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Dynamics of a Two Competing Prey-One Predator System with Fear Effect and Anti-Predator Behaviour

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Abstract

This paper examines the effect of fear and anti-predator behaviour in a two competing prey and one predator system. Here, one of the competing prey species is affected by the fear of predators while the other competing prey species shows anti-predator behaviour. Basic results on positivity, the boundedness of the solutions, local and global stability of the coexistence equilibrium point, uniform persistence, Hopf bifurcation and the nature of the limit cycle emerging through Hopf bifurcation are investigated. Numerical examples are provided to support our obtained results.

Keywords: Predator-prey model, fear factor, anti-predator behaviour, stability, uniform persistence, bifurcation
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1 Introduction

In population biology, study of dynamic function between predator-prey becomes an important area of research due to its universal existence in nature. Modelling is a basic tool for describing the above interaction in the presence of different ecological factors mathematically. Developing a mathematical model, one can predict the coexistence of interacting species in future time. Most of the mathematical models in predator-prey interaction emphasizes direct killing [1, 2] whereas much can happen to predator-prey behaviour facing each other. Thus, it is reasonable to consider the effect of fear on prey population.

In predator-prey interaction, predator affects prey populations in various ways, with the fear of predation causing large population fitness decreases [3, 4]. This may be the fact when a single prey population is not killed. Prey species that faces fears in the presence of predators will normally forage less because of vigilance.

Fear effect is mainly observed in interactions between vertebrates or avian species. For example, wolf and deer are identified by the investigators. In the Yellowstone ecosystem, there is a decline in deer population growth due to the attack of predator species.

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Keeping the above fact in mind, several studies have been done to assess the effect of fear on the density of the prey population [5, 6, 7, 8, 9, 10, 11, 12, 13, 14]. Wang et al. [5] first introduced the effect of fear in predator-prey system. They showed that fear can affect the birth rate of prey populations. They also commented that the cost of fear has no role in dynamical behaviour for Holling type I functional response while this is not so in Holling type II functional response. Mukherjee [6] discussed the impact of fear in a predator-prey system in the presence of competitor for the prey. Mukherjee [7] investigated the effect of fear in predator-prey system where the fear factor doesn't only affect the birth rate of the prey population but also the death rate. He developed criteria for uniform persistence, global stability and Hopf bifurcation. Xia and Yuan [13] studied a stochastic predator-prey model with prey refuge and fear effect. They found stationary distribution and performed survival analysis of the model. Qi and Meng [15] analyzed the threshold behaviour of a stochastic predator-prey system with prey refuge, fear effect and non-constant mortality rate. Sarkar and Khajanchi [16] examined the impact of fear on the growth of prey in a predator-prey model. They discussed permanence, Hopf bifurcation and stability of the coexistence equilibrium point. Liu and Jiang [17] illustrated the role of fear factor in the dynamics of stochastic predator-prey model. They obtained sufficient criteria for the existence and uniqueness of an ergodic stationary distribution of positive solution to the system.

Another interesting feature of the prey species is the anti-predator behaviour. A well known example of anti-predator behaviour is that many animals use horns, claws and teeth to fight off predators. Ives and Dobson [18] investigated a predator-prey model with anti-predator behaviour and remarked that anti-predator behaviour can increase prey density, reduce the rate of predator-prey density and dams the oscillation of a predator-prey system. Tang and Xiao [19] studied a predator-prey model considering the non-monotonic functional response and anti-predator behaviour so that the adult prey can attack vulnerable predators. They commented that anti-predator behaviour can induce coexistence of the prey and predator and also damps the oscillation in a predator-prey system. Sun et al. [20] analyzed a predator-prey model with anti-predator behaviour when the population size of the prey exceeds a threshold. They discussed stability and bifurcation of the system.

Previous studies [19] and [20] did not consider the fear effect as well as the presence of competitor for prey. The main reason for this, the predator may also change prey dynamics by allowing the prey interactions with the other species. Thus, the species coexistence doesn't only depend on predation but also by the inter-specific competition. On multi-species system, predator usually attacks weak competing prey. In that case, the superior competitor plays an important role in structuring the community. There are some cases, where superior competitor shows an anti-predator behaviour while the birth rate of weak competitor is affected by the fear of predators. Anti-predator behaviour reduces the predation pressure and eventually it promotes the coexistence of all the species.

