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Discrete Multiparameters Potential Boundary Value Problems Involving the $p(k)$-Kirchhoff Equations

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Abstract

This paper is concerned with the existence and multiplicity of solutions of discrete inclusions problems involving the so called $p(k)$-Laplacian operator of Kirchhoff type subjected to potential boundary values conditions.

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1 Introduction

We study in this paper, the existence and multiplicity of solutions to nonlinear discrete problems with inclusions of the form

$$\begin{cases} -M(I(u))\left[\Delta(a(k-1,\Delta u(k-1))+\varphi_{p(k)}(u(k))\right] \in \lambda\partial F(k,u(k))+\mu\partial G(k,u(k)), \\ k\in\mathbb{Z}[1,T], \\ (a(0,\Delta u(0)),-a(T,\Delta u(T)))\in\partial j(u(0),u(T+1)); \end{cases} \tag{1}$$

where

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$$I(u)=\sum_{k=1}^{T+1}A(k-1,\Delta u(k-1))-\sum_{k=1}^{T}\left(\frac{1}{p(k)}|u(k)|^{p(k)}\right).$$

$T\geq 2$ is an integer, $\mathbb{Z}[1,T]$ is the discrete set $\{1,2,...,T\}$,
$\Delta u(k)=u(k+1)-u(k)$ is the forward difference operator; $u(k)\in\mathbb{R}$, for all $k\in\mathbb{Z}[1,T]$. φ is an homomorphism defined by $\varphi_s(y)=|y|^{s-2}y$ and
$M:(0,\infty)\to(0,\infty)$ is a non-decreasing continuous function. λ and μ are positive real parameters such that $(\mu\leq\lambda)$. $j:\mathbb{R}\times\mathbb{R}\to(-\infty,\infty]$ is convex, proper i.e. $D(j):=\{z\in\mathbb{R}\times\mathbb{R}:j(z)<\infty\}\neq\emptyset)$, lower semicontinuous and ∂j denote the subdifferential of j where, for $s'\in\mathbb{R}\times\mathbb{R}$, the set ∂j is defined by

$$\partial j(s')=\{v\in\mathbb{R}\times\mathbb{R}:j(\tau)-j(s')\geq(v,\tau-s'),\ \text{for all}\ \tau\in\mathbb{R}\times\mathbb{R}\},\tag{2}$$

where $(.,.)$ stands for the usual inner product in $\mathbb{R}\times\mathbb{R}$.
F and G belongs to $\mathcal{S}$, where $\mathcal{S}$ is the set of all locally Lipschitz continuous function $H:\mathbb{Z}\times\mathbb{R}\longrightarrow\mathbb{R}$ which satisfy

(s_1) : H is periodic with respect to k, i.e. there exists $m\in\mathbb{Z}$ such that
$H(k+m,u)=H(k,u),\quad$ for all $(k,u)\in\mathbb{Z}\times\mathbb{R}$.
(s_2) : $H(k,0)=0$ for all $k\in\mathbb{Z}$.

$\partial F(k,u)$ and $\partial G(k,u)$ are the Clarke subdifferential of the functions F and G with respect to the second variable.

Difference equations arise in many areas of applied mathematics and for this reason, the study of this type of equations has received great attention by now (see [1, 3, 6, 28, 31, 36, 37]).
The presence of the non-local term $I(u)$ is an important feature of this article. Indeed, Kirchhoff in 1876 (see [21]) suggested a model defined by the following equation

$$\rho\frac{\partial^2 u}{\partial t^2}=\left(T_0+\frac{Ea}{2L}\int_0^L\left|\frac{\partial u}{\partial x}\right|^2dx\right)\frac{\partial^2 u}{\partial x^2},\tag{3}$$

where $\rho>0$ is the mass per unit length, T_0 is the base tension, E is the Young modulus, a is the area of cross section and L is the initial length of the string.

Equation (3) takes into account the change of the tension on the string which is caused by the change of its length during the vibration. After that, several physicists also considered such equations for their research in the theory of nonlinear vibrations theoretically or experimentally [7, 8, 26, 27]. On the other hand, Kirchhoff's equation received much attention only after Lions in 1978 (see [24]) established an abstract framework for the problem related to the stationary analog of the equation of Kirchhoff type. Some important and interesting results can be found in [2, 9]. In recent years, discrete Kirchhoff-type problems have been widely investigated. We refer the readers to papers [10, 18, 22, 28, 29, 33, 34, 35] and references therein.

To our knowledge, very few papers can be found in the literature dealing with the multiplicity of discrete inclusions involving the $p(k)$-Laplacian operator of Kirchhoff type subjected to potential boundary values conditions. We refer the readers to the work done by Ouaro and Zoungrana and Dianda and Ouaro (see [13, 28]) and the references therein.
For instance, in [19] Heidarkhani and Moghadam dealed with the following problem.

$$\begin{cases}-\Delta\left(\varphi_p\left(\Delta u(k-1)\right)\right)+q_k\varphi_p(u(k))=\lambda f(k,u(k))+\mu g(k,u(k)),\ k\in\mathbb{Z}[1,T]\\ u(0)=u(T+1)=0,\end{cases}\tag{4}$$

where T is a fixed positive integer, $\mathbb{Z}[1,T]$ is the discrete set $\{1,2,...,T\}$;
$\Delta u(k) = u(k+1) - u(k)$ is the forward difference operator. $f,g : \mathbb{Z}[1,T] \times \mathbb{R} \to \mathbb{R}$ are two continuous functions; $\lambda, \mu \geq 0$ are two parameters, $q_k \in \mathbb{R}_0^+$ for all $k \in \mathbb{Z}[1,T]$ and $\varphi_p(s) = |s|^{p-2}s$, $1 < p < \infty$. They used two kind of three critical points theorems obtained in [4] and [5], to ensure the existence of at least three solutions for problem (4).
If we take $\mu = 0$ and $\varphi_p = 0$, problem (1) is reduced to the problem (5) studied by Dianda and Ouaro in [13] under inclusion and potential boundary conditions. In [13], firstly, the authors applied the direct method of the calculus of variations and the Mountain pass technic in order to prove the existence of at least one non-trivial solution. Secondly, they applied three critical points theorem for locally lipschitz fonctionnal established by Cabada, Iannizzotto and Tersian and also by Galewski and Glab (see [15, 20]) to prove the existence of multiple solutions for the following problem.

$$\begin{cases} -M\left(A\left(k-1,\Delta u(k-1)\right)\right)\Delta a\left(k-1,\Delta u(k-1)\right) \in \lambda\partial F(k,u(k)),\ k \in \mathbb{Z}[1,T] \\ \\ (a(0,\Delta u(0)), -a(T,\Delta u(T))) \in \partial j(u(0),u(T+1)). \end{cases} \tag{5}$$

By the way, the main operator in problem (1.1) can be seen as the discrete counterpart of the following anisotropic operator

$$\sum_{i=1}^{N} \frac{\partial}{\partial x_i} a_i \left(x, \frac{\partial}{\partial x_i} u \right).$$

Motivated by the above mentioned results, we study the problem (1). The paper is organized as follows. Section 2 provides some necessary background materials for the use of critical points theory. Section 3 and Section 4 are devoted to some classical results of existence of critical points and the relationship between critical point of the energy functional associated and the solution to problem (1). In the last section, we prove the multiplicity of solutions to problem (1).

2 Mathematical Background

Let $(X, \|.\|)$ be a real Banach space. We denote the dual space of X by X^*. A function $J : X \longrightarrow \mathbb{R}$ is called locally Lipschitz when for every $u \in X$, there exists a neighbourhood U of u and a constant $K \geq 0$ such that

$$|J(z) - J(w)| \leq K\|z - w\|, \quad \text{for all } z, w \in U.$$

For all $u, z \in X$, we write $J^0(u;z)$ for the generalized directional derivative of J at the point u along the direction z, i.e.

$$J^0(u;z) := \lim_{w \to u} \sup_{t \to 0^+} \frac{J(w+tz) - J(w)}{t}.$$

• If $\varphi, \psi : X \longrightarrow \mathbb{R}$ are locally Lipschitz functions, then

$$(\varphi + \psi)^0 (x,v) \leq \varphi^0(x,v) + \psi^0(x,v), \text{ for any } x, v \in X. \tag{6}$$

The generalized gradient of the function J in u, denoted by $\partial J(u)$, is the set

$$\partial J(u) := \{u^* \in X^* : \langle u^*, z\rangle \leq J^0(u;z), \text{ for all } z \in X\}.$$

The basic properties of generalized directional derivative and generalized gradient were studied in [11] and [12].
Inequality (6) lead to the following result.
• If $\varphi, \psi : X \longrightarrow \mathbb{R}$ are locally Lipschitz functions, then.

i. $\partial(\varphi+\psi)(x) \subset \partial\varphi(x)+\partial\psi(x)$ for all $x \in X$, with equality if one of the sets $\partial\varphi(x), \partial\psi(x)$ is a singleton.
ii. $\partial(\alpha\psi)(x) = \alpha\partial\psi(x)$ for all $\alpha \in \mathbb{R}, x \in X$.

We recall that if J is continuously Gâteaux differentiable at u, then J is locally Lipschitz at u and $\partial J(u) = \{J^{'}(u)\}$, where $J^{'}(u)$ stands for the first derivative of J at u.
Furthermore, a point u is called a critical point of the locally Lipschitz continuous function J, if $0_{X^*} \in \partial J(u)$ i.e.

$$J^0(u;z) \geq 0, \quad \text{for every } z \in X.$$

Clearly, if J is continuously Gâteaux differentiable at u, then u becomes a (classical) critical point of J, that is $J'(u) = 0_{X^*}$.

• A locally Lipschitz continuous functional $J : X \longrightarrow \mathbb{R}$ satisfy the Palais-Smale (PS) condition if every sequence $\{u_n\}$ in X such that $\{J(u_n)\}$ is bounded and

$$J^0(u_n; u-u_n) \geq -\epsilon_n \|u-u_n\|$$

for all $u \in X$, where $\epsilon_n \longrightarrow 0^+$ as $n \longrightarrow \infty$, possesses a convergent subsequence.

Definition 2.1. A function Φ defined on a normed space X into $\overline{\mathbb{R}}$ is said to be coercive over an unbounded part P of X if $\lim\limits_{\|x\|\longrightarrow\infty} \Phi(x) = \infty$, where $x \in P$.
Φ is said to be anti-coercive if $(-\Phi)$ is coercive.

Theorem 2.2. *[14, 32] Let X be a real Banach space. Assume that $X := X_1 \oplus X_2$, with X_2 a finite-dimensional subspace of X. Let $J : X \longrightarrow \mathbb{R}$ be a locally Lipschitz continuous functional satisfying the (PS) condition and such that.*

$(2.1.1) : J(u) \leq 0, \left(\forall u \in \overline{B}(0,\tau) \cap X_2\right),$

$(2.1.2) : J(u) \geq 0, \left(\forall u \in \overline{B}(0,\tau) \cap X_1\right),$

for some $\tau > 0$. Assume also that J is bounded from below and

$(2.1.3) : \inf\limits_{u \in X} J(u) < 0\,.$
Then, J has at least two nonzero critical points.

Theorem 2.3. *[20] Let $(X, \|.\|)$ be a banach space, $dim(X) < \infty$ and $J \in C^1(X, \mathbb{R})$ be a coercive functional having at least two strict local minimizers x_0, $x_1 \in X$. Then, J has a critical point $x_2 \in X \setminus \{x_0, x_1\}$.*

Theorem 2.4. *[25] Let $(X, \|.\|)$ be a real Banach space and let $J : X \longrightarrow \mathbb{R}$ be a locally Lipschitz continuous functional satisfying (PS) condition. If there exist $u_1, u_2 \in X$, $u_1 \neq u_2$ and $s \in (0, \|u_2 - u_1\|)$ such that*

$$\inf\{J(u) : \|u-u_1\| = s\} \geq \max\{J(u_1), J(u_2)\}.$$

Then, $c := \inf\limits_{\gamma\in\Gamma} \max\limits_{x\in[0,1]} J(\gamma(x)) \geq \max\{J(u_1), J(u_2)\}$ is a critical value for X and $K_c - \{u_1, u_2\} \neq \emptyset$, where K_c is the set of critical points at the level c and Γ the family of continuous paths $\gamma : [0,1] \longrightarrow X$ joining u_1 and u_2.

3 General multiplicity result

In this section we are concerned with the existence of multiple solutions. We recall the tools used in [14, 15, 17, 20].

Theorem 3.1. *[17] Let (X,τ) be an Hausdorff space and $\Phi, J : X \longrightarrow \mathbb{R}$ be functionals. Let M be the set (possibly empty) of all global minimizers of J and define*

$$\alpha := \inf_{x\in X} \Phi(x),$$

$$\beta := \begin{cases} \inf_{x\in M} \Phi(x) & \text{if } M \neq \emptyset \\ \sup_{x\in X} \Phi(x) & \text{if } M = \emptyset. \end{cases}$$

Let $\alpha < \beta$ and assume that the set

$$\{x \in X : \Phi(x) + \sigma J(x) \leq \rho\},$$

for every $\sigma > 0$ and every $\rho \in \mathbb{R}$, is sequentially compact (if it is non empty). Then, at least one of the following conditions holds.
(a) There exists a continuous mapping $h : (\alpha,\beta) \longrightarrow X$ with the following property: for every $t \in (\alpha,\beta)$,

$$\Phi(h(t)) = t$$

and for every $x \in \Phi^{-1}(t)$ with $x \neq h(t)$,

$$J(x) > J(h(t)).$$

(b) There exists $\lambda > 0$ such that the functional $\Phi + \lambda J$ admits at least two global minimizers in X.

Theorem 3.2. *(see [15, 20]) Let X be a finite dimensional real Banach space. Let $\mu : X \longrightarrow \mathbb{R}$ be a coercive C^1-functional such that $\mu(0) = 0$ and let $J : X \longrightarrow \mathbb{R}$ be locally Lipschitz. Let $s > 0$ and $0 < r < s$ be fixed. Assume that.*
(d_1) : $\lim_{\mu(u)\longrightarrow\infty} \inf \dfrac{J(u)}{\mu(u)} \geq 0;$
(d_2) : $\inf_{u\in X} J(u) < \inf_{\mu(u)\leq s} J(u);$
(d_3) : $J(0) \leq \inf_{r\leq\mu(u)\leq s} J(u).$
Then, there exists $\lambda > 0$ such that the functional $\mu + \lambda J$ has at least three critical points in X, at least two of which are non-trivial.

Theorem 3.3. *[14] Let X be a finite dimensional real Banach space. Let $\mu : X \longrightarrow \mathbb{R}$ be a coercive C^1-functional such that $\mu(0) = 0$ and let $J : X \longrightarrow \mathbb{R}$ be a locally Lipschitz functional bounded from below. Let $s > 0$ and $0 < r < s$ be fixed constants. Assume moreover that conditions (d_2) and (d_3) hold. Then, there exists $\lambda^* > 0$ such that the functional $\mu + \lambda J$ has at least three critical points in X.*

4 Assumptions and preliminary

Define the space

$$\mathcal{H}_T = \Big\{u = \{u(k)\}_{k\in\mathbb{Z}} : u(k) \in \mathbb{R}, u(k+T) = u(k), k \in \mathbb{Z}\Big\},$$

which, endowed with the Euclidean norm

$$\|u\| := \left(\sum_{k=1}^{T} |u(k)|^2\right)^{\frac{1}{2}}$$

is an Hilbert space.
Let $W_b := \left\{u = \{u(k)\}_{k\in\mathbb{Z}} : u(k) = b,\ k \in \mathbb{Z}\right\}$ and $Z_b := W_b^{\perp}$, with $b \in \mathbb{R}$.
From [28], one has the decomposition

$$\mathcal{H}_T = Z_b \oplus W_b,\ \text{for any}\ \ b \in \mathbb{R}.$$

In addition, we introduce the functional $J : \mathcal{H}_T \longrightarrow (-\infty, \infty)$ by

$$J(u) = j(u(0); u(T+1)),\ \ \forall u \in \mathcal{H}_T \tag{7}$$

where j is proper, convex, l.s.c, $(0,0) \in \partial j(0,0)$ and $j(0,0) = 0$ (see [3]).
Note that as $(0,0) \in \partial j(0,0)$ and $j(0,0) = 0$, then $J(u) \geq J(0) = 0$, for all $u \in D(J)$.

We assume the following conditions on the data.

$$a(k,.) : \mathbb{R} \longrightarrow \mathbb{R} \text{ is continuous } \forall k \in \mathbb{Z}[0,T]$$

and there exists a mapping $A : \mathbb{Z}[0,T] \times \mathbb{R} \longrightarrow \mathbb{R}$ which satisfies

$$a(k,x) = \frac{\partial}{\partial x} A(k,x), \qquad A(k,0) = 0, \quad \forall k \in \mathbb{Z}[0,T]. \tag{8}$$

We also assume that there exists a positive constant C_1 such that

$$|a(k,x)| \leq C_1\left(1 + |x|^{p(k)-1}\right) \tag{9}$$

and

$$|x|^{p(k)} \leq a(k,x)x \leq p(k)A(k,x), \quad \forall x \in \mathbb{R}. \tag{10}$$

The following relations hold true for all $k \in \mathbb{Z}[0,T]$.

$$(a(k,x) - a(k,y))(x-y) > 0, \forall x,y \in \mathbb{R},\ x \neq y. \tag{11}$$

$$p \text{ is a } T\text{-periodic function i.e. } p(k+T) = p(k),\ \text{for all } k \in \mathbb{Z}[0,T]. \tag{12}$$

a is a T-periodic function with respect to k i.e.

$$a(k+T,x) = a(k,x) \text{ for all } (k,x) \in \mathbb{Z}[0,T] \times \mathbb{R}. \tag{13}$$

For the function $M : (0,\infty) \longrightarrow (0,\infty)$, we suppose that it is continuous, non-decreasing and there exist positive numbers R_1, R_2 with $R_1 \leq R_2$ and δ an even number such that

$$R_1 t^{\delta-1} \leq M(t) \leq R_2 t^{\delta-1},\ \text{for } t > 0. \tag{14}$$

Example 4.1. As functions which satisfies assumptions (8) – (14), we can give the following.

- $A(k,x) = \dfrac{1}{p(k)}|x|^{p(k)}$; $a(k,x) = |x|^{p(k)-2}x;\ x \in \mathbb{R};\ k \in \mathbb{Z},$ with $p(k) \geq 2$.
- $M(t) = ct^{\delta-1} + d$, where c and d are two positive constants.

In this paper, we assume that the function $p : \mathbb{Z} \longrightarrow [2, \infty)$.

One denotes by

$$p^- := \min_{k\in\mathbb{Z}[0,T]} p(k) \quad \text{and} \quad p^+ := \max_{k\in\mathbb{Z}[0,T]} p(k).$$

Lemma 4.2. *(see [16]) The following properties hold true.*

(a_1) *For every* $m > 0$,

$$\sum_{k=1}^{T} |u(k)|^m \leqslant T\|u\|^m, \quad \text{for all} \quad u \in \mathcal{H}_T.$$

(a_2) *For every* $m \geqslant 2$,

$$\sum_{k=1}^{T} |u(k)|^m \geqslant T^{\frac{(2-m)}{2}} \|u\|^m, \quad \text{for all} \quad u \in \mathcal{H}_T.$$

(a_3) *For all* $u \in \mathcal{H}_T$,

$$\sum_{k=1}^{T} |\Delta u(k-1)|^{p(k-1)} \leqslant T\Big(2^{p^+}\|u\|^{p^+} + 1\Big).$$

We define the energy functional associated to problem (1) by $\Gamma_T : \mathcal{H}_T \longrightarrow \mathbb{R}$ such that

$$\Gamma_T(u) = \widehat{M}\Big(\sum_{k=1}^{T+1} A(k-1, \Delta u(k-1)) - \sum_{k=1}^{T} \frac{1}{p(k)}|u(k)|^{p(k)}\Big) - \lambda \sum_{k=1}^{T} \Psi(k, u(k)) + J(u), \tag{15}$$

where $\widehat{M}(t) = \int_0^t M(s)ds$ and $\Psi(k, u(k)) = F(k, u(k)) + \frac{\mu}{\lambda} G(k, u(k))$.

Definition 4.3. A solution of problem (1) is a function $u \in \mathcal{H}_T$ such that

$$\begin{cases} -M\left(I(u)\right)\left[\sum_{k=1}^{T} \Delta a(k-1, \Delta u(k-1))v(k) + \sum_{k=1}^{T} |u(k)|^{p(k)-2}u(k)v(k)\right] + \\ j'((u(0), u(T+1)), (v(0), v(T+1))) - \lambda\left(\sum_{k=1}^{T} F^0(k, u(k)) + \frac{\mu}{\lambda}\sum_{k=1}^{T} G^0(k, u(k))\right) v(k) \geq 0, \end{cases}$$

for all $v \in \mathcal{H}_T$ with $v(k) \geq 0$ and for all $k \in \mathbb{Z}[1, T]$; which in turn is a classical solution of problem (1).

Proposition 4.4. *Suppose that* (14) *hold. Then,* Γ_T *is locally Lipschitz continuous.*

Proof. Taking $y, z \in \mathcal{H}_T$ and recall that

$$\Gamma_T(u) \;=\; \widehat{M}\Big(\sum_{k=1}^{T+1} A(k-1, \Delta u(k-1)) - \sum_{k=1}^{T} \frac{1}{p(k)}|u(k)|^{p(k)}\Big) - \lambda \sum_{k=1}^{T} \Psi(k, u(k)) + J(u)$$

or

$$\Gamma_T(u) \;=\; \widehat{M}(I(u)) - \lambda \sum_{k=1}^{T} \Psi(k, u(k)) + J(u), \tag{16}$$

where

$$I(u) = \Big(\sum_{k=1}^{T+1} A(k-1, \Delta u(k-1)) - \sum_{k=1}^{T} \frac{1}{p(k)}|u(k)|^{p(k)}\Big).$$

From (16), we deduce that

$$\begin{aligned}
\left|\Gamma_T(y)-\Gamma_T(z)\right| &= \left|\widehat{M}\Big(I(y)\Big)-\widehat{M}\Big(I(z)\Big)-\lambda\sum_{k=1}^{T}\Psi(k,y(k))+\lambda\sum_{k=1}^{T}\Psi(k,z(k))+J(y)-J(z)\right|\\
&\leq \left|\widehat{M}\Big(I(y)\Big)-\widehat{M}\Big(I(z)\Big)\right|+\lambda\sum_{k=1}^{T}\Big|\Psi(k,z(k))-\Psi(k,y(k))\Big|+|J(y)-J(z)|\\
&\leq \left|\int_{I(y)}^{I(z)}M(s)\mathrm{d}s\right|+\lambda\sum_{k=1}^{T}\Big|\Psi(k,z(k))-\Psi(k,y(k))\Big|+|J(y)-J(z)|\\
&\leq \left|\int_{I(y)}^{I(z)}R_2 s^{\delta-1}ds\right|+\lambda\sum_{k=1}^{T}\Big|\Psi(k,z(k))-\Psi(k,y(k))\Big|+|J(y)-J(z)|\\
&\leq \frac{R_2}{\delta}\Big|I(z)^{\delta}-I(y)^{\delta}\Big|+\lambda\sum_{k=1}^{T}\Big|\Psi(k,y(k))-\Psi(k,z(k))\Big|+|J(y)-J(z)|.
\end{aligned}$$

Firstly, we suppose that $y=z$. We deduce that there exists a constant $C_2>0$ such that

$$|\Gamma_T(y)-\Gamma_T(z)|\leq C_2\|y-z\|. \tag{17}$$

Secondly, we suppose that $y\neq z$ then, $\|y-z\|\neq 0$ and since the quantity

$$\left(\sum_{k=1}^{T+1}A(k-1,\Delta z(k-1))-\sum_{k=1}^{T}\frac{1}{p(k)}|z(k)|^{p(k)}\right)^{\delta}$$

and

$$-\left(\sum_{k=1}^{T+1}A(k-1,\Delta y(k-1))-\sum_{k=1}^{T}\frac{1}{p(k)}|y(k)|^{p(k)}\right)^{\delta}$$

are finite, then there exists a constant $C_3>0$ such that

$$\Big|I(z)^{\delta}-I(y)^{\delta}\Big|\leq C_3\|y-z\|.$$

In addition, from [3], $\Psi(k,.)$ and J are locally Lipschitz continuous, then, for all $u\in\mathcal{H}_T$, there exist a neighbourhood V_u of u and a constant $C_4>0$ such that $|\Psi(k,z(k))-\Psi(k,y(k))|\leq C_4\|y-z\|$, and a neighbourhood V'_u of u and a constant $C_5>0$ such that $|J(y)-J(z)|\leq C_5\|y-z\|$.

Therefore,

$$\Big|\Gamma_T(y)-\Gamma_T(z)\Big|\leq K\|y-z\|,$$

where $K=\dfrac{R_2}{\mu}C_3+\lambda T C_4+C_5$. Thus, Γ_T is locally Lipschitz continuous.

□

Lemma 4.5. *Let $u\in\mathcal{H}_T$ be a critical point of Γ_T. Then, u is a classical solution of the problem* (1.1).

Proof. Suppose that $u\in\mathcal{H}_T$ is a critical point of Γ_T. Then, for any $v\in\mathcal{H}_T$, $\Gamma_T^0(u,v)\geq 0$. Using

(15), we obtain

$$
\begin{aligned}
\Gamma_T(w+sv) &= \widehat{M}\Big(\sum_{k=1}^{T+1} A(k-1,\Delta(w+sv)(k-1)) - \sum_{k=1}^{T} \frac{1}{p(k)}|w+sv|^{p(k)}\Big) \\
&- \lambda \sum_{k=1}^{T} \Psi(k,(w+sv)(k)) + J(w+sv) \\
&= \widehat{M}\Big(\sum_{k=1}^{T+1} A(k-1,\Delta w(k-1)+s\Delta v(k-1)) - \sum_{k=1}^{T} \frac{1}{p(k)}|w+sv|^{p(k)}\Big) \\
&- \lambda \sum_{k=1}^{T} F(k,w(k)+sv(k)) - \mu \sum_{k=1}^{T} G(k,w(k)+sv(k)) + J(w+sv)
\end{aligned}
$$

and

$$
\begin{aligned}
\Gamma_T(w+sv)-\Gamma_T(w) &= \widehat{M}\Big(\sum_{k=1}^{T+1} A(k-1,\Delta w(k-1)+s\Delta v(k-1)) - \sum_{k=1}^{T} \frac{1}{p(k)}|w+sv|^{p(k)}\Big) \\
&\quad -\widehat{M}\Big(\sum_{k=1}^{T+1} A(k-1,\Delta(w(k-1)) - \sum_{k=1}^{T} \frac{1}{p(k)}|w|^{p(k)}\Big) \\
&\quad -\lambda\left[\sum_{k=1}^{T} F(k,w(k)+sv(k)) - \sum_{k=1}^{T} F(k,w(k))\right] + J(w+sv) - J(w) \\
&\quad -\mu\left[\sum_{k=1}^{T} G(k,w(k)+sv(k)) - \sum_{k=1}^{T} G(k,w(k))\right]. \qquad (18)
\end{aligned}
$$

Dividing (18) by s and letting $s \longrightarrow 0$ and $w \longrightarrow u$, we get

$$
\begin{aligned}
\Gamma_T^0(u,v) &= M(I(u))\left[\sum_{k=1}^{T+1} a(k-1,\Delta u(k-1))\Delta v(k-1) - \sum_{k=1}^{T} |u(k)|^{p(k)-2}u(k)v(k)\right] \\
&- \lambda \sum_{k=1}^{T} F^0(k,u(k))v(k) - \mu \sum_{k=1}^{T} G^0(k,u(k))v(k) + j'(u,v) \geq 0, \qquad (19)
\end{aligned}
$$

where $I(u) = \Big(\sum_{k=1}^{T+1} A(k-1,\Delta u(k-1)) - \sum_{k=1}^{T} \frac{1}{p(k)}|u(k)|^{p(k)}\Big)$.

Otherwise, we use Abel's summation as in [14] and [23] to obtain

$$
\begin{aligned}
\sum_{k=1}^{T+1} a(k-1,\Delta u(k-1))\Delta v(k-1) &= \sum_{k=1}^{T} a(k-1,\Delta u(k-1))v(k) - \sum_{k=1}^{T} a(k,\Delta u(k))v(k) \\
&\quad -a(0,\Delta u(0))v(0) + a(T,\Delta u(T))v(T+1),
\end{aligned}
$$

which gives

$$
\begin{aligned}
\sum_{k=1}^{T+1} a(k-1,\Delta u(k-1))\Delta v(k-1) &= -\sum_{k=1}^{T} \Delta a(k-1,\Delta u(k-1))v(k) - a(0,\Delta u(0))v(0) \\
&\quad +a(T,\Delta u(T))v(T+1).
\end{aligned}
$$

As $v \in \mathcal{H}_T$ is arbitrary chosen, thus if $v(0) = v(T+1) = 0$ and $\Gamma_T^0(u,v) \geq 0$, it follows that

$$M\left(I(u)\right)\left[\sum_{k=1}^{T} -\Delta(a(k-1,\Delta u(k-1))v(k) - \sum_{k=1}^{T} |u(k)|^{p(k)-2}u(k)v(k)\right]$$
$$-\lambda \sum_{k=1}^{T} F^0(k,u(k))v(k) - \mu \sum_{k=1}^{T} G^0(k,u(k))v(k) \geq 0. \tag{20}$$

Thus, $0_{\mathcal{H}_T^*} \in \partial\Gamma_T(u)$.
We now prove that $(a(0,\Delta u(0)), -a(T,\Delta u(T))) \in \partial j(u(0),u(T+1))$. Indeed, one use (19) and (20), to obtain

$$-M\left(I(u)\right)\left[\sum_{k=1}^{T} \Delta(a(k-1,\Delta u(k-1))v(k) + \sum_{k=1}^{T} |u(k)|^{p(k)-2}u(k)v(k)\right] +$$

$$j'((u(0),u(T+1)),(v(0),v(T+1))) + M(I(u))(a(T,\Delta u(T))v(T+1) - a(0,\Delta u(0))v(0))$$
$$-\lambda \sum_{k=1}^{T} F^0(k,u(k))v(k) - \mu \sum_{k=1}^{T} G^0(k,u(k))v(k) \geq 0.$$

