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The Fractional Landweber Method for Identifying Unknown Source for the Fractional Elliptic Equations

Doan Vuong Nguyen[1], **Nguyen Hoang Tuan**[2] **and Vo Viet Tri**[3,*]

1. Faculty of Fundamental Science, Industrial University of Ho Chi Minh City, Ho Chi Minh City, Vietnam
2. University of Health Sciences, Dong Hoa Ward, Di An City, Binh Duong Province, Vietnam
3. Division of Applied Mathematics, Thu Dau Mot University, 06 Tran Van On, Thu Dau Mot City, Binh Duong Province, Vietnam

[*]Corresponding author

Abstract

The article addresses the inverse problem of identifying an unknown source term in a fractional elliptic equation defined in a bounded domain. The approach to solving the problem under consideration, the Landweber fractional method is used. This method involves constructing a regularization algorithm. A posteriori and a priori lapses estimates areobtain obtained, and final data with random data is regard.

Keywords: Convergence estimates, Regularization method, Ill-posed problem, Nonlocal problem
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1 Introduction

Fractional models are a key area in mathematical analysis that focus on the study and application of integrals and derivatives of arbitrary orders. Due to their wide-ranging applications across multiple scientific and engineering fields, they have become an area of significant interest for researchers [2, 3, 4, 5, 6, 7, 8, 9, 10]. This paper examines the problem of identifying a source function in a fractional elliptic partial differential equation, expressed as:

$$\mathcal{D}_z^\gamma \mathcal{D}_z^\gamma \theta(z) - \mathcal{A}\theta(z) = \mathcal{G}, \quad z > 0, \tag{1}$$

where $0 < \gamma < 1$, and $\mathcal{D}_z^\gamma$ represents the fractional Liouville-Caputo derivative of order γ, as defined in [11]:

$$\mathcal{D}_z^\gamma \theta(z,x) = \frac{1}{\Gamma(1-\gamma)} \int_0^z (z-s)^{-\gamma} \theta_s(s,x) ds, \quad 0 < \gamma < 1. \tag{2}$$

Contact: Doan Vuong Nguyen ✉ doanvuongnguyen@iuh.edu.vn; Nguyen Hoang Tuan ✉ nhtuan@uhsvnu.edu.vn; Vo Viet Tri ✉ trivv@tdmu.edu.vn

Our objective is to determine the unknown function f from interior data, given by:

$$\theta(L) = f \in \mathcal{H}, \quad 0 < L < \infty. \tag{3}$$

where:

- $\mathcal{A} : D(\mathcal{A}) \in \mathcal{H} \to \mathcal{H}$ is a positive self-adjoint operator with a compact resolvent.
- $\mathcal{H}$ represents a separable Hilbert space with an inner product $(.,.)$ and norm $||.||$.

Consider the set (λ_n, en), where $\lambda_n n \geq 1$ is an increasing unbounded sequence, and $enn \geq 1$ are the eigenvalues and eigenfunctions of $\mathcal{A}$, forming an orthonormal basis in $\mathcal{H}$. At $z = \mathcal{L}$, additional data ρ is observed, which may include measurement inaccuracies. Inverse problems often suffer from ill-posedness, making it essential to compensate for errors and establish reliable error estimates. Various studies have tackled ill-posed fractional inverse problems using different techniques. In [17, 18], the quasi-boundary value method was employed to derive a regularized solution, while [19] adopted the quasireversibility method. Other approaches [15, 16] utilized alternative regularization methods to mitigate instability in inverse problems. The Fourier truncation method was introduced in [20], whereas [21] proposed a non-stationary iterative Landweber regularization approach, incorporating a finite-dimensional approximation to reconstruct a stable source term. Meanwhile, [12] explored an inverse problem involving the elliptic fractional operator $(\mathcal{D}_z^\gamma \mathcal{D}_z^\gamma - \mathcal{A})$ over an infinite domain and suggested a modified Kozlov-Maz'ya iteration method with preconditioning techniques to recover missing data under a complementary condition. This paper aims to address the inverse problem of recovering the source function in the fractional elliptic diffusion equation over a general domain using the fractional Landweber method. Initially, this method was introduced to solve the Cauchy problem for the Helmholtz equation. Later, Li and Xiong recommended [14] for an inverse heat conduction problem. Unlike conventional cases where the a-priori bound of the exact solution is known, our problem does not allow for straightforward estimation. Thus, instead of using a-priori parameter selection rules, we derive convergence rates using both a-priori and a-posteriori parameter selection approaches. Our goal is to recover the source function $\mathcal{G}(x)$ from indirect observable data $\theta(L) = f$ at the final state $z = L$. The observed data $f(x)$, which may contain measurement errors, satisfy:

$$\left\| f_\delta - f \right\|_{\mathcal{H}} \leq \delta. \tag{4}$$

Unless explicitly stated, $| \cdot |_{\mathcal{H}}$ denotes the L^2 norm, and $\delta > 0$ represents the noise level. The structure of the paper is as follows. Section 2 introduces key mathematical tools relevant to the problem. Section 3 establishes the mild solution. Section 4 discusses the ill-posed nature of problem (1) and presents convergence approximations using the fractional Landweber method in Section 5, incorporating both a-priori and a-posteriori parameter selection strategies. The final section examines the convergence rate when the function ρ involves random data.

2 Preliminary

The classical Mittag-Leffler function is defined by

$$E_{\gamma,1}(z) = \sum_{n=0}^{\infty} \frac{z^n}{\Gamma(1+n\gamma)}, \beta > 0, z \in \mathbb{C}. \tag{5}$$

Theorem 2.1. *For every $\gamma \in (0,1)$, we have*

$$\frac{1}{1+\Gamma(1-\gamma)x} \leq E_{\gamma,1}(-x) \leq \frac{1}{1+\Gamma(1+\gamma)^{-1}x}, x \geq 0. \tag{6}$$

From (6), we deduce that

$$\frac{1}{1+\Gamma(1-\gamma)\sqrt{\xi_k}L^\gamma} \leq E_{\gamma,1}\big(-\sqrt{\xi_k}L^\gamma\big) \leq \frac{1}{1+\Gamma(1+\gamma)^{-1}\sqrt{\xi_k}L^\gamma}, L>0. \tag{7}$$

3 The mild solution

For $0<\gamma<1$, let us consider the following well-posed system equations

$$\begin{cases} \mathcal{D}_z^\gamma \mathcal{D}_z^\gamma \theta(z) - \mathcal{A}\theta(z) = \mathcal{G}, & z \in (0,\infty), \\ \theta(0) = 0. \end{cases} \tag{8}$$

Theorem 3.1. *Let $\mathcal{G} \in \mathcal{H}$, then the problem (1) admits a unique generalized solution given by*

$$\begin{aligned} \theta(z) &= -\big(I - E_{\gamma,1}\big(-z^\gamma\sqrt{\mathcal{A}}\big)\big)\mathcal{A}^{-1}\mathcal{G} = -K_{\gamma,1}(z)\mathcal{G} \\ &= -\sum_{k=1}^{\infty} \frac{\big(1-E_{\gamma,1}\big(-\sqrt{\xi_k}\, z^\gamma\big)\big)}{\xi_k}\langle \mathcal{G}, e_k\rangle e_k. \end{aligned} \tag{9}$$

Let $z=L$ in (9), we obtain

$$\theta(L) = -\big(I - E_{\gamma,1}\big(-L^\gamma\sqrt{\mathcal{A}}\big)\big)\mathcal{A}^{-1}\mathcal{G} = -K_{\gamma,1}(L)\mathcal{G} = f. \tag{10}$$

$K_{\gamma,1}(L)$ is a self-adjoint compact linear operator and $\sup_{z\geq 0}\|K_{\gamma,1}(z)\| \leq \xi_1^{-1}$. For $f \in H$, the space H^1 is defined by

$$\mathcal{H}^1 = \Big\{ f \in \mathcal{H} : \|\mathcal{A}f\|_{\mathcal{H}} < \infty \Big\}. \tag{11}$$

The operator equation (10) admits a unique solution if and only if $g \in \mathcal{H}^1$.

4 Ill-posedness of the inverse problem

To determine the unknown function $\mathcal{G}$, we just need to solve The operator equation (10), then we have the following

$$\mathcal{G} = \sum_{k=1}^{\infty} \frac{-\xi_k}{1-\big(E_{\gamma,1}\big(-L^\gamma\sqrt{\xi_k}\big)\big)}\langle f, e_k\rangle e_k, \tag{12}$$

From (12) that the term $\dfrac{\xi_k}{(1-E_{\gamma,1}(-L^\gamma\sqrt{\xi_k}))}$ is the instability cause, and we get

$$\frac{1}{\sqrt{\xi_k}}\,\mathcal{C}_1(\gamma) \leq E_{\gamma,1}\big(-L^\gamma\sqrt{\xi_k}\big) \leq \mathcal{C}_2(\gamma), \tag{13}$$

which implies that

$$1-\mathcal{C}_2(\gamma) \leq 1-E_{\gamma,1}\big(-L^\gamma\sqrt{\xi_k}\big) \leq 1-\mathcal{C}_1(\gamma)\frac{1}{\sqrt{\xi_k}}, \tag{14}$$

and so

$$\xi_k \leq \frac{\xi_k}{1-\mathcal{C}_1(\gamma)\xi_k^{-\frac{1}{2}}} \leq \frac{\xi_k}{1-E_{\gamma,1}\big(-L^\gamma\sqrt{\xi_k}\big)} \leq \frac{\xi_k}{1-\mathcal{C}_2(\gamma)}, \tag{15}$$

and therefore

$$\xi_k \leq \frac{\xi_k}{1-E_{\gamma,1}\big(-L^\gamma\sqrt{\xi_k}\big)} \leq \frac{\xi_k}{1-\mathcal{C}_2(\gamma)} \to \infty \text{ as } k\to\infty, \tag{16}$$

Theorem 4.1. *[11] Let the following condition holds*

$$\|\mathcal{G}\|^2_{\mathcal{H}^s} = \sum_{k=1}^{\infty} \xi_k^{2s} |\langle \mathcal{G}, e_k \rangle|^2 \leq \mathcal{E}^2, s > 0, \tag{17}$$

then

$$\|\mathcal{G}\|_{L^2} \leq \mathcal{C}_{\gamma,s} \mathcal{E}^{\frac{1}{1+s}} \|f\|_{L^2}^{\frac{s}{1+s}}, \text{ where } \mathcal{C}_{\gamma,s} = (1 - \mathcal{C}_2(\gamma))^{-\frac{s}{1+s}}. \tag{18}$$

5 The Fractional Landweber method

In this paragraph, a fractional Landweber regularization method has proposed to solve the ill-posed problem (1). Moreover, we provide convergence estimate under the a-priori regularization parameter choice rule. For noisy data and exact data, the solutions of fractional Landweber regularization method are given. To solve the unposed ill-posed backward problem (1)-(3), we propose a fractional Landweber regularization method. Next, according to the a-priori regularization parameter selection rule, we derive the convergence estimate. A repetitive execution of the fractional Landweber method can be found in [1]. For noisy data, Denote the fractional Landweber regularization solution by

$$\mathcal{G}_m^{\delta}(x) = \sum_{k=1}^{\infty} \left[1 - \left(1 - \beta |\mathcal{D}_k^{\gamma,1}(L)|^2\right)^m\right]^{\alpha} \frac{\langle f^{\delta}, e_k \rangle}{|\mathcal{D}_k^{\gamma,1}(L)|} e_k(\cdot), \quad \frac{1}{2} < \alpha \leq 1, \tag{19}$$

and the fractional Landweber regularization solution with the exact data by

$$\mathcal{G}_m(x) = \sum_{k=1}^{\infty} \left[1 - \left(1 - \beta |\mathcal{D}_k^{\gamma,1}(L)|^2\right)^m\right]^{\alpha} \cdot \frac{\langle f, e_k \rangle}{|\mathcal{D}_k^{\gamma,1}(L)|} e_k(\cdot), \frac{1}{2} < \alpha \leq 1, \tag{20}$$

where $m > 0$ plays the role of regularization parameter and α is called the fractional parameter.