In view of above, we propose a two prey one predator system where the preys are in competition where the birth rate of weak prey is affected by fear of predators while the other prey exhibits anti-predator behaviour. The predator consumes both the prey species. For example, Uganda kobs and buffalo are two competing prey species. Lions consume both the prey species, whereas buffalo shows anti-predator behaviour.

The paper is structured as follows. In the next section, we propose our model. The biological validity of the model is examined in Section 3. We demonstrate some dynamical properties of the model (1) in Section 4.2. Computer simulations are given in Section 5. A brief discussion concludes the paper in Section 6.

2 Mathematical model

We consider a habitat where the two preys are in competition and the predator consumes both the prey population, birth rate of one of the competing prey species designated as x_1 is affected by the fear of predator while the other prey species x_2 shows anti-predator behaviour. Due to the presence of predator, fear affects the the birth rate of the prey so the production term r_1x_1 is multiplied by a factor $f(k,y) = \frac{1}{1+ky}$ where k is the level of fear. The function $f(k,y)$ satisfy the following conditions:

(i) $f(0,y) = 1$ i.e., in absence of fear, there is no reduction of prey species birth rate.

(ii) $f(k,0) = 1$ i.e., in absence of predation, there is no reduction of prey species birth rate.

(iii) $\lim_{k\to\infty} f(k,y) = 0$ i.e, the birth rate of prey species become zero for massive increase of fear.

(iv) $\lim_{y\to\infty} f(k,y) = 0$ i.e, the birth rate of prey species become zero when there is a large number of predators present in the system.

(v) $\frac{\partial}{\partial k}(f(k,y)) = -\frac{y}{(1+ky)^2} < 0$ i.e., the birth rate of prey decreases due to increase of the level of fear.

(vi) $\frac{\partial}{\partial y}(f(k,y)) = -\frac{k}{(1+ky)^2} < 0$ i.e., the birth rate of prey decreases due to increase of the predator.

To introduce anti-predator efficiency, we consider the response function $g(y) = \frac{\alpha y}{1+\beta y}$ where $\frac{1}{\beta}$ is the half saturation constant. $\frac{\alpha}{\beta}$ is the maximal anti-predator efficiency of the strong prey. α is called the anti-predator rate parameter that does not directly benefit the prey population but reduces the growth rate of the predator population since the prey population does not feed chiefly on the predator population.. Clearly, $g(y)$ increasing function of the predator density. There is a limit of the anti-predator efficiency of the prey x_2. That is, the anti-predator efficiency of the prey x_2 cannot infinitely increase as the density of predator increases. For more justification on anti-predator behaviour can be found in [20]. The proposed model is applicable to elk-bison-wolf in Yellowstone National Park. The model is described by:

$$\begin{aligned} \frac{dx_1}{dt} &= x_1\left(\frac{r_1}{1+ky} - m - a_{11}x_1 - a_{12}x_2 - p_1y\right), \\ \frac{dx_2}{dt} &= x_2(r_2 - a_{21}x_1 - a_{22}x_2 - p_2y), \\ \frac{dy}{dt} &= y\left(-d + c_1p_1x_1 + c_2p_2x_2 - \frac{\alpha x_2}{1+\beta y} - hy\right) \end{aligned} \tag{1}$$

with $x_i(0) = x_{i0} > 0, i = 1,2, y(0) = y_0 > 0$. Here the variables x_1, x_2 and y denote the densities of two competing prey and predator population at time t respectively. All the parameters are assumed to be positive. r_1 and m are the birth and death rate of the weak competing prey species respectively. r_2 is the intrinsic growth rate of the strong competing prey species. a_{ii} represents the intra-specific competition among the prey species i. a_{ij} measures the the action of species j upon the growth rate of species i. p_1 and p_2 stand for the per capita predator consumption rate. c_1 and c_2 are the conversion rate of prey biomass to predator biomass. h represents the intra-specific competition coefficient among the predator species.