Then,

$$j'((u(0),u(T+1)),(v(0),v(T+1))) \geq M(I(u))\left(-a(T,\Delta u(T))v(T+1) + a(0,\Delta u(0))v(0)\right).$$

From (14), we have

$$j'((u(0),u(T+1)),(v(0),v(T+1))) \quad \geq$$

$$K'\left(\sum_{k=1}^{T+1} A(k-1,\Delta u(k-1)) - \sum_{k=1}^{T} \frac{1}{p(k)}|u(k)|^{p(k)}\right)^{\delta-1} \left(-a(T,\Delta u(T))v(T+1) + a(0,\Delta u(0))v(0)\right)$$

with

$$K' = \begin{cases} R_1 & \text{if} \quad -a(T,\Delta u(T)) + a(0,\Delta u(0)) \geq 0 \\ R_2 & \text{if} \quad -a(T,\Delta u(T)) + a(0,\Delta u(0)) \leq 0. \end{cases}$$

Taking $w = K'\left(\sum_{k=1}^{T+1} A(k-1,\Delta u(k-1)) - \sum_{k=1}^{T} \frac{1}{p(k)}|u(k)|^{p(k)}\right)^{\delta-1}$, we obtain

$$j'((u(0),u(T+1)),(v(0),v(T+1))) \quad \geq \quad \left[-a(T,\Delta u(T))(wv)(T+1) + a(0,\Delta u(0))(wv)(0)\right].$$

We put $(wv)(0) = p'$ and $(wv)(T+1) = q'$, for all $v \in \mathcal{H}_T$ where $p', q' \in \mathbb{R}$ are arbitrarily chosen, to obtain

$$j'((u(0),u(T+1)),(v(0),v(T+1))) \geqslant -a(T,\Delta u(T))q' + a(0,\Delta u(0))p'.$$

By ([30],Theorem 23.2), we deduce that

$$(a(0,\Delta u(0)), -a(T,\Delta u(T))) \in \partial j(u(0),u(T+1)).$$

□

5 Existence of multiple solutions

This section is devoted to the study of the existence of T-periodic solutions for the problem (1). For that, we make the following additional assumptions on the data.

(H_1): There exist three T-periodic functions $r : \mathbb{Z} \longrightarrow [2,\infty)$, $a_1 : \mathbb{Z} \longrightarrow (0,\infty)$ and a function $b_1 : \mathbb{Z} \longrightarrow \mathbb{R}$ such that

$$\min\{F(k,u), -|G(k,u)|\} \geq a_1(k)|u|^{r(k)} + b_1(k), \text{ for all } k \in \mathbb{Z} \text{ and } u \in \mathbb{R}. \tag{21}$$

We define $r^- := \min\limits_{k\in\mathbb{Z}[1,T]} r(k);\ a_1^- := \min\limits_{k\in\mathbb{Z}[1,T]} a_1(k); b_1^- := \min\limits_{k\in\mathbb{Z}[1,T]} b_1(k)$.

Assume further that

$$(H_2) : \lim_{|u|\to 0} \frac{F(k,u)}{|u|^{r^-}} = 0 \text{ and } \lim_{|u|\to 0} \frac{G(k,u)}{|u|^{r^-}} = 0 \text{ uniformly in } k \in \mathbb{Z}. \tag{22}$$

(H_3): There exist three constants $\eta_0, \eta_1, \eta_2 > 0$, with $\eta_2 > \eta_1 > \eta_0$, such that for all $k \in \mathbb{Z}$,
$(t_2) : \min\{F(k,u), G(k,u)\} \geq 0$ with $|u| \leq \eta_0$,
$(t_3) : \max\{F(k,u), G(k,u)\} < 0$ with $\eta_1 \leq |u| \leq \eta_2$.

Example 5.1. As example of nonlinear terms which respect assumptions $\mathcal{S}$ and $(H_1) - (H_3)$, we can give the following.

$$H(k,u) := \begin{cases} 16u^4 & \text{if} \quad |u| \leq \dfrac{1}{2} \\ -4|u| + 3 & \text{if} \quad |u| \in \left]\dfrac{1}{2}, 1\right[\\ u^4 + |u-1| - 2 & \text{if} \quad |u| \geq 1 \end{cases} \tag{23}$$

and function $r : \mathbb{Z} \longrightarrow [2,\infty)$ such that

$$r(k) := \begin{cases} 4 & \text{if} \quad k = 2n \\ 2 & \text{if} \quad k = 2n+1 \text{ with } n \in \mathbb{Z}. \end{cases} \tag{24}$$

Lemma 5.2. *Suppose that (8) − (14) and (21) are satisfied with $r^- > \delta p^+$. Then, the functional Γ_T is anti-coercive on $\mathcal{H}_T$.*

Proof. Recall that

$$\begin{aligned} \Gamma_T(u) &= \widehat{M}\Big(\sum_{k=1}^{T+1} A(k-1,\Delta u(k-1)) - \sum_{k=1}^{T} \frac{1}{p(k)}|u(k)|^{p(k)}\Big) - \lambda \sum_{k=1}^{T} \Psi(k,u(k)) + J(u), \\ &= \widehat{M}(I(u)) - \lambda \sum_{k=1}^{T} \Psi(k,u(k)) + J(u). \end{aligned}$$

From (8) − (11), we deduce that

$$\begin{aligned} \sum_{k=1}^{T+1} A(k-1,\Delta u(k-1)) - \sum_{k=1}^{T} \frac{1}{p(k)}|u(k)|^{p(k)} &\leq C_1 \sum_{k=1}^{T+1} |\Delta u(k-1)| + C_1 \sum_{k=1}^{T+1} \frac{|\Delta u(k-1)|^{p(k-1)}}{p(k-1)} \\ &+ \frac{1}{p-}\left(\sum_{k=1}^{T} |u(k)|^{p^+} + \sum_{k=1}^{T} |u(k)|^{p^-}\right) \\ &\leq 2C_1\sqrt{T+1}\|u\| + \frac{C_1}{p^-} \sum_{k=1}^{T+1} |\Delta u(k-1)|^{p(k-1)} \\ &+ \frac{2}{p^-} \sum_{k=1}^{T} |u(k)|^{p^+}. \end{aligned}$$

We use inequality (14) and Lemma 4.2, to obtain

$$\begin{aligned}
\widehat{M}(I(u)) &\leq \frac{R_2}{\delta}\left(\sum_{k=1}^{T+1} A(k-1,\Delta u(k-1)) - \sum_{k=1}^{T} \frac{1}{p(k)}|u(k)|^{p(k)}\right)^{\delta} \\
&\leq \frac{R_2}{\delta}\left[\frac{C_1}{p^-}\sum_{k=1}^{T+1}|\Delta u(k-1)|^{p(k-1)} + \left(\frac{2T}{p^-}+2C_1\sqrt{T+1}\right)\|u\|^{p^+}\right]^{\delta} \\
&\leq \frac{R_2}{\delta}\left[(T+1)C_1 + \left(\frac{2T}{p^-}+\frac{2^{p^+}C_1(T+1)}{p^-}+2C_1\sqrt{T+1}\right)\|u\|^{p^+}\right]^{\delta}.
\end{aligned}$$

According to (21), we have

$$\min\{F(k,u), -|G(k,u)|\} \geq a_1(k)|u|^{r(k)} + b_1(k).$$

Therefore,

$$\begin{aligned}
-\lambda\sum_{k=1}^{T}\Psi(k,u) &\leq -\lambda\sum_{k=1}^{T}F(k,u) - \mu\sum_{k=1}^{T}G(k,u) \\
&\leq -(\lambda+\mu)\sum_{k=1}^{T}\left[a_1(k)|u|^{r(k)} + b_1(k)\right] \\
&\leq -(\lambda+\mu)\left(a_1^- T^{\frac{2-r^-}{2}}\|u\|^{r^-} + b_1^- T\right).
\end{aligned}$$

From [13], there exists a positive constant $C_5 > 0$ such that $J(u) \leq C_5\|u\|^{\delta p^+}$. Then, we obtain

$$\begin{aligned}
\Gamma_T(u) &\leq \frac{R_2}{\delta}\left[(T+1)C_1 + \left(\frac{2T}{p^-}+\frac{2^{p^+}(T+1)}{p^-}+2C_1\sqrt{T+1}\right)\|u\|^{p^+}\right]^{\delta} \\
&- (\lambda+\mu)\left(a_1^- T^{\frac{2-r^-}{2}}\|u\|^{r^-} + b_1^- T\right) + C_5\|u\|^{\delta p^+}.
\end{aligned}$$

We use the convexity of the functional $x \mapsto x^{\delta}$, to obtain

$$\begin{aligned}
\Gamma_T(u) &\leq \frac{R_2}{\delta}\left(\frac{1}{2^{\delta}}\right)\left[((T+1)C_1)^{\delta} + \left(\frac{2T}{p^-}+\frac{2^{p^+}(T+1)}{p^-}+2C_1\sqrt{T+1}\right)^{\delta}\|u\|^{\delta p^+}\right] \\
&- (\lambda+\mu)\left(a_1^- T^{\frac{2-r^-}{2}}\|u\|^{r^-} + b_1^- T\right) + C_5\|u\|^{\delta p^+}, \qquad (25)
\end{aligned}$$

since $r^- > \delta p^+$, then Γ_T is anti-coercive on $\mathcal{H}_T$. □

Lemma 5.3. *Suppose that (8) − (14) and (21) are satisfied with $r^- = \delta p^+$. Then, there exists $\lambda_0 > 0$ such that for any $\lambda > \lambda_0$, the functional Γ_T is anti-coercive on $\mathcal{H}_T$.*

Proof. By reasoning as in the proof of Lemma 5.2, using inequality (25) we obtain

$$\begin{aligned}
\Gamma_T(u) &\leq \left[C_5 + \frac{R_2}{\delta}\left(\frac{1}{2^{\delta}}\right)\left(\frac{2T}{p^-}+\frac{2^{p^+}(T+1)}{p^-}+2C_1\sqrt{T+1}\right)^{\delta} - (\lambda+\mu)a_1^- T^{\frac{2-r^-}{2}}\right]\|u\|^{\delta p^+} \\
&+\frac{R_2}{\delta}\left(\frac{C_1(T+1)}{2}\right)^{\delta} - (\lambda+\mu)b_1^- T.
\end{aligned}$$

We choose

$$\lambda_0 = \frac{T^{\frac{r^- - 2}{2}}}{a_1^-}\left(C_5 + \frac{R_2}{\delta}\left(\frac{1}{2^\delta}\right)\left(\frac{2T}{p^-} + \frac{2^{p^+}(T+1)}{p^-} + 2C_1\sqrt{T+1}\right)^\delta\right),$$

to get that Γ_T is anti-coercive on $\mathcal{H}_T$ for any $\lambda \in (\lambda_0, \infty)$. □

Note that for any $p^+ \geq 1$, the functional $\|.\|_{p^+} : Z_b \longrightarrow \mathbb{R}$ defined by

$$\|u\|_{p^+} := \left(\sum_{k=1}^{T+1} |\Delta u(k-1)|^{p^+}\right)^{\frac{1}{p^+}}$$

is a norm on Z_b, while it is obviously not a norm on $\mathcal{H}_T$. As all norms in Z_b are equivalent then, there exists a constant $C'' > 0$ small enough with $\frac{C''}{T} < \frac{p^+}{p^-}$
such that

$$\sum_{k=1}^{T+1} |\Delta u(k-1)|^{p^+} \geq C''\|u\|^{p^+}, \quad \text{for all } u \in Z_b. \tag{26}$$

We can state the first main result of this section.

Theorem 5.4. *Suppose that* $(8)-(14)$ *and* $(H_1)-(H_3)$ *are satisfied with* $r^- > \delta p^+$. *Then, for all* $\lambda > 0$, *the problem* (1) *has at least three T-periodic solutions such that at least two of which are non-trivial.*

Proof. According to (H_2) and inequality (t_2) of assumption (H_3), there exists $\tau \in (0, \eta_0) \cap (0, 1)$ such that

$$\min\{F(k,u), G(k,u)\} \leq \varrho |u|^{r^-} \text{ for } |u| \leq \tau, \tag{27}$$

with ϱ a positive number. If $u \in Z_b$ with $\|u\| \leq \tau$, then $|u(k)| \leq \tau$ for all $k \in \mathbb{Z}$.
From (10) and (14), one has

$$\begin{aligned}
\widehat{M}\left(I(u)\right) &\geq \frac{R_1}{\delta}\left(\sum_{k=1}^{T+1} A(k-1, \Delta u(k-1)) - \sum_{k=1}^{T} \frac{1}{p(k)}|u(k)|^{p(k)}\right)^\delta \\
&\geq \frac{R_1}{\delta}\left(\sum_{k=1}^{T+1} \frac{|\Delta u(k-1)|^{p(k-1)}}{p(k-1)} - \sum_{k=1}^{T} \frac{1}{p(k)}|u(k)|^{p(k)}\right)^\delta.
\end{aligned} \tag{28}$$

For all $k \in \mathbb{Z}$, we have $|u(k)|^{p(k)} \leq |u(k)|^{p^+} + |u(k)|^{p^-}$. From inequality (a_1) of Lemma (4.2), we obtain

$$\sum_{k=1}^{T} |u(k)|^{p^+} \leq T\|u\|^{p^+} \text{ and } \sum_{k=1}^{T} |u(k)|^{p^-} \leq T\|u\|^{p^-}, \forall u \in Z_b. \tag{29}$$

Then, according to (26) and (29), it follows that

$$\widehat{M}\left(I(u)\right) \geq \frac{R_1}{\delta}\left(\frac{1}{p^+}\sum_{k=1}^{T+1} |\Delta u(k-1)|^{p^+} - \frac{2T}{p^-}\|u\|^{p^-}\right)^\delta \tag{30}$$

$$\widehat{M}\left(I(u)\right) \geq \frac{R_1}{\delta}\left(\frac{C''}{p^+}\right)^\delta \|u\|^{\delta p^+}\left(1 - \frac{2Tp^+\|u\|^{p^-}}{p^- C''\|u\|^{p^+}}\right)^\delta \tag{31}$$

$$\widehat{M}\left(I(u)\right) \geq \frac{R_1}{\delta}\left(\frac{C''}{p^+}\right)^\delta \|u\|^{\delta p^+}\left(1 - \frac{2Tp^+}{p^- C''}\|u\|^{p^- - p^+}\right)^\delta. \tag{32}$$

As $\|u\| \leq \tau < 1$, then $1 - \dfrac{2Tp^+}{p^- C''}\|u\|^{p^- - p^+} < -1$. Thus,

$$\left(1 - \frac{2Tp^+}{p^- C''}\|u\|^{p^- - p^+}\right)^{\delta} > 1. \tag{33}$$

Therefore,

$$\widehat{M}\Big(I(u)\Big) \geq \frac{R_1}{\delta}\left(\frac{C''}{p^+}\right)^{\delta}\|u\|^{\delta p^+}. \tag{34}$$

Using (22), one has

$$-\lambda \sum_{k=1}^{T} \Psi(k,u) \geq -(\lambda+\mu)\varrho \sum_{k=1}^{T} |u(k)|^{r^-}.$$

Since $J(u) \geq 0$, we deduce that

$$\begin{aligned}
\Gamma_T(u) &\geq \frac{R_1}{\delta}\left(\frac{C''}{p^+}\right)^{\delta}\|u\|^{\delta p^+} - (\lambda+\mu)\varrho \sum_{k=1}^{T}|u(k)|^{r^-} \\
&\geq \frac{R_1}{\delta}\left(\frac{C''}{p^+}\right)^{\delta}\|u\|^{r^-} - (\lambda+\mu)\varrho T\|u\|^{r^-} \\
&\geq \|u\|^{r^-}\left(\frac{R_1}{\delta}\left(\frac{C''}{p^+}\right)^{\delta} - (\lambda+\mu)\varrho T\right).
\end{aligned}$$

Taking $\varrho \leq \dfrac{\frac{R_1}{\delta}\left(\frac{C''}{p^+}\right)^{\delta}}{(\lambda+\mu)T}$, we obtain $\Gamma_T(u) \geq 0$. Thus,

$$\Gamma_T(u) \geq \Gamma_T(0) \text{ for all } u \in Z_b \text{ with } \|u\| \leq \tau.$$

Hence, we get condition (2.1.2) of Theorem 2.2.

To show condition (2.1.1), note that for all $u \in W_b$, $u(k) = u(k-1)$, $\forall k \in \mathbb{Z}$. Using (8) with $F,\ G \in \mathcal{S}$, one obtains

$$\Gamma_T(u) = -\lambda \sum_{k=1}^{T} F(k,u(k)) - \mu \sum_{k=1}^{T} G(k,u(k)) + J(u),$$

for all $u \in W_b$, $\forall b \in \mathbb{R}$. If $u \in W_b$ with $\|u\| \leq \tau$, then $|u(k)| \leq \tau$ for all $k \in \mathbb{Z}$. By (s_2), (H_1) and as J is bounded from below, it follows that

$$\Gamma_T(u) \leq \Gamma_T(0) \text{ for all } u \in W_b, \text{ with } \|u\| \leq \tau, \text{ where } b \in \mathbb{R}.$$

To get (2.1.3), let $\Upsilon_T = -\Gamma_T(u)$. From (5.2), we deduce that Υ_T satisfies the Palais-Smale (PS) condition. In addition, Υ_T is coercive and bounded from below, then it admits a minimizer. We use (t_3) to obtain

$$\inf_{u \in \mathcal{H}_T} \Upsilon_T(u) = \inf_{u \in \mathcal{H}_T} (-\Gamma_T(u)) = -\sup_{u \in \mathcal{H}_T} \Gamma_T(u) < 0.$$

From theorems 2.2 and 2.4, we deduce that Γ_T has at least three critical points such that at least two of them are non-zero critical points. By Lemma 4.5, the critical points are non-trivial T-periodic solutions of problem (1.1). $\square$

Remark 5.5. In the special case $r^- = \delta p^+$. Arguing again as above, but taking into account Lemma 5.3 and the proof of Theorem 5.4, we can show that for every $\lambda \in \left(\frac{T^{\frac{r^- -2}{2}}}{a_1^-} \left(C_5 + \frac{R_2}{\delta} \left(\frac{1}{2^\delta}\right) \left(\frac{2T}{p^-} + \frac{2^{p^+}(T+1)}{p^-} + 2C_1\sqrt{T+1} \right)^\delta \right), \infty \right)$, the same conclusion still holds without any additional assumption. Hence, the problem (1) has at least three T-periodic solutions, at least two of which are non-trivial.

6 Existence of multiple solutions by three critical points theorem

In this section, we apply the Theorem 3.3, to investigate the existence of multiple solution of problem (1).

Let functionals $\varphi, v : \mathcal{H}_T \longrightarrow \mathbb{R}$ be defined by

$$\varphi(u) := \widehat{M}\left(\sum_{k=1}^{T+1} A(k-1, \Delta u(k-1)) - \sum_{k=1}^{T} \frac{1}{p(k)} |u(k)|^{p(k)} \right) \quad and \quad v(u) := -\sum_{k=1}^{T} \Psi(k, u(k)),$$

where $\Psi(k,u) = F(k,u) + \frac{\mu}{\lambda} G(k,u)$.

Recall that $\Gamma_T(u) = \tilde{\varphi}(u) + \lambda v(u)$ where $\tilde{\varphi} = \varphi(u) + J(u)$.

Assume that

(E_1) : There exists a constant $C \in \mathbb{R}$ such that

$$\max\{F(k,u), G(k,u)\} \leq C \quad \text{for all} \quad (k,u) \in \mathbb{Z} \times \mathbb{R}.$$

(E_2) : There exist numbers $\sigma_1, \sigma_2, \sigma_3 > 0$, $\sigma_3 > \sigma_2 > \sigma_1$ such that for all $k \in \mathbb{Z}$:
$(E_2^{'})$: $\max\{F(k,u), G(k,u)\} > 0$ with $0 < |u| \leq \sigma_1$,
$(E_2^{''})$: $\min\{F(k,u), G(k,u)\} < 0$ with $\sigma_2 \leq |u| \leq \sigma_3$.

In the sequel we will consider the problem (1) on Z_b.

Remark 6.1. The functional $\tilde{\varphi}$ is not coercive on $\mathcal{H}_T$, but it is coercive on Z_b, for $b \in \mathbb{R}$. Indeed, given a sequence $\{u_n\}_{n\in\mathbb{N}} \subset W_b$, we see that

$$\begin{aligned} \tilde{\varphi}(u_n) &= \widehat{M}\left(\sum_{k=1}^{T+1} A(k-1, \Delta u_n(k-1)) - \sum_{k=1}^{T} \frac{1}{p(k)} |u_n(k)|^{p(k)} \right) + j(u(0), u(T+1)) \\ &= \widehat{M}\left(\sum_{k=1}^{T+1} A(k-1, 0) - \sum_{k=1}^{T} \frac{1}{p(k)} |b|^{p(k)} \right) + J(u) \\ &= \widehat{M}\left(-\sum_{k=1}^{T} \frac{1}{p(k)} |b|^{p(k)} \right) + J(u). \end{aligned}$$

Letting $b \longrightarrow \infty$, we obtain that $\tilde{\varphi}$ is not coercive on $\mathcal{H}_T$.
Let us take $u \in Z_b$ with $\|u\| > 1$, we have

$$\widehat{M}\Big(I(u) \Big) \geq \frac{R_1}{\delta} \left(\frac{C''}{p^+} \right)^\delta \|u\|^{\delta p^+} \left(1 - \frac{2Tp^+}{p^- C''} \right)^\delta. \tag{35}$$

Then, $\tilde{\varphi}$ is coercive on Z_b.

Theorem 6.2. *Suppose that* (8) − (14) *and* $(E_1) - (E_2)$ *are satisfied. Then, there exists* $\lambda^* > 0$ *such that problem* (1.1) *has at least three solutions in* Z_b*, at least two of which are necessarily non-zero.*

Proof. Based on results above, $\tilde{\varphi}$ is coercive on Z_b and it is of class C^1.

Recall that $v(u) := -\sum_{k=1}^{T} \Psi(k, u(k))$. From (E_1), we deduce that

$$v(u) \geq -TC\left(1 + \frac{\mu}{\lambda}\right) > -\infty. \tag{36}$$

Then, v is bounded from below.

Using (E''_2), we obtain

$$-\Psi(u) < 0.$$

Then, for all $u \in Z_b$ $(b \in \mathbb{R})$ such that $\sigma_2 \leq |u(k)| \leq \sigma_3$ for all $k \in \mathbb{Z}$, we deduce that there exists $u \in Z_b$ such that $v(u) < 0$. From Weierstrass theorem, we obtain

$$\inf_{u \in Z_b} v(u) < 0.$$

Using (11), $(E_2^{'})$ and since $\tilde{\varphi}$ is continuous, coercive, convex, non-negative and $\tilde{\varphi}(0) = 0$, we get that there are $s, r > 0$, such that $v(u) > 0$ for $r \leq \tilde{\varphi}(u) \leq s$ (see [14, 15]). Hence, (d_2) follows. Since $F, G \in \mathcal{S}$, we use inequality $(E_2^{''})$ to obtain

$$\inf_{r \leq \tilde{\varphi}(u) \leq s} v(u) > v(0) = 0.$$

Thus, we get (d_3).

Using the relation between critical points of Γ_T and problem (1), we deduce by Theorem 3.3 that there exists $\lambda^* > 0$ such that problem (1) has at least three solutions in Z_b, at least two of which are necessarily non-zero.

7 Conclusion

In this work, one has studied some discrete inclusions problems involving the $p(k)-$Laplacian operator of Kirchhoff type subjected to potential boundary conditions. One has proved the existence of multiple solutions by using a three critical points theorem using a variational approach.

□

8 Declarations

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References

[1] R.P. Agarwal, *Difference equations and inequalities : Theory, methods and applications*, 2 ed., Marcel Dekker, Inc., New York, 2000.

[2] A. Arosio and S. Pannizi, *On the well-posedness of the Kirchhoff string*, Trans. Amer. Math. Soc, **348** (1996), no. 1, 305-330.

[3] C. Bereanu, P. Jebelean and C. Serban, *Ground state and mountain pass solutions for discrete $p(k)$-Laplacian*, Bound. Value Probl, **104**(2012), no. 1, 1-13.

[4] G. Bonanno and P. Candito, *Infinitely many solutions for a class of discrete nonlinear boundary value problems*, Appl. Anal,**88**(2009), no. 4, 605-616.

[5] G. Bonanno and S.A. Marano, *On the structure of the critical set of non-differentiable functions with a weak compactness condition*, Appl. Anal, **89**(2010), 1-10.

[6] X. Cai and J. Yu, *Existence theorems for second-order discrete boundary value problems*, J. Math. Anal. Appl, **320**(2006), 649-661.

[7] G.F. Carrier, A note on the vibrating string, Quart. Appl. Math, **7**(1949), 97-101.

[8] G.F. Carrier, *On the nonlinear vibration problem of the elastic string*, Quart. Appl. Math, **3**(1945), 157-165.

[9] M.M. Cavalcanti, V.N. Cavalcanti and J.A. Soriano, *Global existence and uniform decay rates for the Kirchhoff-Carrier equation with nonlinear dissipation*, Adv. Differ. Equ. **6**(2001), 701-730.

[10] O. Chakrone, E.M. Hssini, M. Rahmani and O. Darhouche; *Multiplicity results for a p-Laplacian discrete problems of Kirchhoff type*, Appl. Math. Comput, **276**(2016), 310-315.

[11] K.C. F. Chang, *Variational methods for non-differentiable functionals and their applications to partial differential equations*, J. Math. Anal. Appl, **80**(1981), no. 1, 102-129.

[12] F.H. Clarke, *Optimization and nonsmooth analysis, Classics in Applied Mathematics*, SIAM, Philadelphia, Vol.5, 1990.

[13] A. A. K. Dianda and S. Ouaro, *Existence and multiplicity of solutions to discrete inclusions with the $p(k)$-Laplacian problem of Kirchhoff type*, Asia Pac. J. Math., **6**(2020), 1–19.

[14] M. Galewski, G.M. Bisci and R. Wieteska, *Existence and multiplicity of solutions to discrete inclusions with the $p(k)$-Laplacian problem*, J. Difference Equ. Appl, **21**(2015), no.10, 887–903.

[15] M. Galewski and Sz. Glab, *On the discrete boundary value problem for anisotropic equation*, J. Math. Anal. Appl, **386**(2012), no. 2, 956–965.

[16] M. Galewski, Sz. Glab and R. Wieteska, *Positive solutions for anisotropic discrete boundary-value problems*, Electron. J. Differ. Equ, **32**(2013), 1–9.

[17] M. Galewski and R. Wieteska, *Multiple solutions for periodic problems with the discrete $p(k)$-Laplacian*, Discrete Contin. Dyn. Syst. Ser., **19**(2014), no. 8, 2535-2547.

[18] S. Heidarkhani, G. Caristi and A. Salari, *Perturbed Kirchhoff-type p-Laplacian discrete problems*, Collect. Math, **68**(2017), No. 3, 401-418.

[19] S. Heidarkhani and M.K. Moghadam, *Existence of three solutions for perturbed nonlinear difference equations*, Op Math, **34**(2014), no. 4, 747–761.

[20] A. Iannizzotto, A. Cabada and S. Tersian, *Multiple solutions for discrete boundary value problems*, J. Math. Anal. Appl, **32**(2009), 418–428.

[21] G. Kirchhoff, *Vorlesungen uber mathematische Physik: Mechanik*, Teubner, Leipzig, 1876.

[22] B. Koné, I. Nyanquini and S. Ouaro, *Weak solutions to discrete nonlinear two-point boundary value problems of Kirchhoff type*, Electron. J. Differ. Equ. **105**(2015), 1-10.

[23] L. Kong, *Homoclinic solutions for a higher order difference equation with p-Laplacian*, Indag. Math, **27**(2016), 124–146.

[24] J.L. Lions, *On some questions in boundary value problems of mathematical physics. In Contemporary Developpements in Continuum Mechanics and Partial Differential Equations*, Proc. Internat. Sympos. Inst. Mat. Univ. Fed. Rio de Janeiro, North-Holland Math Stud. **30**(1978), 284-345.

[25] D. Motreanu and Cs. Varga, *Some critical point result for locally Lipschitz functionals*, Comm. Appl. Nonlinear Anal, **4**(1997), 17–33.

[26] R. Narashima, *Nonlinear vibration of an elastic string*, J. Sound Vibration. **8**(1968), 134-146.

[27] D.W. Oplinger, *Frequency response of a nonlinear stretched string*, J. Acoustic Soc. Amer. **32**(1960), 1529-1538.

[28] S. Ouaro and M. Zougrana, *Multiplicity of solutions to discrete inclusions with the $p(k)$-laplace Kirchhoff type equations*, Asia Pac. J. Math, **5**(2018), no. 1, 27-49.

[29] A. Ourraoui and A. Ayoujil, *On a class of non-local discrete boundary value problem*, Arab J. Math. Sci, **28**(2022), No. 2, 130-141.

[30] R.T. Rockafellar, *Convex analysis*, Princeton University Press, 1972.

[31] R. Sanou, I. Ibrango and B. Koné, *Weak nontrivial solutions to discrete nonlinear two-point boundary-value problems of Kirchhoff type*, J. Adv. Appl. Math, **6**(2021), no. 1.

[32] X. Wu, *A new critical point theorem for locally lipschitz functionals with applications to differential equations*, Nonlinear Anal, **66**(2007), no. 3, 624–638.

[33] J. Yang and J. Liu, *Nontrivial solutions for discrete Kirchhoff type problems with resonance via critical groups*, Adv. Differ. Equ, **1**(2013), 1-14.

[34] Z. Yücedag, *Solutions for a discrete boundary value problem involving Kirchhoff type equation via variational methods*, TWMS J. App. Eng. Math. **8**(2018), no. 1, 144-154.

[35] Z. Yücedag, *Existence of solutions for anisotropic discrete boundary value problems of Kirchhoff type*, Int. J. Differ. Equ. Appl. **13**(2014), no. 1, 1-15.

[36] G. Zhang and S. Liu, *On a class of semipositone discrete boundary value problem*, J. Math. Anal. Appl, **325**(2007), 175-182.

[37] V. Zhikov, *Averaging of functionals in the calculus of variations and elasticity*, Mathematics of the USSR-Izvestiya, **29**(1987), 33-66.