Lemma 5.1. *For $\xi_k > 0$, and combining the estimation of (16), we get*

$$\frac{1 - \mathcal{C}_2(\gamma)}{\xi_k} \leq \mathcal{D}_k^{\gamma,1}(L) = \frac{1 - E_{\gamma,1}(-L^{\gamma}\sqrt{\xi_k})}{\xi_k} \leq \frac{1}{\xi_k}. \tag{21}$$

Lemma 5.2. *For $0 < \varsigma < 1, p > 0, m \in \mathbb{N}$, let $r_m(\varsigma) := (1 - \varsigma)^m$, the following inequality holds:*

$$r_m(\varsigma)\varsigma^p \leq \theta_p (m+1)^{-p}, \tag{22}$$

where

$$\theta_p = \begin{cases} 1, & 0 \leq p \leq 1 \\ p^p, & p > 1 \end{cases} \tag{23}$$

Lemma 5.3. *For $\xi_k > 0$ and $\frac{1}{2} < \alpha < 1, 0 < \beta |\mathcal{D}_k^{\gamma,1}(L)|^2 < 1, n \geq 1$, it gives*

$$\sup_{\xi_k > 0} \left[1 - \left(1 - \beta |\mathcal{D}_k^{\gamma,1}(L)|^2\right)^n\right]^{\alpha} \frac{1}{|\mathcal{D}_k^{\gamma,1}(L)|} \leq \sqrt{\beta}\sqrt{n}. \tag{24}$$

Lemma 5.4. *For $\frac{1}{2} < \alpha < 1, n \geq 1, \xi_k > 0, 0 < \beta |\mathcal{D}_k^{\gamma,1}(L)|^2 < 1$, one has*

$$\sup_{\xi_k > 0} \left(1 - \beta |\mathcal{D}_k^{\gamma,1}(L)|^2\right)^n |\mathcal{D}_k^{\gamma,1}(L)|^{\frac{p}{2}} \leq \rfloor(\beta, s) n^{-\frac{s}{2}}. \tag{25}$$

where the constant $\mathcal{C}(\beta, s) = \left(\frac{s}{2\beta}\right)^{\frac{s}{2}}$, where $n \geq 1$ acts as a regularization parameter, $0 < \beta < \left|\mathcal{D}_k^{\gamma,1}(L)\right|^{-2}$, and α is called the fractional parameter. The following theorem gives a result about convergence.

Theorem 5.5. *Let $n = \left\lfloor \left(\frac{E}{\delta}\right)^{\frac{2}{s+1}} \right\rfloor$. If the a-priori condition (17) and the noise assumption (4) hold, we have*

$$\|\mathcal{G}_n^\delta - \mathcal{G}\|_{\mathcal{H}} \text{ is of order } \delta^{\frac{s}{s+1}}. \tag{26}$$

where
$\lfloor n \rfloor \leqslant s$ and it also is the largest integer
$\mathcal{C}_1, \mathcal{C}_2$ are depending on $\beta, s, \alpha, \mathcal{C}$ and positive constants.

Proof. Applying triangle inequality, we have

$$\|\mathcal{G}_n^\delta - \mathcal{G}\|_{\mathcal{H}} \leq \|\mathcal{G}_n^\delta - \mathcal{G}_n\|_{\mathcal{H}} + \|\mathcal{G} - \mathcal{G}_n\|_{\mathcal{H}} = \mathcal{I}_1 + \mathcal{I}_2. \tag{27}$$

From Lemma 5.3 and (5.3), we have

$$\begin{aligned}
\mathcal{I}_1 &= \|\mathcal{G}_n^\delta - \mathcal{G}_n\|_{\mathcal{H}} \\
&= \left\| \sum_{k=1}^{\infty} \left[1 - \left(1 - \beta|\mathcal{D}_k^{\gamma,1}(L)|^2\right)^n\right]^\alpha \cdot \frac{1}{|\mathcal{D}_k^{\gamma,1}(L)|} \langle f^\delta - f, e_k \rangle e_k \right\|_{\mathcal{H}} \\
&\leq \delta \sup_{\xi_k > 0} \left[1 - \left(1 - \beta|\mathcal{D}_k^{\gamma,1}(L)|^2\right)^n\right]^\alpha \cdot \frac{1}{|\mathcal{D}_k^{\gamma,1}(L)|} \leq \sqrt{\beta}\sqrt{n}\,\delta.
\end{aligned} \tag{28}$$

From Lemma 5.4 and the a-priori bound condition (17), we can deduce that

$$\begin{aligned}
\mathcal{I}_2 &= \|\mathcal{G} - \mathcal{G}_n\|_{\mathcal{H}} \\
&= \left\| \sum_{k=1}^{\infty} [1 - \left[1 - \left(1 - \beta|\mathcal{D}_k^{\gamma,1}(L)|^2\right)^n\right]^\alpha] \cdot \frac{\langle f, e_k \rangle}{|\mathcal{D}_k^{\gamma,1}(L)|} e_k \right\|_{\mathcal{H}} \\
&\leq \left\| \sum_{k=1}^{\infty} \left(1 - \beta|\mathcal{D}_k^{\gamma,1}(L)|^2\right)^n \xi_k^{-s} \cdot \frac{\xi_k^s \langle f, e_k \rangle}{|\mathcal{D}_k^{\gamma,1}(L)|} e_k \right\|_{\mathcal{H}} \\
&= \left\| \sum_{k=1}^{\infty} \left(1 - \beta|\mathcal{D}_k^{\gamma,1}(L)|^2\right)^n \xi_k^{-s} \xi_k^s \langle f, e_k \rangle e_k \right\|_{\mathcal{H}} \\
&\leq \mathcal{E} \sup_{\xi_k > 0} \left(1 - \beta|\mathcal{D}_k^{\gamma,1}(L)|^2\right)^n \xi_k^{-s} \\
&\leq \mathcal{E} \sup_{\xi_k > 0} \left(1 - \beta|\mathcal{D}_k^{\gamma,1}(L)|^2\right)^n \mathcal{C}^{-s} |\mathcal{D}_k^{\gamma,1}(L)|^s \\
&\leq \mathcal{E}\mathcal{C}_1(\beta, s)\mathcal{C}^{-s} n^{-\frac{s}{2}}.
\end{aligned} \tag{29}$$

From (28) and (29), we obtain

$$\|\mathcal{G}_n^\delta - \mathcal{G}\|_{\mathcal{H}} \leq \beta^{\frac{1}{2}} m^{\frac{1}{2}} \delta + Ec(\beta, p)\mathcal{C}^{-s} n^{-\frac{s}{2}}, \tag{30}$$

n is chosen by

$$n = \left\lfloor \left(\frac{\mathcal{E}}{\delta}\right)^{\frac{2}{s+1}} \right\rfloor, \tag{31}$$

we then obtain the following convergence estimate

$$\|\mathcal{G}_n^\delta - \mathcal{G}\|_{\mathcal{H}} \leq (\mathcal{C}_1 + \mathcal{C}_2)\, \mathcal{E}^{\frac{1}{s+1}} \delta^{\frac{s}{s+1}}. \tag{32}$$

□

6 Convergence analysis and a-posteriori parameter choice rule

We need a reliable stopping rule to detect the significant change from convergence to divergence because of the semi-convergent nature of repetitive regularization methods for ill-posed problems. In this section, following the parameter selection rule, we give an a-posteriori parameter selection rule from which the rate of convergence for the regularization solution (1) to this fractional Landweber method can be inferred. We can be formulated the Morozov's discrepancy principle is the general a-posteriori rule is according [1] as:

$$\|\mathcal{D}_k^{\gamma,1}(L)\mathcal{G}_n^\delta - f^\delta\| \leq \lambda\delta. \tag{33}$$

where $\lambda > 1$ is a user-supplied constant independent of $\delta, n > 0$ is the regularization parameter which makes (33) hold at the first iteration time, and $\mathcal{K}$ is the forward operator defined by (16).

Lemma 6.1. *Let* $\rho(m) = \|\mathcal{D}_k^{\gamma,1}(L)\mathcal{G}_n^\delta - f^\delta\|_{\mathcal{H}}$, *we have*

- $\rho(n)$ *is a continuous function;*
- $\lim_{n\to 0}\rho(n) = \|f^\delta\|_{\mathcal{H}}$;
- $\lim_{n\to\infty}\rho(n) = 0$;
- $\rho(n)$ *is a strictly decreasing function over* $(0,\infty)$.

Proof. From (33), we obtain

$$\rho(n) = \left(\sum_{k=1}^{\infty}\left[1-\left[1-\left(1-\beta|\mathcal{D}_k^{\gamma,1}(L)|^2\right)^n\right]^\alpha\right]^2 |\langle f^\delta, e_k\rangle|^2\right)^{\frac{1}{2}} \tag{34}$$

Since

$$\lim_{n\to 0}\rho(n) = \left(\sum_{k=1}^{\infty}|\langle f^\delta, e_k\rangle|^2\right)^{\frac{1}{2}} = \|f^\delta\|_{\mathcal{H}}. \tag{35}$$

We assume that the noisy data $\|f^\delta\|_{\mathcal{H}}$ is large enough such that $0 < \lambda\delta < \|f^\delta\|_{\mathcal{H}}$. According to Lemma 5.4, there exists a unique minimal solution for the inequality (36). □

Lemma 6.2. *If n make (33) hold at the first time, we have*

$$(n\beta)^{\frac{1}{2}} \leq \left(\frac{\Theta_{\frac{s+1}{2}}}{\mathcal{C}^s(\lambda-1)}\right)^{\frac{1}{s+1}}\left(\frac{\mathcal{E}}{\delta}\right)^{\frac{1}{s+1}}, \tag{36}$$

where

$$\Theta_{\frac{s+1}{2}} = \begin{cases} 1, & 0 \leq s \leq 1, \\ \left(\frac{s+1}{2}\right)^{\frac{s+1}{2}}, & s > 1. \end{cases} \tag{37}$$

Proof. From the definition of n and Lemma 5.2, we obtain

$$\begin{aligned}
\lambda\delta &\leq \left\|\mathcal{D}_k^{\gamma,1}(L)\mathcal{G}_{n-1}^\delta - f^\delta\right\|_{\mathcal{H}} \\
&= \left\|\sum_{k=1}^{\infty}\left[1-\left[1-\left(1-\beta|\mathcal{D}_k^{\gamma,1}(L)|^2\right)^{n-1}\right]^\gamma\right]\langle f^\delta, e_k\rangle e_k\right\|_{\mathcal{H}} \\
&\leq \left\|\sum_{k=1}^{\infty}\left(1-\beta|\mathcal{D}_k^{\gamma,1}(L)|^2\right)^{n-1}\langle f^\delta - f, e_k\rangle e_k\right\|_{\mathcal{H}} \\
&\quad + \left\|\sum_{k=1}^{\infty}\left(1-\beta|\mathcal{D}_k^{\gamma,1}(L)|^2\right)^{n-1}\langle f, e_k\rangle e_k\right\|_{\mathcal{H}}
\end{aligned} \tag{38}$$

Hence, we have

$$\begin{aligned}\lambda\delta &\le \delta + \mathcal{E}\sup_{\xi_k>0}\left(1-\beta|\mathcal{D}_k^{\gamma,1}(L)|^2\right)^{n-1}|\mathcal{D}_k^{\gamma,1}(L)|\xi_k^{-s}\\ &\le \delta + \mathcal{C}^{-s}\mathcal{E}\sup_{\xi_k>0}\left(1-\beta|\mathcal{D}_k^{\gamma,1}(L)|^2\right)^{n-1}\left(\beta|\mathcal{D}_k^{\gamma,1}(L)|^2\right)^{\frac{s+1}{2}}\beta^{-\frac{s+1}{2}}\\ &\le \delta + \mathcal{C}^{-s}\mathcal{E}\Theta_{\frac{s+1}{2}}(n\beta)^{-\frac{s+1}{2}}.\end{aligned} \tag{39}$$

This yields

$$(n\beta)^{\frac{1}{2}} \le \left(\frac{\Theta_{\frac{s+1}{2}}}{\mathcal{C}^s(\lambda-1)}\right)^{\frac{1}{s+1}}\left(\frac{\mathcal{E}}{\delta}\right)^{\frac{1}{s+1}}. \tag{40}$$

□

Theorem 6.3. *If the noise assumption 4 hold and the a-priori condition (17) then*

$$\|\mathcal{G}_n^\delta-\mathcal{G}\|_{\mathcal{H}} \le \left[\left(\frac{\lambda+1}{\mathcal{C}}\right)^{\frac{s}{s+1}}+\left(\frac{\Theta_{\frac{s+1}{2}}}{\mathcal{C}^s(\lambda-1)}\right)^{\frac{1}{s+1}}\right]\mathcal{E}^{\frac{1}{s+1}}\delta^{\frac{s}{s+1}}. \tag{41}$$

Proof. By the triangle inequality, we have

$$\|\mathcal{G}_n^\delta-\mathcal{G}\|_{\mathcal{H}} \le \|\mathcal{G}_n^\delta-\mathcal{G}_n\|_{\mathcal{H}}+\|\mathcal{G}-\mathcal{G}_n\|_{\mathcal{H}} = \mathcal{I}_3+\mathcal{I}_4. \tag{42}$$

From 28, we have

$$\mathcal{I}_3 = \|\mathcal{G}_n^\delta-\mathcal{G}_n\|_{\mathcal{H}} \le \sqrt{\beta}\sqrt{n}\,\delta \le \left(\frac{\Theta_{\frac{s+1}{2}}}{\mathcal{C}^s(\lambda-1)}\right)^{\frac{1}{s+1}}\mathcal{E}^{\frac{1}{s+1}}\delta^{\frac{s}{s+1}}. \tag{43}$$

Applying the a-priori bound condition (17), we know that

$$\begin{aligned}\mathcal{I}_4 &= \|\mathcal{G}-\mathcal{G}_n\|_{\mathcal{H}}\\ &= \left\|\sum_{k=1}^{\infty}\left[1-\left[1-\left(1-\beta|\mathcal{D}_k^{\gamma,1}(L)|^2\right)^n\right]^\alpha\right]\langle f,e_k\rangle e_k\right\|_{\mathcal{H}}\\ &\le \left\|\sum_{k=1}^{\infty}\left(1-\beta|\mathcal{D}_k^{\gamma,1}(L)|^2\right)^n\langle f,e_k\rangle e_k\right\|_{\mathcal{H}}\\ &\le \left\|\sum_{k=1}^{\infty}\left(1-\beta|\mathcal{D}_k^{\gamma,1}(L)|^2\right)^n\langle f,e_k\rangle e_k\right\|_{\mathcal{H}}^{\frac{s}{s+1}}\\ &\qquad\times\left\|\sum_{k=1}^{\infty}\left(1-\beta|\mathcal{D}_k^{\gamma,1}(L)|^2\right)^n\xi_k^{-s}\xi_k^{s}\langle f,e_k\rangle e_k\right\|_{\mathcal{H}}^{\frac{1}{s+1}}.\end{aligned} \tag{44}$$

Thank to Hölder inequality, using the triangle inequality, it gives

$$\begin{aligned}\mathcal{I}_4 &\le \Bigg(\left\|\sum_{k=1}^{\infty}(1-\beta|\mathcal{D}_k^{\gamma,1}(L)|^2)^n\frac{1}{|\mathcal{D}_k^{\gamma,1}(L)|}\langle f^\delta-f,e_k\rangle e_k\right\|_{\mathcal{H}}\\ &\quad+\left\|\sum_{k=1}^{\infty}\left(1-\beta|\mathcal{D}_k^{\gamma,1}(L)|^2\right)^n\frac{1}{|\mathcal{D}_k^{\gamma,1}(L)|}\langle f^\delta,e_k\rangle e_k\right\|\Bigg)^{\frac{s}{s+1}}\left\|\sum_{k=1}^{\infty}\xi_k^{-s}\xi_k^{s}\langle f,e_k\rangle e_k\right\|_{\mathcal{H}}^{\frac{1}{s+1}}\\ &\le (\delta+\lambda\delta)^{\frac{s}{s+1}}\mathcal{E}^{\frac{1}{s+1}}\sup_{\xi_k>0}\left(\frac{1}{\xi_k|\mathcal{D}_k^{\gamma,1}(L)|}\right)^{\frac{s}{s+1}}\\ &\le \left(\frac{\lambda+1}{\mathcal{C}}\right)^{\frac{s}{s+1}}\mathcal{E}^{\frac{1}{s+1}}\delta^{\frac{s}{s+1}}.\end{aligned} \tag{45}$$

Combining (43), (44) and (45), we obtain the convergence estimate. Thus, formula (41) of Theorem 6.3 has been proven. □

7 Declarations

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The authors declare that they have no competing interests.

Ethical Approval

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Authors's Contributions

All authors contributed equally. All the authors read and approved the final manuscript.

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