)})\right]^2 \times \overline{\mathscr{C}}^2(\gamma'-\gamma),\\
&\qquad + 3\left[\mathscr{V}^{\epsilon_1,\epsilon_2,\epsilon_3,\xi}_{\sigma_1,\beta_2,\beta_3,d}(\gamma_0,\gamma_1,T,C_{m,n},A,C_F)\right]^2 \times \int_0^t \left(s^{\gamma'\epsilon_3-\gamma\epsilon_1+\epsilon_1\epsilon_2}\|\mathrm{u}_{\mathrm{fl}'}-u_\gamma\|_{H^n(\Omega)}\right)^2\mathrm{d}s,
\end{aligned}$$

where we denote that

$$\begin{aligned}
&\left[\mathscr{K}^{\epsilon_1,\epsilon_2,\epsilon_3,\xi}_{\sigma_1,\beta_2,\beta_3,d}(\gamma_0,\gamma_1,T,C_{m,n},A,C_F,\|g\|_{H^n(\Omega)})\right]^2,\\
&\qquad := 3\left[C(\gamma_0,\epsilon_1,\epsilon_2,\epsilon_3,\sigma_1,T)\|g\|_{H^n(\Omega)}\right]^2 + 3\left[\mathscr{W}^{\epsilon_1,\epsilon_2,\epsilon_3,\xi}_{\sigma_1,\beta_2,\beta_3,d}(\gamma_0,\gamma_1,T,C_{m,n},A,C_F)\right]^2.
\end{aligned}$$

Finally, using Grönwall's inequality, we can show that

$$\begin{aligned}
&\left[t^{\gamma'\epsilon_3-\gamma\epsilon_1+\epsilon_1\epsilon_2}\|u_{\gamma'}(t)-u_\gamma(t)\|_{H^n(\Omega)}\right]^2,\\
&\qquad\le\left[\mathscr{K}^{\epsilon_1,\epsilon_2,\epsilon_3,\xi}_{\sigma_1,\beta_2,\beta_3,d}(\gamma_0,\gamma_1,T,C_{m,n},A,C_F,\|g\|_{H^n(\Omega)})\right]^2\times\overline{\mathscr{C}}^2(\gamma'-\gamma),\\
&\qquad\qquad\times\exp\left(\left[\mathscr{V}^{\epsilon_1,\epsilon_2,\epsilon_3,\xi}_{\sigma_1,\beta_2,\beta_3,d}(\gamma_0,\gamma_1,T,C_{m,n},A,C_F)\right]^2 t\right).
\end{aligned}$$

Therefore we obtain the following result. And the proof is completed.

$$\begin{aligned}
&\|u_{\gamma'}(t)-u_\gamma(t)\|_{\mathbb{C}^{\gamma'\epsilon_3-\gamma\epsilon_1+\epsilon_1\epsilon_2}((0,T],H^n(\Omega)'}\\
&\qquad\le\mathscr{K}^{\epsilon_1,\epsilon_2,\epsilon_3,\xi}_{\sigma_1,\beta_2,\beta_3,d}(\gamma_0,\gamma_1,T,C_{m,n},A,C_F,\|g\|_{H^n(\Omega)})\times\overline{\mathscr{C}}(\gamma'-\gamma),\\
&\qquad\qquad\times\exp\left(\mathscr{V}^{\epsilon_1,\epsilon_2,\epsilon_3,\xi}_{\sigma_1,\beta_2,\beta_3,d}(\gamma_0,\gamma_1,T,C_{m,n},A,C_F)t^{\frac{1}{2}}\right).
\end{aligned}$$

□

5 Declarations

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Not applicable.

Authors's Contributions

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