3 Positivity and boundedness of solutions

In this section, we first examine positivity and boundedness of solutions of system (1) which justify the validity and well behaved nature of the model. We first show positivity.

Lemma 1. All solutions $(x_1(t), x_2(t), y(t))$ of system (1) with initial values $(x_{10}, x_{20}, y_0) \in \mathbb{R}^3_+$ remains positive for all $t > 0$.

Proof. The positivity of $x_1(t), x_2(t), y(t)$ can be verified by the equations.

$$\begin{aligned}
x_1(t) &= x_{10}\exp\{\int_0^t [\frac{r_1}{1+ky(s)} - m - a_{11}x_1(s) - a_{12}x_2(s) - p_1y(s)]ds\}, \\
x_2(t) &= x_{20}\exp\{\int_0^t [r_2 - a_{21}x_1(s) - a_{22}x_2(s) - p_2y(s)]ds\}, \\
y(t) &= y_0\exp\{\int_0^t [-d + c_1p_1x_1(s) + c_2p_2x_2(s) - \frac{\alpha x_2(s)}{1+\beta y(s)} - hy(s)]ds\}
\end{aligned}$$

with $x_{10}, x_{20}, y_0 > 0$. As $x_{10} > 0$ then $x_1(t) > 0$ for all $t > 0$. The same argument is valid for component $x_2(t)$ and $y(t)$. Hence the interior of $\mathbb{R}^3_+$, is an invariant set of system (1).

Lemma 2. All the solutions of system (1) will lie in the region

$$B = \{(x_1, x_2, y) \in \mathbb{R}^3_+ : 0 \leq c_1x_1 + c_2x_2 + y \leq \frac{c_1(r_1+\mu)^2}{4a_{11}} + \frac{c_2(r_2+\mu)^2}{4a_{22}}\}$$

as $t \to \infty$ for all positive initial values $(x_{10}, x_{20}, y_0) \in \mathbb{R}^3_+$, where $\mu < d$.

Proof. Let us consider the function

$$W(t) = c_1x_1 + c_2x_2 + y.$$

The time derivative along a solution of (1) is

$$\frac{dW(t)}{dt} = c_1x_1(\frac{r_1}{1+ky} - m - a_{11}x_1 - a_{12}x_2) + c_2x_2(r_2 - a_{21}x_1 - a_{22}x_2) - y(d + hy + \frac{\alpha x_2}{1+\beta y}).$$

For each $\mu > 0$, the following inequality is satisfied

$$\begin{aligned}
\frac{dW}{dt} + \mu W &\leq c_1x_1(\frac{r_1}{1+ky} - m - a_{11}x_1 + \mu) + c_2x_2(r_2 - a_{22}x_2 + \mu) + y(\mu - d) \qquad (2) \\
&\leq -c_{11}\{(x_1 - \frac{r_1+\mu}{2a_{11}})^2 - (\frac{r_1+\mu}{2a_{22}})^2\} - c_{22}\{(x_2 - \frac{r_2+\mu}{2a_{22}})^2 - (\frac{r_2+\mu}{2a_{22}})^2\} + y(\mu - d) \\
&\leq \frac{c_1(r_1+\mu)^2}{4a_{11}} + \frac{c_2(r_2+\mu)^2}{4a_{22}}. \qquad (3)
\end{aligned}$$

Now choose μ such that $\mu < d$. Thus (3) can be written as

$$\frac{dW}{dt} + \mu W \leq \frac{c_1(r_1+\mu)^2}{4a_{11}} + \frac{c_2(r_2+\mu)^2}{4a_{22}}.$$

By using the comparison theorem [21], we get

$$0 \leq W(x_1(t), x_2(t), y(t)) \leq \frac{c_1(r_1+\mu)^2}{4a_{11}} + \frac{c_2(r_2+\mu)^2}{4a_{22}} + W(x_1(0), x_2(0), y(0))/e^{\lambda t}.$$

Taking limit when $t \to \infty$, we have $0 < W(t) \leq \frac{c_1(r_1+\mu)^2}{4a_{11}} + \frac{c_2(r_2+\mu)^2}{4a_{22}}$. