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# Weak Mean Attractors of Fractional Stochastic Lattice Systems Driven by Nonlinear Delay Noise

**Ailin Bai** [1], **Pengyu Chen**[1,*]

**1.** Department of Mathematics, Northwest Normal University, Lanzhou 730070, China

[*]Corresponding author

## Abstract

This paper deals with the existence and uniqueness of weak pullback mean random attractors for fractional stochastic lattice systems driven by *nonlinear* delay noise defined on the entire integer set $\mathbb{Z}$. We first establish the global well-posedness to stochastic lattice system in $C([\tau,\infty), L^2(\Omega,\ell^2))$ when the nonlinear diffusion terms and drift terms are locally Lipschitz continuous functions. Then we define a mean random dynamical system through the solution operators and prove the existence and uniqueness of weak pullback mean random attractors in $L^2(\Omega,\mathcal{F};\ell^2)\times L^2(\Omega,\mathcal{F};L^2((-\rho,0),\ell^2))$ under certain conditions.

**Keywords**: Fractional stochastic lattice system, nonlinear delay noise, weak mean random attractor
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## 1 Introduction

This paper is concerned with the existence and uniqueness of weak mean attractors for fractional stochastic lattice systems driven by *nonlinear* delay noise defined on the entire integer set $\mathbb{Z}$

$$\begin{cases} \frac{du_i(t)}{dt} + (-\Delta_d)^\alpha u_i(t) + \lambda u_i(t) = f_i(u_i(t)) + g_i(t) \\ \qquad\qquad + \sum\limits_{k=1}^{\infty} \left(a_{k,i}(t) + \sigma_{k,i}(u_i(t), u_i(t-\rho))\right) \frac{dW_k(t)}{dt}, \quad t > \tau, \\ u_i(\tau) = u_{0,i}, \quad u_i(s+\tau) = \varphi_i(s), \quad s \in (-\rho, 0), \end{cases} \tag{1}$$

where $u = (u_i)_{i\in\mathbb{Z}}$ is an unknown sequence, $i \in \mathbb{Z}$, $\tau \in \mathbb{R}$, $\lambda > 0$, $(-\Delta_d)^\alpha$ with $\alpha \in (0,1)$ is the fractional discrete Laplacian, $g(t) = (g_i(t))_{i\in\mathbb{Z}}$ and $a_k(t) = (a_{k,i}(t))_{i\in\mathbb{Z}}$ are time-dependent random sequences for each $k \in \mathbb{N}$, $f_i$ is continuously differentiable polynomial growth nonlinearities, $\sigma_{k,i}$ is local Lipschitz continuous function for every $i \in \mathbb{Z}$ and $k \in \mathbb{N}$, $\rho > 0$ is a

**Contact:** Ailin Bai bailin1376@163.com; Pengyu Chen chpengyu123@163.com

time delay parameter, and $(W_k)_{k\in\mathbb{N}}$ is a sequence of independent two-side real-valued Wiener processes defined on a complete filtered probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\in\mathbb{R}}, \mathbb{P})$ satisfying the usual condition.

The fractional discrete Laplacian $(-\Delta_d)^\alpha$ reduces to the discrete Laplacian $(-\Delta_d)$ if $\alpha = 1$. The discrete Laplacian means nearest neighbor interaction (or called local interaction) of nodes. Fractional discrete Laplacian $(-\Delta_d)^\alpha$ is the fractional powers of the discrete Laplacian and has been considered in [3, 4, 6, 7, 15]. Lattice systems have been widely considered in many fields such as biology, chemical reaction, physics and so on, see [12]. Recently, there are many works on lattice systems, like global attractors, the chaotic properties of solutions [11], traveling waves and invariant measures [8, 9, 10, 16, 17, 21].

As is well known, pathwise random attractors of stochastic lattice systems/PDEs driven by linear multiplicative noise or additive noise were investigated in [2, 5, 18, 22]. However, if the stochastic lattice system is driven by nonlinear noise, then there is no approach available in the literature on convert a stochastic system into a pathwise deterministic one, and thereby the existence of pathwise random attractors cannot be studied. In order to deal with the nonlinear diffusion coefficients of noise, Kloeden [14] and Wang [20, 21] have introduced the concept of mean random attractors. The well-posedness and the dynamics of these equations have been studied by many experts.

However, it seems that there is not a result available in the literature on the existence of weak mean attractors for fractional stochastic lattice systems driven by *nonlinear* delay noise. Inspired by reference [23], the main purpose of the present paper is to establish the global well-posedness as well as existence of weak pullback mean random attractors for system (1) driven by *nonlinear* delay noise. To achieve this goal, we will first establish the global existence as well as uniqueness of solutions to (1) in $L^2(\Omega, C([\tau,\infty), \ell^2)) \bigcap L^2(\Omega, L^2((\tau-\rho,\tau), \ell^2))$ when the nonlinear drift function $f_i$ and diffusion functions $\sigma_{k,i}$ are locally Lipschitz continuous. It is worth mentioning that the weak mean random attractors are established on reflexive Banach spaces, and cannot be applied to stochastic processes taking values in general Banach space $C([-\rho,0], \ell^2)$ which is usually chosen as a phase space to study the pathwise random dynamics of delayed lattice systems. Our idea of solving this problem is to choose the reflexive Banach space $L^2(\Omega, \mathcal{F}; \ell^2) \times L^2\big(\Omega, \mathcal{F}; L^2((-\rho,0), \ell^2)\big)$ as a phase space. Based on the well-posedness, we will define a mean random dynamical system in $L^2(\Omega, \mathcal{F}; \ell^2) \times L^2\big(\Omega, \mathcal{F}; L^2((-\rho,0), \ell^2)\big)$ and prove the existence and uniqueness of weak pullback mean random attractor.

This article is organized as follows. In Section 2, we show the global well-posedness of solutions for the fractional stochastic lattice system (1) driven by nonlinear delay noise. In the last section, we prove the existence and uniqueness of weak pullback mean random attractors for the system (1) in $L^2(\Omega, \mathcal{F}; \ell^2) \times L^2\big(\Omega, \mathcal{F}; L^2((-\rho,0), \ell^2)\big)$.

## 2 Well-posedness of fractional stochastic lattice system

In this section, we prove the global existence and uniqueness of solutions to problem (1) when the nonlinear diffusion terms and drift terms are locally Lipschitz continuous functions, which is a necessary step for studying the weak mean random attractors of the stochastic lattice systems.

We denote by $\ell^p$ $(1 \le p \le \infty)$ with the following Banach space:

$$\ell^p = \Big\{u = (u_i)_{i\in\mathbb{Z}} : \sum_{i\in\mathbb{Z}} |u_i|^p < +\infty\Big\}, \quad p \ge 1,$$

endowed with the norm

$$\|u\|_p = \Big(\sum_{i\in\mathbb{Z}} |u_i|^p\Big)^{\frac{1}{p}}.$$

If $p = 2$, then $\ell^2$ is a Hilbert space, and we use $\|\cdot\|$ and $(\cdot,\cdot)$ to denote the norm and inner product of $\ell^2$, respectively.

In order to establish the definition of fractional discrete Laplacian, for $0 \le \alpha \le 1$, we define $\ell_\alpha$ by

$$\ell_\alpha = \left\{ u : \mathbb{Z} \to \mathbb{R} \mid \|u\|_{\ell_\alpha} = \sum_{m\in\mathbb{Z}} \frac{|u_m|}{(1+|m|)^{1+2\alpha}} < \infty \right\}.$$

Obviously, $\ell^p \subset \ell^q \subset \ell_\alpha$ if $1 \le p \le q \le \infty$ and $0 \le \alpha \le 1$.

In this paper, we assume that $g = (g_i)_{i\in\mathbb{Z}}$ and $a_k = (a_{k,i})_{k\in\mathbb{N},i\in\mathbb{Z}}$ are $\ell^2$-valued progressively measurable processes such that

$$\int_\tau^{\tau+T} \mathbb{E}\Big(\|g(r)\|^2 + \sum_{k\in\mathbb{N}} \|a_k(r)\|^2\Big) dr < \infty, \quad \forall \tau \in \mathbb{R},\ T > 0. \tag{2}$$

For the nonlinear term in (1), we assume that the nonlinear drift function $f_i$ satisfies, there exist positive numbers $\beta$ and $\gamma_0$ such that

$$|f_i(x)| \le \gamma_{0,i}|x|^{p-1} + \beta_i, \ \ \gamma_0 = (\gamma_{0,i})_{i\in\mathbb{Z}} \in \ell^\infty, \ \beta = (\beta_i)_{i\in\mathbb{Z}} \in \ell^2. \tag{3}$$

In addition, for every $i \in \mathbb{Z}$,

$$|f_i'(x)| \le \gamma_{1,i}|x|^{p-2} + \beta_{1,i}, \ \ \gamma_1 = (\gamma_{1,i})_{i\in\mathbb{Z}} \in \ell^\infty, \ \beta_1 = (\beta_{1,i})_{i\in\mathbb{Z}} \in \ell^\infty. \tag{4}$$

For the diffusion coefficients in (1), we assume that $\sigma_{k,i} : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is locally Lipschitz continuous uniformly with respect to $i \in \mathbb{Z}$, that is, for every $k \in \mathbb{N}$ and any compact interval $I \subseteq \mathbb{R}$, there exists a constant $L_k > 0$ such that

$$|\sigma_{k,i}(x_1, x_1^*) - \sigma_{k,i}(x_2, x_2^*)| \le L_k(|x_1 - x_2| + |x_1^* - x_2^*|), \qquad \forall\, x_1, x_1^*, x_2, x_2^* \in I. \tag{5}$$

We also assume that $\sigma_{k,i} : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ grows linearly, that is, for every $k \in \mathbb{N}$ and $i \in \mathbb{Z}$, there exist positive numbers $\delta_{k,i}$ and $\gamma_k$ with $(\delta_{k,i})_{k\in\mathbb{N},i\in\mathbb{Z}} \in \ell^2$ and $(\gamma_k)_{k\in\mathbb{N}} \in \ell^2$ such that

$$|\sigma_{k,i}(x, x^*)| \le \gamma_k(|x| + |x^*|) + \delta_{k,i}, \qquad \forall\, x, x^* \in \mathbb{R},\ k \in \mathbb{N}. \tag{6}$$

In the following, we formulate system (1) as an abstract stochastic ordinary differential equation in $\ell^2$. To this end, we introduce the definitions of fractional discrete Laplace operator $(-\Delta_d)^\alpha$ with $\alpha \in (0,1)$. As the fractional Laplace operator given in [13], there are numerous approaches to define the fractional discrete Laplace operator. For simplicity, we consider the case that the size of mesh equals 1 in this paper. For a function $u : \mathbb{Z} \to \mathbb{R}$, we use the notation $u_i = u(i)$ to denote the value of $u$ at the mesh point $i \in \mathbb{Z}$. The discrete Laplacian $-\Delta_d$ is given by

$$-\Delta_d u_i = 2u_i - u_{i-1} - u_{i+1}.$$

For $0 < \alpha < 1$ and $u_j \in \mathbb{R}$, the fractional discrete Laplacian $(-\Delta_d)^\alpha$ is defined with the semigroup method (or Bochner subordination) as

$$(-\Delta_d)^\alpha u_j = \frac{1}{\Gamma(-\alpha)} \int_0^\infty (e^{t\Delta_d} u_j - u_j) \frac{dt}{t^{1+\alpha}}, \tag{7}$$

where $\Gamma(\cdot)$ denotes the Gamma function with $\Gamma(-\alpha) = \int_0^\infty (e^{-r} - 1) \frac{dr}{r^{1+\alpha}} < 0$ and $w_j(t) = e^{t\Delta_d} u_j$ is the solution to the semidiscrete heat equation

$$\begin{cases} \partial_t w_j = \Delta_d w_j, & \text{in} \quad \mathbb{Z} \times (0,\infty), \\ w_j(0) = u_j, & \text{on} \quad \mathbb{Z}. \end{cases} \tag{8}$$

It follows from the semidiscrete Fourier transform that the solution of (8) can be written as

$$e^{t\Delta_d}u_j = \sum_{m\in\mathbb{Z}} G(j-m,t)u_m = \sum_{m\in\mathbb{Z}} G(m,t)u_{j-m}, \quad t \geq 0, \tag{9}$$

where the semidiscrete heat kernel $G(m,t) = e^{-2t}I_m(2t)$ and $I_v$ is the modified Bessel function of order $v$. By (7) and (9), we have the pointwise formula for $(-\Delta_d)^\alpha$ presented in the following statement.

**Theorem 2.1.** *[3, Theorem 1.1] Let $0 < \alpha < 1$ and $u = (u_j)_{j\in\mathbb{Z}} \in \ell_\alpha$. Then we have*

$$(-\Delta_d)^\alpha u_j = \sum_{\substack{m\in\mathbb{Z}\\ m\neq j}} (u_j - u_m)\widetilde{K}_\alpha(j-m), \tag{10}$$

*where the discrete kernel $\widetilde{K}_\alpha$ is given by*

$$\widetilde{K}_\alpha(m) = \begin{cases} \frac{4^\alpha\Gamma(\frac{1}{2}+\alpha)}{\sqrt{\pi}|\Gamma(-\alpha)|}\cdot\frac{\Gamma(|m|-\alpha)}{\Gamma(|m|+1+\alpha)}, & m \in \mathbb{Z}\backslash\{0\}, \\ 0, & m = 0. \end{cases}$$

*In addition, there exist positive constants $\check{c_\alpha}$ and $\hat{c_\alpha}$ with $\check{c_\alpha} \leq \hat{c_\alpha}$ such that for any $m \in \mathbb{Z}\backslash\{0\}$,*

$$\frac{\check{c_\alpha}}{|m|^{1+2\alpha}} \leq \widetilde{K}_\alpha(m) \leq \frac{\hat{c_\alpha}}{|m|^{1+2\alpha}}.$$

It follows from the above that the fractional discrete Laplacian is a nonlocal operator on $\mathbb{Z}$. Meanwhile, $(-\Delta_d)^\alpha u$ is a well defined bounded function whenever $u \in \ell^p$ $(1 \leq p \leq \infty)$. In particular, we also find that, for $0 < \alpha < 1$, if $u \in \ell^2$, then

$$(-\Delta_d)^\alpha u \in \ell^2, \quad \text{satisfying } \|(-\Delta_d)^\alpha u\| \leq 4^\alpha\|u\|. \tag{11}$$

On the other hand, fractional discrete Laplacian can be given by means of the discrete Fourier transform. To introduce this method, we denote the following function of order $\sigma > 0$:

$$R_\sigma := \frac{1}{2\pi}\int_{-\pi}^{\pi}\left(4\sin^2\left(\frac{\theta}{2}\right)\right)^\sigma e^{-in\theta}d\theta, \quad n \in \mathbb{Z}. \tag{12}$$

A computation shows that the function $R_\sigma(n)$ can be explicitly written as

$$R_\sigma(n) = \frac{(-1)^n\Gamma(2\sigma+1)}{\Gamma(1+\sigma+n)\Gamma(1+\sigma-n)}, \quad n \in \mathbb{Z},\ \sigma \in (0,\infty)\backslash\mathbb{N}.$$

Therefore, by using the semidiscrete Fourier transform and (12), we have the following expression for the fractional discrete Laplacian (please see [15] for more details).

**Theorem 2.2.** *Let $0 < \alpha < 1$. If $u \in \ell^2$ then*

$$\begin{aligned}(-\Delta_d)^\alpha u_j &= \left(\mathcal{F}_{\mathbb{Z}}^{-1}\left(4\sin^2\left(\frac{\theta}{2}\right)\right)^\alpha * u\right)_j \\ &= \sum_{m\in\mathbb{Z}} R_\alpha(j-m)u_m, \end{aligned} \tag{13}$$

*where $\mathcal{F}_{\mathbb{Z}}^{-1}$ denotes the inverse semidiscrete Fourier transform and $*$ means convolution. Furthermore, $\widetilde{K}_\alpha(j) = -R_\alpha(j)$ for all $j \in \mathbb{Z}\backslash\{0\}$ and $\widetilde{K}_\alpha(0) = \sum_{m\in\mathbb{Z}} R_\alpha(m) = 0$.*

In view of (11) and (13), by using Fubini's Theorem, we have the following result of $(-\Delta_d)^\alpha$. More detailed information can be found in [3, Lemma 6.2].

**Lemma 2.3.** *Let $u, v \in \ell^2$. Then for every $\alpha \in (0,1)$,*

$$
\begin{aligned}
((-\Delta_d)^\alpha u, v) &= ((-\Delta_d)^{\frac{\alpha}{2}} u, (-\Delta_d)^{\frac{\alpha}{2}} v) \\
&= \frac{1}{2} \sum_{j\in\mathbb{Z}} \sum_{\substack{m\in\mathbb{Z}\\ m\neq j}} (u_j - u_m)(v_j - v_m)\widetilde{K}_\alpha(j-m).
\end{aligned}
$$

Obviously, by Lemma 2.3 we have

$$
\Big((-\Delta_d)^{\frac{\alpha}{2}} u, (-\Delta_d)^{\frac{\alpha}{2}} u\Big) = \|(-\Delta_d)^{\frac{\alpha}{2}} u\|^2 = \frac{1}{2} \sum_{j\in\mathbb{Z}} \sum_{\substack{m\in\mathbb{Z}\\ m\neq j}} |u_j - u_m|^2 \widetilde{K}_\alpha(j-m) \ \text{ for } \ u \in \ell^2. \tag{14}
$$

In what follows, we will define the fractional discrete Laplacian $(-\Delta_d)^\alpha : \ell^2 \to \ell^2$, then it yields from (11), Lemma 2.3 and (14) that for all $u = (u_i)_{i\in\mathbb{Z}} \in \ell^2$, $v = (v_i)_{i\in\mathbb{Z}} \in \ell^2$,

$$
\begin{aligned}
&\sum_{i\in\mathbb{Z}} \Big(((-\Delta_d)^\alpha u_i) - (-\Delta_d)^\alpha v_i)\Big)(u_i - v_i) \\
=&\Big((-\Delta_d)^\alpha u - (-\Delta_d)^\alpha v, u - v\Big) = \Big((-\Delta_d)^\alpha (u-v), u - v\Big) \\
=&\Big((-\Delta_d)^{\frac{\alpha}{2}} (u-v), (-\Delta_d)^{\frac{\alpha}{2}} (u-v)\Big) \\
=&\frac{1}{2} \sum_{i\in\mathbb{Z}} \sum_{\substack{m\in\mathbb{Z}\\ m\neq i}} \Big((u-v)_i - (u-v)_m\Big)\Big((u-v)_i - (u-v)_m\Big)\widetilde{K}_\alpha(i-m) \\
=&\frac{1}{2} \sum_{i\in\mathbb{Z}} \sum_{\substack{m\in\mathbb{Z}\\ m\neq i}} |(u-v)_i - (u-v)_m|^2 \widetilde{K}_\alpha(i-m) \\
=&\|(-\Delta_d)^{\frac{\alpha}{2}} (u-v)\|^2 \\
\leq& 4^\alpha \|(u-v)\|^2.
\end{aligned} \tag{15}
$$

This implies that for every $(-\Delta_d)^\alpha u, (-\Delta_d)^\alpha v \in \ell^2$, we can find a constant $L > 0$ satisfying, for all $u, v \in \ell^2$,

$$
((-\Delta_d)^\alpha u - (-\Delta_d)^\alpha v, u - v) = \|(-\Delta_d)^{\frac{\alpha}{2}} (u-v)\|^2 \leq L\|u - v\|^2. \tag{16}
$$

In addition, it follows from (16) that

$$
((-\Delta_d)^\alpha u - (-\Delta_d)^\alpha v, u - v) \geq 0, \quad \forall\, u, v \in \ell^2. \tag{17}
$$

To simplify the notation, we set

$$
\beta = (\beta_i)_{i\in\mathbb{Z}}, \quad \gamma = (\gamma_k)_{k\in\mathbb{N}}, \quad \|\beta\|^2 = \sum_{i\in\mathbb{Z}} |\beta_i|^2, \quad \|\gamma\|^2 = \sum_{k\in\mathbb{N}} |\gamma_k|^2.
$$

For each $k \in \mathbb{N}$, we define the operators $f : \ell^2 \to \ell^2$, $\sigma_k : \ell^2 \times \ell^2 \to \ell^2$ by

$$
f(u) = \big(f_i(u_i)\big)_{i\in\mathbb{Z}} \ \text{and} \ \sigma_k(u,v) = \big(\sigma_{k,i}(u_i, v_i)\big)_{i\in\mathbb{Z}}, \ \forall\, u = (u_i)_{i\in\mathbb{Z}} \in \ell^2,\, v = (v_i)_{i\in\mathbb{Z}} \in \ell^2.
$$

It follows from (4) and differential mean value theorem for complex numbers, there exists $\mu \in (0,1)$ such that for all $p > 2$ and $u_1, u_2 \in \ell^2$,

$$\begin{aligned}\sum_{i\in\mathbb{Z}} |f_i(u_{1,i}) - f_i(u_{2,i})|^2 &= \sum_{i\in\mathbb{Z}} \left|f_i'(\mu u_{1,i} + (1-\mu)u_{2,i})\right|^2 |u_{1,i} - u_{2,i}|^2 \\ &\le \sum_{i\in\mathbb{Z}} \Big(|\gamma_{1,i}||\kappa u_{1,i} + (1-\mu)u_{2,i}|^{p-2} + |\beta_{1,i}|\Big)^2 |u_{1,i} - u_{2,i}|^2 \\ &\le \sum_{i\in\mathbb{Z}} \Big(2^{2p-4}|\gamma_{1,i}|^2(|u_{1,i}|^{2p-4} + |u_{2,i}|^{2p-4}) + 2|\beta_{1,i}|^2\Big)|u_{1,i} - u_{2,i}|^2 \\ &\le \Big(2^{2p-4}\|\gamma_1\|_\infty^2(\|u_1\|^{2p-4} + \|u_2\|^{2p-4}) + 2\|\beta_1\|_\infty^2\Big)\|u_1 - u_2\|^2. \end{aligned} \tag{18}$$

This together with $f_i(0) \in \ell^2$ by (3) yields $f(u) \in \ell^2$ for all $u \in \ell^2$, one can verify that $f : \ell^2 \to \ell^2$ is well-defined. In addition, we deduce from (18) that $f$ is locally Lipschitz continuous, that is, for any $n > 0$, there exists $L_f(n) > 0$ such that for all $u_1, u_2 \in \ell^2$ with $\|u_1\| \le n$ and $\|u_2\| \le n$,

$$\|f(u_1) - f(u_2)\|^2 \le L_f(n)\|u_1 - u_2\|^2, \tag{19}$$

and by (3) we obtain

$$\|f(u)\|^2 \le 4\gamma_0^2\|u\|^2 + 2\|\beta\|^2, \ \ \forall\, u \in \ell^2. \tag{20}$$

For each $k \in \mathbb{N}$, we get from (5) and (6) that $\sigma_k$: $\ell^2 \times \ell^2 \to \ell^2$ is locally Lipschitz continuous, that is, for any $n > 0$, there exists $L_\sigma(n) > 0$ such that for all $u_1, u_2, v_1, v_2 \in \ell^2$ with $\|u_1\| \le n$, $\|u_2\| \le n$, $\|v_1\| \le n$ and $\|v_2\| \le n$,

$$\sum_{k\in\mathbb{N}} \|\sigma_k(u_1, v_1) - \sigma_k(u_2, v_2)\|^2 \le L_\sigma(n)\big(\|u_1 - u_2\|^2 + \|v_1 - v_2\|^2\big), \tag{21}$$

and

$$\sum_{k\in\mathbb{N}} \|\sigma_k(u, v)\|^2 \le 4\|\gamma\|^2\big(\|u\|^2 + \|v\|^2\big) + 2\|\delta\|^2, \ \ \forall\, u, v \in \ell^2. \tag{22}$$

In light of above notations, we can rewrite system (1) as the following abstract system in $\ell^2$:

$$\begin{cases} \frac{du(t)}{dt} + (-\Delta_d)^\alpha u(t) + \lambda u(t) = f(u(t)) + g(t) \\ \qquad\qquad + \sum\limits_{k=1}^{\infty} \big(a_k(t) + \sigma_k(u(t), u(t-\rho))\big)\frac{dW_k(t)}{dt}, \ \ t > \tau, \\ u(\tau) = u_0, \quad u(s+\tau) = \varphi(s), \quad s \in (-\rho, 0), \end{cases} \tag{23}$$

where $u_0 = (u_{0,i})_{i\in\mathbb{Z}}$, $\varphi = (\varphi_i)_{i\in\mathbb{Z}}$.

In this article, the solutions of system (23) are understanded in the following sense.

**Definition 2.4.** For every $\tau \in \mathbb{R}$, $u_0 \in L^2(\Omega, \mathcal{F}_\tau; \ell^2)$ and $\varphi \in L^2\big(\Omega, \mathcal{F}_\tau; L^2((-\rho, 0), \ell^2)\big)$, a continuous $\ell^2$-valued stochastic process $u(t)$ with $t \in (\tau - \rho, \infty)$ is called a solution of system (23) if $(u_t)_{t\ge 0}$ is $\mathcal{F}_t$-adapted, $u_\tau = \varphi$, $u(\tau) = u_0$, $u \in L^2(\Omega, C([\tau, \tau + T], \ell^2)) \bigcap L^2(\Omega, L^2((\tau - \rho, \tau), \ell^2))$ for all $T > 0$, and for all $t \ge \tau$ and almost all $\omega \in \Omega$:

$$\begin{aligned} u(t) =& u_0 + \int_\tau^t \big(-(-\Delta_d)^\alpha u(s) - \lambda u(s) + f(u(s)) + g(s)\big)ds \\ &+ \sum_{k=1}^{\infty} \int_\tau^t \big(a_k(s) + \sigma_k(u(s), u(s-\rho))\big)dW_k(s). \end{aligned} \tag{24}$$

Note that the fractional discrete Laplacian $(-\Delta_d)^\alpha$ is bounded from $\ell^2$ to $\ell^2$. Moreover, the nonlinear functions $f$ and $\sigma_k$ are only locally Lipschitz continuous from $\ell^2$ to $\ell^2$. To obtain the global existence and uniqueness of solutions of system (23), we need to approximate the locally Lipschitz continuous operators $f$ and $\sigma_k$ by globally Lipschitz continuous operators. To that end, for each $n > 0$, we define a cut-off function $\xi_n : \mathbb{R} \to \mathbb{R}$ by

$$\xi_n(x) = \begin{cases} -n, & \text{for } x \in (-\infty, -n), \\ x, & \text{for } x \in [-n, n], \\ n, & \text{for } x \in (n, +\infty). \end{cases} \tag{25}$$

Then we find that $\xi_n : \mathbb{R} \to \mathbb{R}$ is increasing and has the following properties:

$$\xi_n(0) = 0, \quad |\xi_n(x)| \le n \quad \text{and} \quad |\xi_n(x_2) - \xi_n(x_1)| \le |x_2 - x_1|, \quad \forall\, x, x_1, x_2 \in \mathbb{R}. \tag{26}$$

For $n > 0$, $k \in \mathbb{N}$, $u = (u_i)_{i\in\mathbb{Z}} \in \ell^2$ and $v = (v_i)_{i\in\mathbb{Z}} \in \ell^2$, we denote by

$$f_n(u) = \big(f_i(\xi_n(u_i))\big)_{i\in\mathbb{Z}} \quad \text{and} \quad \sigma_{n,k}(u, v) = \big(\sigma_{k,i}(\xi_n(u_i), \xi_n(v_i))\big)_{i\in\mathbb{Z}}. \tag{27}$$

Similar with the proof of (19) and (20), by (25) and (26) we know that for all $n > 0$, $f_n$: $\ell^2 \to \ell^2$ is globally Lipschitz continuous, that is, for each $n > 0$,

$$\|f_n(u_1) - f_n(u_2)\|^2 \le L_f(n)\big(\|u_1 - u_2\|^2\big), \ \ \forall\, u_1, u_2 \in \ell^2, \tag{28}$$

and

$$\|f_n(u)\|^2 \le 4\gamma_0^2\|u\|^2 + 2\|\beta\|^2, \ \ \forall\, u \in \ell^2. \tag{29}$$

In addition, similar with the proof of (21) and (22), we get from (25) and (26) that for all $n > 0$, $\sigma_{n,k}$: $\ell^2 \times \ell^2 \to \ell^2$ is also globally Lipschitz continuous, more precisely, for each $n > 0$,

$$\sum_{k\in\mathbb{N}} \|\sigma_{n,k}(u_1, v_1) - \sigma_{n,k}(u_2, v_2)\|^2 \le L_\sigma(n)\big(\|u_1 - u_2\|^2 + \|v_1 - v_2\|^2\big), \ \ \forall\, u_1, u_2, v_1, v_2 \in \ell^2, \tag{30}$$

and

$$\sum_{k\in\mathbb{N}} \|\sigma_{n,k}(u, v)\|^2 \le 4\|\gamma\|^2\big(\|u\|^2 + \|v\|^2\big) + 2\|\beta\|^2, \ \ \forall\, u, v \in \ell^2. \tag{31}$$

For every $n \in \mathbb{N}$, we consider the following approximate stochastic equation on $\ell^2$:

$$\begin{cases} \frac{du_n(t)}{dt} + (-\Delta_d)^\alpha u_n(t) + \lambda u_n(t) = f_n(u_n(t)) + g(t) \\ \qquad\qquad + \sum\limits_{k=1}^{\infty} \big(a_k(t) + \sigma_{n,k}(u_n(t), u_n(t-\rho))\big)\frac{dW_k(t)}{dt}, \ \ t > \tau, \\ u_n(\tau) = u_0, \quad u_n(s+\tau) = \varphi(s), \quad s \in (-\rho, 0). \end{cases} \tag{32}$$

By the arguments of proving the existence and uniqueness of solutions of stochastic equations in $\mathbb{R}^n$ (see e.g., [1]), we can prove, under (28)-(29) and (30)-(31) that for every $n \in \mathbb{N}$, $\tau \in \mathbb{R}$, $u_0 \in L^2(\Omega, \mathcal{F}_\tau; \ell^2)$ and $\varphi \in L^2(\Omega, \mathcal{F}_\tau; L^2((-\rho, 0), \ell^2))$, approximate system (32) has a unique solution $u_n$ in the sense of Definition 2.4 with $f$ and $\sigma_k$ are replaced by $f_n$ and $\sigma_{n,k}$, respectively.

In what follows, we prove the existence and uniqueness of solutions of the abstract fractional stochastic lattice system (23) in the sense of Definition 2.4 by considering the limiting behavior of the family of $\{u_n\}_{n=1}^\infty$ of solutions of approximate fractional stochastic lattice system (32) as $n \to \infty$. To that end, for $\tau \in \mathbb{R}$ and $n \in \mathbb{N}$, we define a stopping time by

$$\tau_n = \inf\{t \ge \tau : \|u_n(t)\| > n\} \ \text{ and } \ \tau_n = +\infty \ \text{ if } \ \{t \ge \tau : \|u_n(t)\| > n\} = \emptyset. \tag{33}$$

**Lemma 2.5.** *Suppose that the assumptions (2)-(6) hold. If $u_n$ is the solution of approximate fractional stochastic lattice system (32), then*

$$u_{n+1}(t \wedge \tau_n) = u_n(t \wedge \tau_n) \quad \text{and} \quad \tau_{n+1} \geq \tau_n, \text{ a.s.,} \tag{34}$$

*for all $t \geq \tau$, $n \in \mathbb{N}$ and $\tau_n$ is the stopping time.*

*Proof.* By (32), we have

$$\begin{aligned}
&u_{n+1}(t \wedge \tau_n) - u_n(t \wedge \tau_n) + \int_\tau^{t\wedge\tau_n} \big((-\Delta_d)^\alpha (u_{n+1}(s)) - (-\Delta_d)^\alpha (u_n(s))\big) ds \\
&+ \lambda \int_\tau^{t\wedge\tau_n} \big(u_{n+1}(s) - u_n(s)\big) ds \\
=& \int_\tau^{t\wedge\tau_n} \big(f_{n+1}(u_{n+1}(s)) - f_n(u_n(s))\big) ds \\
&+ \sum_{k=1}^\infty \int_\tau^{t\wedge\tau_n} \big(\sigma_{n+1,k}(u_{n+1}(s), u_{n+1}(s-\rho)) - \sigma_{n,k}(u_n(s), u_n(s-\rho))\big) dW_k(s).
\end{aligned} \tag{35}$$

It follows from (35) and Ito's formula that a.s.,

$$\begin{aligned}
&\|u_{n+1}(t \wedge \tau_n) - u_n(t \wedge \tau_n)\|^2 + 2\lambda \int_\tau^{t\wedge\tau_n} \|u_{n+1}(s) - u_n(s)\|^2 ds \\
&+ 2 \int_\tau^{t\wedge\tau_n} \big((-\Delta_d)^\alpha (u_{n+1}(s)) - (-\Delta_d)^\alpha (u_n(s)), u_{n+1}(s) - u_n(s)\big) ds \\
=&2 \int_\tau^{t\wedge\tau_n} \big(f_{n+1}(u_{n+1}(s)) - f_n(u_n(s)), u_{n+1}(s) - u_n(s)\big) ds \\
&+ \sum_{k=1}^\infty \int_\tau^{t\wedge\tau_n} \|\sigma_{n+1,k}(u_{n+1}(s), u_{n+1}(s-\rho)) - \sigma_{n,k}(u_n(s), u_n(s-\rho))\|^2 ds \\
&+ 2 \sum_{k=1}^\infty \int_\tau^{t\wedge\tau_n} \big(u_{n+1}(s) - u_n(s)\big) \big(\sigma_{n+1,k}(u_{n+1}(s), u_{n+1}(s-\rho)) \\
&- \sigma_{n,k}(u_n(s), u_n(s-\rho))\big) dW_k(s),
\end{aligned} \tag{36}$$

where $u_{n+1}(s) - u_n(s)$ in the last term is identified with an element in the dual space of $\ell^2$ in view of the Riesz representation theorem.

For all $n \in \mathbb{N}$ and $s \in [\tau, \tau_n)$, we have $\|u_n(s)\| \leq n$ for $s \in [\tau, \tau_n)$ and therefore

$$\begin{aligned}
&f_{n+1}(u_n(s)) = f_n(u_n(s)) = f(u_n(s)), \\
&\sigma_{n+1,k}(u_n(s), u_n(s-\rho)) = \sigma_{n,k}(u_n(s), u_n(s-\rho)) = \sigma_k(u_n(s), u_n(s-\rho)).
\end{aligned} \tag{37}$$

For the third term on the left-hand side of (36), we get from (16), (37) and Cauchy-Schwarz inequality that

$$\begin{aligned}
&2\left| \int_\tau^{t\wedge\tau_n} \big((-\Delta_d)^\alpha (u_{n+1}(s)) - (-\Delta_d)^\alpha (u_n(s)), u_{n+1}(s) - u_n(s)\big) ds \right| \\
=&2\left| \int_\tau^{t\wedge\tau_n} \left\| (-\Delta_d)^{\frac{\alpha}{2}} \big(u_{n+1}(s) - u_n(s)\big) \right\|^2 ds \right| \\
\leq&2L \int_\tau^{t\wedge\tau_n} \|u_{n+1}(s) - u_n(s)\|^2 ds.
\end{aligned} \tag{38}$$

By (28), (37) and Cauchy-Schwarz inequality one know that the first term on the right-hand side of (36) satisfies

$$\begin{aligned}
&2\int_{\tau}^{t\wedge\tau_n}\left(f_{n+1}(u_{n+1}(s))-f_n(u_n(s)),u_{n+1}(s)-u_n(s)\right)ds\\
=&2\int_{\tau}^{t\wedge\tau_n}\left(f_{n+1}(u_{n+1}(s))-f_{n+1}(u_n(s)),u_{n+1}(s)-u_n(s)\right)ds\\
\le&2\sqrt{L_f(n+1)}\int_{\tau}^{t\wedge\tau_n}\|u_{n+1}(s)-u_n(s)\|^2ds. \qquad (39)
\end{aligned}$$

By (30), (37) and Cauchy-Schwarz inequality we get that the second term on the right-hand side of (36) satisfies

$$\begin{aligned}
&\sum_{k=1}^{\infty}\int_{\tau}^{t\wedge\tau_n}\|\sigma_{n+1,k}(u_{n+1}(s),u_{n+1}(s-\rho))-\sigma_{n,k}(u_n(s),u_n(s-\rho))\|^2ds\\
=&\sum_{k=1}^{\infty}\int_{\tau}^{t\wedge\tau_n}\|\sigma_{n+1,k}(u_{n+1}(s),u_{n+1}(s-\rho))-\sigma_{n+1,k}(u_n(s),u_n(s-\rho))\|^2ds\\
\le&L_\sigma(n+1)\int_{\tau}^{t\wedge\tau_n}\|u_{n+1}(s)-u_n(s)\|^2ds+L_\sigma(n+1)\int_{\tau}^{t\wedge\tau_n}\|u_{n+1}(s-\rho)-u_n(s-\rho)\|^2ds. \quad (40)
\end{aligned}$$

It follows from (36) and (38)-(40) that

$$\begin{aligned}
&\|u_{n+1}(t\wedge\tau_n)-u_n(t\wedge\tau_n)\|^2\\
\le&\left(2L+2\sqrt{L_f(n+1)}+L_\sigma(n+1)\right)\int_{\tau}^{t\wedge\tau_n}\|u_{n+1}(s)-u_n(s)\|^2ds\\
&+L_\sigma(n+1)\int_{\tau-\rho}^{t\wedge\tau_n}\|u_{n+1}(s)-u_n(s)\|^2ds\\
&+2\sum_{k=1}^{\infty}\int_{\tau}^{t\wedge\tau_n}(u_{n+1}(s)-u_n(s))\left(\sigma_{n+1,k}(u_{n+1}(s),u_{n+1}(s-\rho))\right.\\
&\left.-\sigma_{n,k}(u_n(s),u_n(s-\rho))\right)dW_k(s)\\
\le&c_1\int_{\tau}^{t\wedge\tau_n}\|u_{n+1}(s)-u_n(s)\|^2ds+2\sum_{k=1}^{\infty}\int_{\tau}^{t\wedge\tau_n}(u_{n+1}(s)-u_n(s))\\
&(\sigma_{n+1,k}(u_{n+1}(s),u_{n+1}(s-\rho))-\sigma_{n,k}(u_n(s),u_n(s-\rho)))dW_k(s), \qquad (41)
\end{aligned}$$

where $c_1=2\left(L+\sqrt{L_f(n+1)}+L_\sigma(n+1)\right)$. Then we get form (41) that

$$\begin{aligned}
&\mathbb{E}\left(\sup_{\tau\le s\le t}\|u_{n+1}(s\wedge\tau_n)-u_n(s\wedge\tau_n)\|^2\right)\le c_1\int_{\tau}^{t}\mathbb{E}\left(\sup_{\tau\le s\le r}\|u_{n+1}(s\wedge\tau_n)-u_n(s\wedge\tau_n)\|^2\right)dr\\
&+2\mathbb{E}\left(\sup_{\tau\le s\le t\wedge\tau_n}\left|\sum_{k=1}^{\infty}\int_{\tau}^{s}(u_{n+1}(r)-u_n(r))\right.\right.\\
&\left.\left.(\sigma_{n+1,k}(u_{n+1}(r),u_{n+1}(r-\rho))-\sigma_{n,k}(u_n(r),u_n(r-\rho)))dW_k(r)\right|\right). \qquad (42)
\end{aligned}$$

By (37), (40) and the Burkholder-Davis-Gundy inequality, one can find that there exists a constant $c_2 > 0$ such that the second term on the right-hand side of (42) satisfies

$$
\begin{aligned}
&2\mathbb{E}\left(\sup_{\tau\le s\le t\wedge\tau_n}\left|\sum_{k=1}^{\infty}\int_{\tau}^{s}(u_{n+1}(r)-u_n(r))\left(\sigma_{n+1,k}(u_{n+1}(r),u_{n+1}(r-\rho))\right.\right.\right.\\
&\left.\left.\left.-\sigma_{n,k}(u_n(r),u_n(r-\rho))\right)dW_k(r)\right|\right)\\
\le&2c_2\mathbb{E}\left(\left(\int_{\tau}^{t\wedge\tau_n}\left(\|u_{n+1}(r)-u_n(r)\|^2\right.\right.\right.\\
&\left.\left.\left.\sum_{k=1}^{\infty}\|\sigma_{n+1,k}(u_{n+1}(r),u_{n+1}(r-\rho))-\sigma_{n,k}(u_n(r),u_n(r-\rho))\|^2\right)dr\right)^{\frac{1}{2}}\right)\\
\le&2c_2\mathbb{E}\left(\sup_{\tau\le r\le t}\|u_{n+1}(r\wedge\tau_n)-u_n(r\wedge\tau_n)\|\right.\\
&\left.\left(\int_{\tau}^{t\wedge\tau_n}\sum_{k=1}^{\infty}\|\sigma_{n+1,k}(u_{n+1}(r),u_{n+1}(r-\rho))-\sigma_{n+1,k}(u_n(r),u_n(r-\rho))\|^2dr\right)^{\frac{1}{2}}\right)\\
\le&\ 2c_2\sqrt{L_\sigma(n+1)}\mathbb{E}\left(\sup_{\tau\le r\le t}\|u_{n+1}(r\wedge\tau_n)-u_n(r\wedge\tau_n)\|\right.\\
&\left.\left(\int_{\tau}^{t\wedge\tau_n}(\|u_{n+1}(r)-u_n(r)\|^2+\|u_{n+1}(r-\rho)-u_n(r-\rho)\|^2)dr\right)^{\frac{1}{2}}\right)\\
\le&\ 2\sqrt{2}c_2\sqrt{L_\sigma(n+1)}\mathbb{E}\left(\sup_{\tau\le r\le t}\|u_{n+1}(r\wedge\tau_n)-u_n(r\wedge\tau_n)\|\left(\int_{\tau}^{t\wedge\tau_n}\|u_{n+1}(r)-u_n(r)\|^2dr\right)^{\frac{1}{2}}\right)\\
\le&\ \frac{1}{2}\mathbb{E}\Big(\sup_{\tau\le s\le t}\|u_{n+1}(s\wedge\tau_n)-u_n(s\wedge\tau_n)\|^2\Big)+c_3\int_{\tau}^{t}\mathbb{E}\Big(\sup_{\tau\le s\le r}\|u_{n+1}(s\wedge\tau_n)-u_n(s\wedge\tau_n)\|^2\Big)dr,
\end{aligned}
\tag{43}
$$

where $c_3 = 4c_2^2L_\sigma(n+1)$. By (42)-(43), we obtain

$$
\begin{aligned}
&\mathbb{E}\Big(\sup_{\tau\le s\le t}\|u_{n+1}(s\wedge\tau_n)-u_n(s\wedge\tau_n)\|^2\Big)\\
\le&2(c_1+c_3)\int_{\tau}^{t}\mathbb{E}\Big(\sup_{\tau\le s\le r}\|u_{n+1}(s\wedge\tau_n)-u_n(s\wedge\tau_n)\|^2\Big)dr.
\end{aligned}
\tag{44}
$$

Applying the Gronwall lemma to (44), we get

$$
\mathbb{E}\Big(\sup_{\tau\le s\le t}\|u_{n+1}(s\wedge\tau_n)-u_n(s\wedge\tau_n)\|^2\Big)=0,\ \ \text{for all } t\ge\tau,
$$

which implies that $u_{n+1}(t\wedge\tau_n) = u_n(t\wedge\tau_n)$ for all $t \ge \tau$ almost surely. This along with (33), we find $\tau_{n+1} \ge \tau_n$ as desired. This completes the proof. □

Next, we show the uniform estimates of solutions $u_n$ of approximate fractional stochastic lattice system (32), and then obtain the limit of stopping time $\tau_n$ defined by (33) as $n \to \infty$.

**Lemma 2.6.** *Suppose that the assumptions (2)-(6) hold. Let $\tau_n$ be the stopping time defined by (33). If $u_n$ is the solution of approximate fractional stochastic lattice system (32), then for every $T > 0$, $u_n$ satisfies*

$$\mathbb{E}\Big(\sup_{\tau\le r\le t}\|u_n(r\wedge\tau_n)\|^2\Big) \le Me^{MT}\Big(\mathbb{E}(\|u_0\|^2)+T+\int_{-\rho}^{0}\mathbb{E}(\|\varphi(s)\|^2)ds + \int_{\tau}^{\tau+T}\mathbb{E}\Big(\|g(t)\|^2+\sum_{k\in\mathbb{N}}\|a_k(t)\|^2\Big)dt\Big), \tag{45}$$

*where $M > 0$ is a constant independent of $u_0$, $\varphi$, $\tau$, $\rho$ and $T$. In addition, we have the following limit property,*

$$\tau := \lim_{n\to\infty}\tau_n = \sup_{n\in\mathbb{N}}\tau_n = \infty, \quad \textit{almost surely}. \tag{46}$$

*Proof.* By (32) and Ito's formula, we have

$$\begin{aligned}
&\|u_n(t\wedge\tau_n)\|^2 + 2\int_{\tau}^{t\wedge\tau_n}\big((-\Delta_d)^{\alpha}(u_n(s)),u_n(s)\big)ds + 2\lambda\int_{\tau}^{t\wedge\tau_n}\|u_n(s)\|^2ds\\
=&\|u_n(\tau)\|^2 + 2\int_{\tau}^{t\wedge\tau_n}\big(f(u_n(s)),u_n(s)\big)ds\\
&+2\int_{\tau}^{t\wedge\tau_n}\big(g(s),u_n(s)\big)ds + \sum_{k=1}^{\infty}\int_{\tau}^{t}\|a_k(s)+\sigma_{n,k}(u_n(s),u_n(s-\rho))\|^2ds\\
&+2\sum_{k=1}^{\infty}\int_{\tau}^{t\wedge\tau_n}u_n(s)\big(a_k(s)+\sigma_{n,k}(u_n(s),u_n(s-\rho))\big)dW_k(s).
\end{aligned} \tag{47}$$

We get from Lemma 2.3, (14) and (17) that the second term on the left-hand side of (47) satisfies

$$2\int_{\tau}^{t\wedge\tau_n}\big((-\Delta_d)^{\alpha}(u_n(s)),u_n(s)\big)ds = 2\int_{\tau}^{t\wedge\tau_n}\big\|(-\Delta_d)^{\frac{\alpha}{2}}u_n(s)\big\|^2ds \ge 0. \tag{48}$$

For the second term on the right-hand side of (47), by (20), (37) and Cauchy-Schwarz inequality we obtain that for $t\in[\tau,\tau+T]$,

$$\begin{aligned}
2\int_{\tau}^{t\wedge\tau_n}(f_n(u_n(s)),u_n(s))ds =&2\int_{\tau}^{t\wedge\tau_n}(f(u_n(s)),u_n(s))ds\\
\le&\int_{\tau}^{t\wedge\tau_n}\|f(u_n(s))\|^2ds + \int_{\tau}^{t\wedge\tau_n}\|u_n(s)\|^2ds\\
\le&2\|\beta\|^2(t-\tau)+(4\gamma_0^2+1)\int_{\tau}^{t\wedge\tau_n}\|u_n(s)\|^2ds\\
\le&2\|\beta\|^2T+(8\gamma_0^2+1)\int_{\tau}^{t\wedge\tau_n}\|u_n(s)\|^2ds.
\end{aligned} \tag{49}$$

For the third term on the right-hand side of (47), we get from Cauchy-Schwarz inequality that

$$2\int_{\tau}^{t\wedge\tau_n}(g(s),u_n(s))ds \le \int_{\tau}^{t\wedge\tau_n}\|u_n(s)\|^2ds + \int_{\tau}^{t\wedge\tau_n}\|g(s)\|^2ds. \tag{50}$$

For the fourth term on the right-hand side of (47), by (22) and (37) one gets that for $t \in [\tau, \tau + T]$,

$$\begin{aligned}
&\sum_{k=1}^{\infty} \int_{\tau}^{t\wedge\tau_n} \|a_k(s) + \sigma_{n,k}(u_n(s), u_n(s-\rho))\|^2 ds \\
=&\sum_{k=1}^{\infty} \int_{\tau}^{t\wedge\tau_n} \|a_k(s) + \sigma_{k}(u_n(s), u_n(s-\rho))\|^2 ds \\
\leq &2\int_{\tau}^{t\wedge\tau_n} \sum_{k=1}^{\infty} \|a_k(s)\|^2 ds + 2\int_{\tau}^{t\wedge\tau_n} \sum_{k=1}^{\infty} \|\sigma_k(u_n(s), u_n(s-\rho))\|^2 ds \\
\leq &2\int_{\tau}^{t\wedge\tau_n} \sum_{k=1}^{\infty} \|a_k(s)\|^2 ds + 4\|\delta\|^2 (t-\tau) + 8\|\gamma\|^2 \int_{\tau}^{t\wedge\tau_n} \|u_n(s)\|^2 ds \\
&+ 8\|\gamma\|^2 \int_{\tau}^{t\wedge\tau_n} \|u_n(s-\rho)\|^2 ds \\
\leq &4\|\delta\|^2 T + 16\|\gamma\|^2 \int_{\tau}^{t\wedge\tau_n} \|u_n(s)\|^2 ds + 8\|\gamma\|^2 \int_{-\rho}^{0} \|\varphi(s)\|^2 ds + 2\int_{\tau}^{t\wedge\tau_n} \sum_{k=1}^{\infty} \|a_k(s)\|^2 ds. \qquad (51)
\end{aligned}$$

Therefore, it follows from (47)-(51) one gets that for $t \in [\tau, \tau + T]$,

$$\begin{aligned}
\|u_n(t\wedge\tau_n)\|^2 \leq &\|u_n(\tau)\|^2 + 2(8\|\gamma\|^2 + 4\gamma_0^2 + 1) \int_{\tau}^{t\wedge\tau_n} \|u_n(s)\|^2 ds + 8\|\gamma\|^2 \int_{-\rho}^{0} \|\varphi(s)\|^2 ds \\
&+ 2\sum_{k=1}^{\infty} \int_{\tau}^{t} u_n(s)\big(a_k(s) + \sigma_{n,k}(u_n(s), u_n(s-\rho))\big) dW_k(s) \\
&+ \int_{\tau}^{t\wedge\tau_n} \Big(\|g(s)\|^2 + 2\sum_{k=1}^{\infty} \|a_k(s)\|^2\Big) ds + 2\|\beta\|^2 T + 4\|\delta\|^2 T. \qquad (52)
\end{aligned}$$

We further obtain from (52) that for $t \in [\tau, \tau + T]$,

$$\begin{aligned}
&\mathbb{E}\Big(\sup_{\tau\leq r\leq t} \|u_n(r\wedge\tau_n)\|^2\Big) \\
\leq &\mathbb{E}(\|u_n(\tau)\|^2) + 2(8\|\gamma\|^2 + 4\gamma_0^2 + 1) \int_{\tau}^{t} \mathbb{E}\Big(\sup_{\tau\leq r\leq s} \|u_n(r\wedge\tau_n)\|^2\Big) ds \\
&+ 8\|\gamma\|^2 \int_{-\rho}^{0} \mathbb{E}(\|\varphi(s)\|^2) ds + \int_{\tau}^{\tau+T} \mathbb{E}\Big(\|g(s)\|^2 + 2\sum_{k=1}^{\infty} \|a_k(s)\|^2\Big) ds \\
&+ 2\mathbb{E}\left(\sup_{\tau\leq r\leq t\wedge\tau_n} \left|\sum_{k=1}^{\infty} \int_{\tau}^{r} u_n(s)\big(a_k(s) + \sigma_{n,k}(u_n(s), u_n(s-\rho))\big) dW_k(s)\right|\right) \\
&+ 2(\|\beta\|^2 + 2\|\delta\|^2) T. \qquad (53)
\end{aligned}$$

By the Burkholder-Davis-Gundy inequality, (37) and (51), one find that there exists a constant

$c_4 > 0$ such that the fifth term on the right-hand side of (53) satisfies, for $t \in [\tau, \tau + T]$,

$$
\begin{aligned}
&2\mathbb{E}\left( \sup_{\tau \le r \le t \wedge \tau_n} \left| \sum_{k=1}^{\infty} \int_{\tau}^{r} u_n(s)\big(a_k(s) + \sigma_{n,k}(u_n(s), u_n(s-\rho))\big) dW_k(s) \right| \right) \\
\le& 2c_4 \mathbb{E}\left( \int_{\tau}^{t\wedge\tau_n} \sum_{k=1}^{\infty} \|u_n(s)\|^2 \|a_k(s) + \sigma_{n,k}(u_n(s), u_n(s-\rho))\|^2 ds \right)^{\frac{1}{2}} \\
\le& 2\sqrt{2} c_4 \mathbb{E}\left( \sup_{\tau \le s \le t} \|u_n(s\wedge\tau_n)\| \left( \int_{\tau}^{t\wedge\tau_n} \sum_{k=1}^{\infty} \Big( \|a_k(s)\|^2 + \|\sigma_{n,k}(u_n(s), u_n(s-\rho))\|^2 \Big) ds \right)^{\frac{1}{2}} \right) \\
\le& \frac{1}{2} \mathbb{E}\Big( \sup_{\tau \le s \le t} \|u_n(s\wedge\tau_n)\|^2 \Big) + 4c_4^2 \int_{\tau}^{t\wedge\tau_n} \mathbb{E}\Big( \sum_{k=1}^{\infty} \Big( \|a_k(s)\|^2 + \|\sigma_k(u_n(s), u_n(s-\rho))\|^2 \Big) \Big) ds \\
\le& \frac{1}{2} \mathbb{E}\Big( \sup_{\tau \le r \le t} \|u_n(r\wedge\tau_n)\|^2 \Big) + 32\|\gamma\|^2 c_4^2 \int_{\tau}^{t} \mathbb{E}\Big( \sup_{\tau \le r \le s} \|u_n(r\wedge\tau_n)\|^2 \Big) ds \\
&+ 4c_4^2 \int_{\tau}^{\tau+T} \mathbb{E}\Big( \sum_{k=1}^{\infty} \|a_k(s)\|^2 \Big) ds + 16\|\gamma\|^2 c_4^2 \int_{-\rho}^{0} \mathbb{E}(\|\varphi(s)\|^2) ds + 8\|\delta\|^2 c_4^2 T.
\end{aligned} \tag{54}
$$

Hence, we get from (53)-(54) that for $t \in [\tau, \tau + T]$,

$$
\mathbb{E}\Big( \sup_{\tau \le r \le t} \|u_n(r\wedge\tau_n)\|^2 \Big) \le c_5 \int_{\tau}^{t} \mathbb{E}\Big( \sup_{\tau \le r \le s} \|u_n(r\wedge\tau_n)\|^2 \Big) ds + c_6, \tag{55}
$$

where

$$
c_5 = 4(1 + 4\gamma_0^2 + 8\|\gamma\|^2 + 16\|\gamma\|^2 c_4^2),
$$

$$
\begin{aligned}
c_6 =& 2\mathbb{E}(\|u_0\|^2) + 8(2\|\gamma\|^2 + 4\|\gamma\|^2 c_4^2) \int_{-\rho}^{0} \mathbb{E}(\|\varphi(s)\|^2) ds \\
&+ 2\int_{\tau}^{\tau+T} \mathbb{E}\Big( \|g(s)\|^2 + (2 + 4c_4^2) \sum_{k=1}^{\infty} \|a_k(s)\|^2 \Big) ds + 4(\|\beta\|^2 + 2\|\delta\|^2 + 4\|\delta\|^2 c_4^2) T.
\end{aligned}
$$

Applying the Gronwall inequality to (55), we obtain that

$$
\mathbb{E}\Big( \sup_{\tau \le r \le t} \|u_n(r\wedge\tau_n)\|^2 \Big) \le c_6 e^{c_5 T}, \ \forall\, t \in [\tau, \tau + T],
$$

which yields the desired uniform estimate (45).

Next, we prove the second result (46), which is an immediate consequence of (45) and Chebychev's inequality. Indeed, let $T \in \mathbb{N}$ be an arbitrary number. By (33), we get that

$$
\{\tau_n < \tau + T\} \subseteq \Big\{ \sup_{\tau \le t \le \tau+T} \|u_n(t\wedge\tau_n)\| \ge n \Big\},
$$

which along with (45) and Chebychev's inequality yields that

$$
\begin{aligned}
\mathbb{P}\{\tau_n < \tau + T\} \le& \mathbb{P}\Big\{ \sup_{\tau \le t \le \tau+T} \|u_n(t\wedge\tau_n)\| \ge n \Big\} \\
\le& \frac{1}{n^2} \mathbb{E}\Big( \sup_{\tau \le t \le \tau+T} \|u_n(t\wedge\tau_n)\|^2 \Big) \le \frac{M}{n^2},
\end{aligned} \tag{56}
$$

where $M$ independent of $n$ is the same number as in (45). It follows from (56) that

$$\sum_{n=1}^{\infty} \mathbb{P}\{\tau_n < \tau + T\} \leq M \sum_{n=1}^{\infty} \frac{1}{n^2} < +\infty. \tag{57}$$

Let $\Omega_T = \bigcap\limits_{m=1}^{\infty} \bigcup\limits_{n=m}^{\infty} \{\tau_n < \tau + T\}$. Then we obtain from the Borel-Cantelli lemma and (57) that

$$\mathbb{P}(\Omega_T) = \mathbb{P}\left( \bigcap_{m=1}^{\infty} \bigcup_{n=m}^{\infty} \{\tau_n < \tau + T\} \right) = 0.$$

Then for every $\omega \in \Omega \setminus \Omega_T$, there exists $n_0 = n_0(\omega) > 0$ such that $\tau_n(\omega) \geq \tau + T$ for all $n \geq n_0$. Note that $\tau_n$ is increasing in $n$, we know $\tau(\omega) \geq \tau + T$ for all $\omega \in \Omega \setminus \Omega_T$. Let $\Omega_0 = \bigcup\limits_{T=1}^{\infty} \Omega_T$, we have $\mathbb{P}(\Omega_0) = 0$ and

$$\tau(\omega) \geq \tau + T \text{ for all } \omega \in \Omega \setminus \Omega_0 \text{ and } T \in \mathbb{N}.$$

Therefore, we find that $\tau(\omega) = \infty$ for all $\omega \in \Omega \setminus \Omega_0$, and then (46) follows. This completes the proof. □

In what follows, we prove the main result of this section, i.e., the existence and uniqueness of solutions of system (1).

**Theorem 2.7.** *Suppose that the assumptions (2)-(6) hold. Then for every $\tau \in \mathbb{R}$, $u_0 \in L^2(\Omega, \mathcal{F}_\tau; \ell^2)$ and $\varphi \in L^2(\Omega, \mathcal{F}_\tau; L^2((-\rho, 0), \ell^2))$, fractional stochastic lattice system (1) has a unique solution $u$ in the sense of Definition 2.4. Moreover, for any $T > 0$, $u$ satisfies*

$$\begin{aligned} \mathbb{E}\left( \|u\|^2_{C\left([\tau,\tau+T],\ell^2\right)} \right) \leq M e^{MT} \Bigg( & \mathbb{E}(\|u_0\|^2) + T + \int_{-\rho}^{0} \mathbb{E}(\|\varphi(s)\|^2) ds \\ & + \int_{\tau}^{\tau+T} \mathbb{E}\left( \|g(t)\|^2 + \sum_{k \in \mathbb{N}} \|a_k(t)\|^2 \right) dt \Bigg), \end{aligned} \tag{58}$$

*where $M > 0$ is a constant independent of $u_0$, $\varphi$, $\tau$, $\rho$ and $T$.*

*Proof.* We only need prove the existence and uniqueness of solutions for abstract fractional stochastic lattice system (23). By Lemma 2.5 and Lemma 2.6, we know that there exists $\Omega_1 \subseteq \Omega$ with $\mathbb{P}(\Omega \setminus \Omega_1) = 0$ such that for all $n > 0$, $\omega \in \Omega_1$ and $t \geq \tau$,

$$\tau(\omega) = \lim_{n \to \infty} \tau_n(\omega) = \infty, \tag{59}$$

$$u_{n+1}(t \wedge \tau_n, \omega) = u_n(t \wedge \tau_n, \omega). \tag{60}$$

By (59) and (60), we know that for every $\omega \in \Omega_1$ and $t \geq \tau$, there exists $n_0 = n_0(t, \omega) \geq 1$ such that

$$\tau_n(\omega) > t, \text{ and hence } u_n(t, \omega) = u_{n_0}(t, \omega), \text{ for all } n \geq n_0. \tag{61}$$

One can define a mapping $u : (\tau - \rho, \infty) \times \Omega \to \ell^2$ by

$$u(t, \omega) = \begin{cases} u_n(t, \omega), & \text{if } \omega \in \Omega_1 \text{ and } t \in (\tau - \rho, \tau_n(\omega)], \\ u_\tau(s, \omega), & \text{if } \omega \in \Omega \setminus \Omega_1, s \in (-\rho, 0] \text{ and } t \in (\tau - \rho, \infty). \end{cases} \tag{62}$$

Since $u_n$ is a continuous $\ell^2$-valued process, we infer from (62) that $u$ is almost surely continuous with respect to $t$ in $\ell^2$. By (62), we see

$$\lim_{n\to\infty} u_n(t,\omega) = u(t,\omega), \ \ \forall\, \omega \in \Omega_1, \ t > \tau - \rho. \tag{63}$$

Since $u_n$ is $\mathcal{F}_t$-adapted, we infer from (63) that $u$ is also $\mathcal{F}_t$-adapted. It follows from (45), (63) and Fatou's lemma that for every $T > 0$,

$$\begin{aligned}\mathbb{E}\Big(\|u\|^2_{C\left([\tau,\tau+T],\ell^2\right)}\Big) \leq & Me^{MT}\bigg(\mathbb{E}(\|u_0\|^2) + T + \int_{-\rho}^{0} \mathbb{E}(\|\varphi(s)\|^2)ds \\ & + \int_{\tau}^{\tau+T} \mathbb{E}\Big(\|g(t)\|^2 + \sum_{k\in\mathbb{N}} \|a_k(t)\|^2\Big)dt\bigg),\end{aligned}$$

where $M$ is the same number as in (45). This shows the desired inequality (58).

On the other hand, it follows from (32) that

$$\begin{aligned}u_n(t\wedge\tau_n) = u_0 + & \int_{\tau}^{t\wedge\tau_n} \big(-(-\Delta_d)^{\alpha}u_n(s) - \lambda u_n(s) + f_n(u_n(s)) + g(s)\big)ds \\ & + \sum_{k=1}^{\infty}\int_{\tau}^{t\wedge\tau_n} \big(a_k(s) + \sigma_{n,k}(u_n(s), u_n(s-\rho))\big)dW_k(s),\end{aligned} \tag{64}$$

in $\ell^2$ for all $t \geq \tau$. By (62) we find $u_n(t\wedge\tau_n) = u(t\wedge\tau_n)$ a.s.. Then by the definition of $f_n$ and $\sigma_{n,k}$, we see that a.s.,

$$f_n(u_n(s)) = f(u(s)) \quad \text{and} \quad \sigma_{n,k}(u_n(s), u_n(s-\rho)) = \sigma_k(u(s), u(s-\rho)), \ \ \forall\, s \in [\tau, \tau_n). \tag{65}$$

Therefore, by (64)-(65) we see that a.s.,

$$\begin{aligned}u(t\wedge\tau_n) = u_0 + & \int_{\tau}^{t\wedge\tau_n} \big(-(-\Delta_d)^{\alpha}u(s) - \lambda u(s) + f(u(s)) + g(s)\big)ds \\ & + \sum_{k=1}^{\infty}\int_{\tau}^{t\wedge\tau_n} \big(a_k(s) + \sigma_k(u(s), u(s-\rho))\big)dW_k(s)\end{aligned} \tag{66}$$

in $\ell^2$ for all $t \geq \tau$. Since $\lim_{n\to\infty} \tau_n = \infty$ a.s., we obtain from (66) that a.s.,

$$\begin{aligned}u(t) = u_0 + & \int_{\tau}^{t} \big(-(-\Delta_d)^{\alpha}u(s) - \lambda u(s) + f(u(s)) + g(s)\big)ds \\ & + \sum_{k=1}^{\infty}\int_{\tau}^{t} \big(a_k(s) + \sigma_k(u(s), u(s-\rho))\big)dW_k(s)\end{aligned}$$

in $\ell^2$ for all $t \geq \tau$. This implies that $u$ is a solution of system (23) in the sense of Definition 2.4.

We now prove the uniqueness of solutions of system (23). Let $u_1$ and $u_2$ are two solutions of (23). For every $n > 0$ and $T > 0$, we define another stoping time:

$$T_n = (\tau + T) \wedge \inf\{t \geq \tau : \|u_1(t)\| \geq n \text{ or } \|u_2(t)\| \geq n\}. \tag{67}$$

By (23), we get

$$\begin{aligned}& u_1(t\wedge T_n) - u_2(t\wedge T_n) + \int_{\tau}^{t\wedge T_n} \big((-\Delta_d)^{\alpha}(u_1(s)) - (-\Delta_d)^{\alpha}(u_2(s))\big)ds \\ & + \lambda\int_{\tau}^{t\wedge T_n}(u_1(s) - u_2(s))ds + \int_{\tau}^{t\wedge T_n}\big(f(u_1(s)) - f(u_2(s))\big)ds \\ = & u_1(\tau) - u_2(\tau) + \sum_{k=1}^{\infty}\int_{\tau}^{t\wedge T_n}\big(\sigma_k(u_1(s), u_1(s-\rho)) - \sigma_k(u_2(s), u_2(s-\rho))\big)dW_k(s).\end{aligned} \tag{68}$$

By Ito's formula and (68), we obtain that a.s.,

$$\begin{aligned}
&\|u_1(t\wedge T_n)-u_2(t\wedge T_n)\|^2+2\lambda\int_\tau^{t\wedge T_n}\|u_1(s)-u_2(s)\|^2ds\\
&+2\int_\tau^{t\wedge T_n}\big((-\Delta_d)^\alpha(u_1(s))-(-\Delta_d)^\alpha(u_2(s)),u_1(s)-u_2(s)\big)ds\\
=&2\int_\tau^{t\wedge T_n}\big(f(u_1(s))-f(u_2(s)),u_1(s)-u_2(s)\big)ds\\
&+\sum_{k=1}^{\infty}\int_\tau^{t\wedge T_n}\|\sigma_k(u_1(s),u_1(s-\rho))-\sigma_k(u_2(s),u_2(s-\rho))\|^2ds\\
&+2\sum_{k=1}^{\infty}\int_\tau^{t\wedge T_n}\big(u_1(s)-u_2(s)\big)\big(\sigma_k(u_1(s),u_1(s-\rho))-\sigma_k(u_2(s),u_2(s-\rho))\big)dW_k(s).
\end{aligned}\tag{69}$$

Next, we estimate all terms of (69). By (16), (67) and Cauchy-Schwarz inequality, we get that the second term on the left-hand side of (69) satisfies

$$2\left|\int_\tau^{t\wedge T_n}\big((-\Delta_d)^\alpha(u_1(s))-(-\Delta_d)^\alpha(u_2(s)),u_1(s)-u_2(s)\big)ds\right|\le 2L\int_\tau^{t\wedge T_n}\|u_1(s)-u_2(s)\|^2ds.\tag{70}$$

For the first term on the right-hand side of (69), by (19), (67) and Cauchy-Schwarz inequality, we obtain that

$$\begin{aligned}
&2\int_\tau^{t\wedge T_n}\big(f(u_1(s))-f(u_2(s)),u_1(s)-u_2(s)\big)ds\\
\le&2\sqrt{L_f(n)}\int_\tau^{t\wedge T_n}\|u_1(s)-u_2(s)\|^2ds.
\end{aligned}\tag{71}$$

By (21), (67) and Cauchy-Schwarz inequality, we get that the second term on the right-hand side of (69) satisfies

$$\begin{aligned}
&\sum_{k=1}^{\infty}\int_\tau^{t\wedge T_n}\|\sigma_k(u_1(s),u_1(s-\rho))-\sigma_k(u_2(s),u_2(s-\rho))\|^2ds\\
\le&L_\sigma(n)\int_\tau^{t\wedge T_n}\|u_1(s)-u_2(s)\|^2ds+L_\sigma(n)\int_\tau^{t\wedge T_n}\|u_1(s-\rho)-u_2(s-\rho)\|^2ds\\
\le&L_\sigma(n)\int_\tau^{t\wedge T_n}\|u_1(s)-u_2(s)\|^2ds+L_\sigma(n)\int_{\tau-\rho}^{\tau}\|u_1(s)-u_2(s)\|^2ds\\
&+L_\sigma(n)\int_\tau^{t\wedge T_n}\|u_1(s)-u_2(s)\|^2ds\\
=&2L_\sigma(n)\int_\tau^{t\wedge T_n}\|u_1(s)-u_2(s)\|^2ds.
\end{aligned}\tag{72}$$

Then we obtain from (69)-(72) that

$$\begin{aligned}
&\|u_1(t\wedge T_n)-u_2(t\wedge T_n)\|^2\le c_7\int_\tau^{t\wedge T_n}\|u_1(s)-u_2(s)\|^2ds\\
&+2\sum_{k=1}^{\infty}\int_\tau^{t\wedge T_n}\big(u_1(s)-u_2(s)\big)\big(\sigma_k(u_1(s),u_1(s-\rho))-\sigma_k(u_2(s),u_2(s-\rho))\big)dW_k(s),
\end{aligned}\tag{73}$$

where $c_7 = 2\big(L + \sqrt{L_f(n)} + L_\sigma(n)\big)$. By (73) one gets that

$$\begin{aligned}&\mathbb{E}\Big(\sup_{\tau\le s\le t}\|u_1(s\wedge T_n)-u_2(s\wedge T_n)\|^2\Big)\le c_7\int_\tau^t\sup_{\tau\le s\le r}\mathbb{E}\Big(\|u_1(s\wedge T_n)-u_2(s\wedge T_n)\|^2\Big)dr\\&+2\mathbb{E}\left(\sup_{\tau\le s\le t\wedge T_n}\sum_{k=1}^{\infty}\int_\tau^s(u_1(r)-u_2(r))\big(\sigma_k(u_1(r),u_1(r-\rho))-\sigma_k(u_2(r),u_2(r-\rho))\big)dW_k(r)\right).\end{aligned}\tag{74}$$

For the last term in (74), we get from (72) and the Burkholder-Davis-Gundy inequality that there exists a constant $c_8 > 0$ such that

$$\begin{aligned}&2\mathbb{E}\left(\sup_{\tau\le s\le t\wedge T_n}\sum_{k=1}^{\infty}\int_\tau^s(u_1(r)-u_2(r))\big(\sigma_k(u_1(r),u_1(r-\rho))-\sigma_k(u_2(r),u_2(r-\rho))\big)dW_k(r)\right)\\
\le&2c_8\mathbb{E}\left(\left(\int_\tau^{t\wedge T_n}\left(\|u_1(r)-u_2(r)\|^2\sum_{k=1}^{\infty}\|\sigma_k(u_1(r),u_1(r-\rho))-\sigma_k(u_2(r),u_2(r-\rho))\|^2\right)dr\right)^{\frac12}\right)\\
\le&2c_8\mathbb{E}\left(\sup_{\tau\le r\le t}\|u_1(r\wedge T_n)-u_2(r\wedge T_n)\|\right.\\
&\left.\left(\int_\tau^{t\wedge T_n}\sum_{k=1}^{\infty}\|\sigma_k(u_1(r),u_1(r-\rho))-\sigma_k(u_2(r),u_2(r-\rho))\|^2dr\right)^{\frac12}\right)\\
\le&2c_8\sqrt{L_\sigma(n)}\mathbb{E}\left(\sup_{\tau\le r\le t}\|u_1(r\wedge T_n)-u_2(r\wedge T_n)\|\right.\\
&\left.\left(\int_\tau^{t\wedge T_n}(\|u_1(r)-u_2(r)\|^2+\|u_1(r-\rho)-u_2(r-\rho)\|^2)dr\right)^{\frac12}\right)\\
\le&2\sqrt{2}c_8\sqrt{L_\sigma(n)}\mathbb{E}\left(\sup_{\tau\le r\le t}\|u_1(r\wedge T_n)-u_2(r\wedge T_n)\|\left(\int_\tau^{t\wedge T_n}\|u_1(r)-u_2(r)\|^2dr\right)^{\frac12}\right)\\
\le&\frac12\mathbb{E}\Big(\sup_{\tau\le s\le t}\|u_1(s\wedge T_n)-u_2(s\wedge T_n)\|^2\Big)\\
&+4c_8^2L_\sigma(n)\int_\tau^t\mathbb{E}\Big(\sup_{\tau\le s\le r}\|u_1(s\wedge T_n)-u_2(s\wedge T_n)\|^2\Big)dr.\end{aligned}\tag{75}$$

Therefore, by (74) and (75) we get

$$\begin{aligned}&\mathbb{E}\Big(\sup_{\tau\le s\le t}\|u_1(s\wedge T_n)-u_2(s\wedge T_n)\|^2\Big)\\
\le&2\big(c_7+4c_8^2L_\sigma(n)\big)\int_\tau^t\mathbb{E}\Big(\sup_{\tau\le s\le r}\|u_1(s\wedge T_n)-u_2(s\wedge T_n)\|^2\Big)dr,\end{aligned}$$

which combined with Gronwall lemma yields

$$\mathbb{E}\Big(\sup_{\tau\le s\le \tau+T}\|u_1(s\wedge T_n)-u_2(s\wedge T_n)\|^2\Big)=0.\tag{76}$$

This implies that $\|u_1(t\wedge T_n)-u_2(t\wedge T_n)\|=0$ for all $t\in[\tau,\tau+T]$ almost surely. Note that $T_n=\tau+T$ for large enough $n$ due to the continuity of $u_1$ and $u_2$ in $t$.

Therefore, we obtain that $\|u_1(t) - u_2(t)\| = 0$ for all $t \in [\tau, \tau + T]$ almost surely. This yields, for every $T > 0$,

$$\mathbb{P}\Big(\|u_1(t) - u_2(t)\| = 0, \ \forall\, t \in [\tau, \tau + T]\Big) = 1. \tag{77}$$

Since $T$ is an arbitrary number, we obtain from (77) that

$$\mathbb{P}\Big(\|u_1(t) - u_2(t)\| = 0, \ \forall\, t \geq \tau\Big) = 1.$$

Then the uniqueness of the solutions follows. This completes the proof. □

## 3 Weak pullback mean random attractors

In this section, we establish the existence and uniqueness of weak pullback mean random attractors of the fractional stochastic lattice systems driven by *nonlinear* delay noise (1) in $L^2(\Omega, \mathcal{F}; \ell^2) \times L^2\big(\Omega, \mathcal{F}; L^2((-\rho, 0), \ell^2)\big)$. We first define a mean random dynamical system and then derive uniform estimates of the solutions. Note that the fractional stochastic lattice systems driven by *nonlinear* delay noise (1) can be rewritten as abstract system (23) in $\ell^2$, we will consider the abstract system (23) in what follows.

It follows from Theorem 2.7 that for every $\tau \in \mathbb{R}$, $u_0 \in L^2(\Omega, \mathcal{F}_\tau; \ell^2)$ and $\varphi \in L^2\big(\Omega, \mathcal{F}_\tau; L^2((-\rho, 0), \ell^2)\big)$, the abstract system (23) has a unique solution $u \in L^2(\Omega, C([\tau, \tau + T], \ell^2))$ for all $T > 0$, which along with the uniform estimates similar to (45) and the Lebesgue dominated convergence theorem implies that $u \in C([\tau, \infty), L^2(\Omega, \ell^2))$. Based upon this fact, for every $t \in \mathbb{R}^+$ and $\tau \in \mathbb{R}$, we define a mapping $\Phi(t, \tau) : L^2(\Omega, \mathcal{F}_\tau; \ell^2) \times L^2\big(\Omega, \mathcal{F}_\tau; L^2((-\rho, 0), \ell^2)\big) \to L^2(\Omega, \mathcal{F}_{\tau+t}; \ell^2) \times L^2\big(\Omega, \mathcal{F}_{\tau+t}; L^2((-\rho, 0), \ell^2)\big)$ by

$$\Phi(t, \tau)(u_0, \varphi) = \big(u(t + \tau, \tau, u_0, \varphi), u_{t+\tau}(\cdot, \tau, u_0, \varphi)\big), \tag{78}$$

for $(u_0, \varphi) \in L^2(\Omega, \mathcal{F}_\tau; \ell^2) \times L^2\big(\Omega, \mathcal{F}_\tau; L^2((-\rho, 0), \ell^2)\big)$, where $u(t, \tau, u_0, \varphi)$ is the solution of the system (23) with initial data $u_0$ and $\varphi$, $u_{t+\tau}(s, \tau, u_0, \varphi) = u(t + \tau + s, \tau, u_0, \varphi)$ for $s \in (-\rho, 0)$. Then $\Phi$ defines a mean random dynamical system for the abstract system (23) on $L^2(\Omega, \mathcal{F}; \ell^2) \times L^2\big(\Omega, \mathcal{F}; L^2((-\rho, 0), \ell^2)\big)$ over $\{\mathcal{F}_t\}_{t\in\mathbb{R}}$ in the sense of [20, Definition 2.9].

For every $\tau \in \mathbb{R}$, we denote by

$$H_\tau = L^2(\Omega, \mathcal{F}_\tau; \ell^2) \times L^2\big(\Omega, \mathcal{F}_\tau; L^2((-\rho, 0), \ell^2)\big). \tag{79}$$

Then $H_\tau$ is a Hilbert space with inner product

$$\big((u_0, \varphi), (v_0, \psi)\big)_{H_\tau} = \mathbb{E}(u_0, v_0) + \mathbb{E}\Big(\int_{-\rho}^{0} (\varphi(s), \psi(s))ds\Big), \quad \forall\, (u_0, \varphi), (v_0, \psi) \in H_\tau \tag{80}$$

and norm

$$\|(u_0, \varphi)\|_{H_\tau} = \Big(\mathbb{E}(\|u_0\|^2) + \int_{-\rho}^{0} \mathbb{E}(\|\varphi(s)\|^2)ds\Big)^{\frac{1}{2}}, \quad \forall\, (u_0, \varphi) \in H_\tau. \tag{81}$$

In the sequel, we assume that

$$\lambda > \gamma_0 + 8\|\gamma\|^2. \tag{82}$$

(82) implies that there exists a constant $\kappa > 0$ such that

$$\kappa - \lambda + \gamma_0 + 4\|\gamma\|^2(1 + e^{\kappa\rho}) < 0. \tag{83}$$

Let $D = \{D(\tau) \subseteq H_\tau : \tau \in \mathbb{R}\}$ be a family of bounded nonempty subsets of $H_\tau$ such that

$$\lim_{\tau \to -\infty} e^{\kappa\tau} \|D(\tau)\|_{H_\tau} = 0, \tag{84}$$

where $\|D(\tau)\|_{H_\tau} = \sup_{(u_0,\varphi)\in H_\tau} \|(u_0,\varphi)\|_{H_\tau}$. In the following, we will prove that $\Phi$ has a unique weak $\mathcal{D}$-pullback mean random attractor in $H_\tau$, where $\mathcal{D}$ is the collection of all families of all bounded nonempty subsets of $H_\tau$ satisfying (84). To this end, we further make the following assumption

$$\int_{-\infty}^{\tau} e^{\kappa r}\mathbb{E}\Big(\sum_{k\in\mathbb{N}} \|a_k(r)\|^2 + \|g(r)\|^2\Big)dr < \infty, \quad \forall\, \tau \in \mathbb{R}, \tag{85}$$

where $\kappa$ is the positive constant as in (83).

Hereafter, we prove the $\mathcal{D}$-uniform estimates in expectation of solutions of system (23) in $L^2(\Omega,\mathcal{F};\ell^2) \times L^2\big(\Omega,\mathcal{F};L^2((-\rho,0),\ell^2)\big)$.

**Lemma 3.1.** *Suppose that the assumptions (2)-(6), (82) and (85) hold. Then for every $\tau \in \mathbb{R}$ and $D = \{D(\tau) : \tau \in \mathbb{R}\} \in \mathcal{D}$, there exists a constant $T := T(\tau, D) \geq \rho$ such that for all $t \geq T$, the solutions of system (23) satisfies the following uniform estimates,*

$$\begin{aligned}\mathbb{E}\big(\|u(\tau,\tau-t,u_0,\varphi)\|^2\big) &+ \int_{-\rho}^{0} \mathbb{E}\big(\|u_\tau(r,\tau-t,u_0,\varphi)\|^2\big)dr \\ &\leq M_1 + M_1\int_{-\infty}^{\tau} e^{\kappa(r-\tau)}\mathbb{E}\Big(\sum_{k\in\mathbb{N}} \|a_k(r)\|^2 + \|g(r)\|^2\Big)dr,\end{aligned} \tag{86}$$

*where $\kappa$ is the same positive constant as in (83), $(u_0,\varphi) \in D(\tau - t)$ and $M_1 > 0$ is a positive constant independent of $\tau$ and $D$.*

*Proof.* Applying Ito's formula to the system (23), we obtain

$$\begin{aligned}d\|u(t)\|^2 + 2\lambda\|u(t)\|^2 + 2\big((-\Delta_d)^\alpha(u(t)),u(t)\big) &= 2\big(f(u(t))),u(t)\big)dt + 2\big(g(t),u(t)\big)dt \\ &+ \sum_{k=1}^{\infty} \|a_k(t) + \sigma_k(u(t),u(t-\rho))\|^2 dt + 2\sum_{k=1}^{\infty} \big(a_k(t) + \sigma_k(u(t)),u(t)\big)dW_k.\end{aligned} \tag{87}$$

(87) and (48) implies

$$\begin{aligned}\frac{d}{dt}\mathbb{E}\big(\|u(t)\|^2\big) + \kappa\mathbb{E}\big(\|u(t)\|^2\big) &\leq (\kappa - 2\lambda)\mathbb{E}\big(\|u(t)\|^2\big) + 2\mathbb{E}\big(f(u(t)),u(t)\big) \\ &+ 2\mathbb{E}\big(g(t),u(t)\big) + 2\sum_{k=1}^{\infty}\mathbb{E}\big(\|a_k(t)\|^2\big) + 2\sum_{k=1}^{\infty}\mathbb{E}\big(\|\sigma_k(u(t),u(t-\rho))\|^2\big).\end{aligned} \tag{88}$$

For the second term on the right-hand side of (88), we get from (20) that

$$\begin{aligned}2\mathbb{E}\big(f(u(t)),u(t)\big) &\leq 2\gamma_0\mathbb{E}\big(\|u(t)\|^2\big) + \frac{1}{2\gamma_0}\Big(2\|\beta\|^2 + 4\gamma_0^2\big(\mathbb{E}(\|u(t)\|^2)\big)\Big) \\ &= 4\gamma_0\mathbb{E}\big(\|u(t)\|^2\big) + \frac{\|\beta\|^2}{\gamma_0}.\end{aligned} \tag{89}$$

For the third term on the right-hand side of (88), by Young's inequality, we have

$$2\mathbb{E}(g(t),u(t)) \leq \kappa\mathbb{E}(\|u(t)\|^2) + \frac{1}{\kappa}\mathbb{E}(\|g(t)\|^2). \tag{90}$$

For the last term on the right-hand side of (88), by (22) and Young's inequality we get that

$$2\sum_{k=1}^{\infty}\mathbb{E}\big(\|\sigma_k(u(t),u(t-\rho))\|^2\big)\leq 4\|\delta\|^2+8\|\gamma\|^2\big(\mathbb{E}(\|u(t)\|^2)+\mathbb{E}(\|u(t-\rho)\|^2)\big). \quad (91)$$

Therefore, we obtain from (88)-(91) that

$$\begin{aligned}\frac{d}{dt}\mathbb{E}\big(\|u(t)\|^2\big)+\kappa\mathbb{E}\big(\|u(t)\|^2\big)\leq&\, 2(\kappa-\lambda+\gamma_0+4\|\gamma\|^2)\mathbb{E}\big(\|u(t)\|^2\big)\\&+8\|\gamma\|^2\mathbb{E}\big(\|u(t-\rho)\|^2\big)+\mathbb{E}\Big(2\sum_{k=1}^{\infty}\|a_k(t)\|^2+\frac{1}{\kappa}\|g(t)\|^2\Big)+\frac{\|\beta\|^2}{\gamma_0}+4\|\delta\|^2. \quad (92)\end{aligned}$$

Then for any $t>0$ and $s\in(\tau-t,\tau]$, multiplying (92) by $e^{\kappa t}$ and integrating the resulting inequality over $(\tau-t,s)$, we have

$$\begin{aligned}&\mathbb{E}\big(\|u(s,\tau-t,u_0,\varphi)\|^2\big)\\\leq&e^{\kappa(\tau-t-s)}\mathbb{E}\big(\|u_0\|^2\big)+2(\kappa-\lambda+\gamma_0+4\|\gamma\|^2)\int_{\tau-t}^{s}e^{\kappa(r-s)}\mathbb{E}\big(\|u(r)\|^2\big)dr\\&+8\|\gamma\|^2e^{\kappa\rho}\int_{\tau-t-\rho}^{s-\rho}e^{\kappa(r-s)}\mathbb{E}\big(\|u(r)\|^2\big)dr+\int_{\tau-t}^{s}e^{\kappa(r-s)}\mathbb{E}\Big(2\sum_{k=1}^{\infty}\|a_k(r)\|^2+\frac{1}{\kappa}\|g(r)\|^2\Big)dr\\&+\frac{\|\beta\|^2}{\kappa\gamma_0}+\frac{4\|\delta\|^2}{\kappa}\\\leq&e^{\kappa(\tau-t-s)}\mathbb{E}\big(\|u_0\|^2\big)+2\big(\kappa-\lambda+\gamma_0+4\|\gamma\|^2(1+e^{\kappa\rho})\big)\int_{\tau-t}^{s}e^{\kappa(r-s)}\mathbb{E}\big(\|u(r)\|^2\big)dr\\&+8\|\gamma\|^2e^{\kappa\rho}e^{\kappa(\tau-t)}\int_{-\rho}^{0}e^{\kappa(r-s)}\mathbb{E}\big(\|\varphi(r)\|^2\big)dr\\&+\int_{\tau-t}^{s}e^{\kappa(r-s)}\mathbb{E}\Big(2\sum_{k=1}^{\infty}\|a_k(r)\|^2+\frac{1}{\kappa}\|g(r)\|^2\Big)dr+\frac{\|\beta\|^2}{\kappa\gamma_0}+\frac{4\|\delta\|^2}{\kappa}. \quad (93)\end{aligned}$$

By (82), (83) and (93), we know that for $s\in(\tau-t,\tau]$,

$$\begin{aligned}&\mathbb{E}\big(\|u(s,\tau-t,u_0,\varphi)\|^2\big)\\\leq&e^{\kappa(\tau-t-s)}\mathbb{E}\big(\|u_0\|^2\big)+8\|\gamma\|^2e^{\kappa\rho}e^{\kappa(\tau-t)}\int_{-\rho}^{0}e^{\kappa(r-s)}\mathbb{E}\big(\|\varphi(r)\|^2\big)dr\\&+\int_{\tau-t}^{s}e^{\kappa(r-s)}\mathbb{E}\Big(2\sum_{k=1}^{\infty}\|a_k(r)\|^2+\frac{1}{\kappa}\|g(r)\|^2\Big)dr+\frac{\|\beta\|^2}{\kappa\gamma_0}+\frac{4\|\delta\|^2}{\kappa}. \quad (94)\end{aligned}$$

We get from (94) that

$$\begin{aligned}&\mathbb{E}\big(\|u(\tau,\tau-t,u_0,\varphi)\|^2\big)\\\leq&e^{-\kappa t}\mathbb{E}\big(\|u_0\|^2\big)+8\|\gamma\|^2e^{\kappa\rho}e^{-\kappa t}\int_{-\rho}^{0}\mathbb{E}\big(\|\varphi(r)\|^2\big)dr\\&+\int_{\tau-t}^{\tau}e^{\kappa(r-\tau)}\mathbb{E}\Big(2\sum_{k=1}^{\infty}\|a_k(r)\|^2+\frac{1}{\kappa}\|g(r)\|^2\Big)dr+\frac{\|\beta\|^2}{\kappa\gamma_0}+\frac{4\|\delta\|^2}{\kappa} \quad (95)\end{aligned}$$

and

$$
\begin{aligned}
&\sup_{\tau-\rho\le s\le\tau}\mathbb{E}\left(\|u(s,\tau-t,u_0,\varphi)\|^2\right)\\
\le& e^{\kappa(\rho-t)}\mathbb{E}\left(\|u_0\|^2\right)+8\|\gamma\|^2e^{\kappa(2\rho-t)}\int_{-\rho}^{0}\mathbb{E}\left(\|\varphi(r)\|^2\right)dr\\
&+e^{\kappa\rho}\int_{\tau-t}^{\tau}e^{\kappa(s-\tau)}\mathbb{E}\Big(2\sum_{k=1}^{\infty}\|a_k(r)\|^2+\frac{1}{\kappa}\|g(r)\|^2\Big)dr+\frac{\|\beta\|^2}{\kappa\gamma_0}+\frac{4\|\delta\|^2}{\kappa},\quad\forall t\ge\rho. \qquad (96)
\end{aligned}
$$

By (95) and (96), we know that for $t\ge\rho$,

$$
\begin{aligned}
&\mathbb{E}\left(\|u(\tau,\tau-t,u_0,\varphi)\|^2\right)+\int_{\tau-\rho}^{\tau}\mathbb{E}\left(\|u(r,\tau-t,u_0,\varphi)\|^2\right)dr\\
\le&(1+\rho e^{\kappa\rho})\Bigg[e^{-\kappa t}\mathbb{E}\left(\|u_0\|^2\right)+8\|\gamma\|^2e^{\kappa(\rho-t)}\int_{-\rho}^{0}\mathbb{E}\left(\|\varphi(r)\|^2\right)dr\\
&+\int_{\tau-t}^{\tau}e^{\kappa(r-\tau)}\mathbb{E}\Big(2\sum_{k=1}^{\infty}\|a_k(r)\|^2+\frac{1}{\kappa}\|g(r)\|^2\Big)dr+\frac{\|\beta\|^2}{\kappa\gamma_0}+\frac{4\|\delta\|^2}{\kappa}\Bigg]. \qquad (97)
\end{aligned}
$$

By (84) and $(u_0,\varphi)\in D(\tau-t)$ with $D\in\mathcal{D}$, we know that as $t\to+\infty$,

$$
\begin{aligned}
&e^{-\kappa t}\mathbb{E}\left(\|u_0\|^2\right)+8\|\gamma\|^2e^{\kappa(\rho-t)}\int_{-\rho}^{0}\mathbb{E}\left(\|\varphi(r)\|^2\right)dr\\
\le&\left(e^{-\kappa\tau}+8\|\gamma\|^2e^{\kappa(\rho-\tau)}\right)e^{\kappa(\tau-t)}\left(\|D(\tau-t)\|^2_{H_{\tau-t}}\right)\to0.
\end{aligned}
$$

This along with (97) implies that there exists $T:=T(\tau,D)\ge\rho$ such that for all $t\ge T$, $\|u_0\|^2+8\|\gamma\|^2e^{\kappa(\rho-t)}\int_{-\rho}^{0}\mathbb{E}\left(\|\varphi(r)\|^2\right)dr\le1$. Then for all $t\ge T$,

$$
\begin{aligned}
&\mathbb{E}\left(\|u(\tau,\tau-t,u_0,\varphi)\|^2\right)+\int_{-\rho}^{0}\mathbb{E}\left(\|u(r+\tau,\tau-t,u_0,\varphi)\|^2\right)dr\\
\le&(1+\rho e^{\kappa\rho})\Bigg[\int_{-\infty}^{\tau}e^{\kappa(r-\tau)}\mathbb{E}\Big(2\sum_{k=1}^{\infty}\|a_k(r)\|^2+\frac{1}{\kappa}\|g(r)\|^2\Big)dr+1+\frac{\|\beta\|^2}{\kappa\gamma_0}+\frac{4\|\delta\|^2}{\kappa}\Bigg],
\end{aligned}
$$

which combined with (85) deduce the desired result (86). □

Based on Lemma 3.1, we now prove the main results of this section as follows.

**Theorem 3.2.** *Suppose that the assumptions (2)-(6), (82) and (85) hold. Then the mean random dynamical system $\Phi$ associated with the fractional stochastic lattice system (1) driven by nonlinear delay noise has a unique weak $\mathcal{D}$-pullback mean random attractor $\mathcal{A}=\{\mathcal{A}(\tau):\tau\in\mathbb{R}\}\in\mathcal{D}$ in $L^2(\Omega,\mathcal{F};\ell^2)\times L^2(\Omega,\mathcal{F};L^2((-\rho,0),\ell^2))$, that is,*

*(i) For every $\tau\in\mathbb{R}$, $\mathcal{A}(\tau)$ is a weakly compact subset of $L^2(\Omega,\mathcal{F};\ell^2)\times L^2(\Omega,\mathcal{F};L^2((-\rho,0),\ell^2))$;*
*(ii) $\mathcal{A}$ is a $\mathcal{D}$-pullback weakly attracting set of $\Phi$;*
*(iii) $\mathcal{A}$ is the minimal element of $\mathcal{D}$ with properties (i) and (ii).*

*Proof.* Firstly, we show that the mean random dynamical system $\Phi$ has a weakly compact $\mathcal{D}$-pullback absorbing set. Let $K=\{K(\tau):\tau\in\mathbb{R}\}\in\mathcal{D}$ with

$$K(\tau):=\left\{(u,\varphi)\in H_\tau:\|(u,\varphi)\|^2_{H_\tau}\le R(\tau)\right\}, \qquad (98)$$

where

$$R(\tau) = M_1 + M_1 \int_{-\infty}^{\tau} e^{\kappa(r-\tau)} \mathbb{E}\Big( \sum_{k=1}^{\infty} \|a_k(r)\|^2 + \|g(r)\|^2 \Big) dr, \tag{99}$$

and $M_1$ being the same number as in (86). Then $K(\tau)$ is a bounded closed convex subset of $H_\tau$, and therefore $K(\tau)$ is weakly compact in $H_\tau$. It follows from (85), (98) and (99) we have

$$\lim_{\tau \to -\infty} e^{\kappa\tau} \|K(\tau)\|^2_{H_\tau} = \lim_{\tau \to -\infty} e^{\kappa\tau} R(\tau) = 0, \tag{100}$$

which means that $K \in \mathcal{D}$. Furthermore, it follows from Lemma 3.1 that for every $\tau \in \mathbb{R}$, $D \in \mathcal{D}$, there exists $T := T(\tau, D) \geq \rho$ such that for all $(u_0, \varphi) \in D(\tau - t)$ and $t \geq T$,

$$\mathbb{E}\big(\|u(\tau, \tau - t, u_0, \varphi)\|^2\big) + \int_{-\rho}^{0} \mathbb{E}\big(\|u_\tau(r, \tau - t, u_0, \varphi)\|^2\big) dr \leq R(\tau).$$

This implies that for all $t \geq T$,

$$\Phi(t, \tau - t) D(\tau - t) \subseteq K(\tau).$$

This fact along with previous discussions shows that $K$ is a weakly compact $\mathcal{D}$-pullback absorbing set for the mean random dynamical system $\Phi$ defined by (78) in $L^2(\Omega, \mathcal{F}; \ell^2) \times L^2\big(\Omega, \mathcal{F}; L^2((-\rho, 0), \ell^2)\big)$. Therefore, Theorem 3.2 follows from the abstract result [20, Theorem 2.13]. □

## 4 Declarations

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### Competing Interests

The authors declare that they have no competing interests.

### Ethical Approval

Not applicable.

### Authors's Contributions

All authors contributed equally. All the authors read and approved the final manuscript.

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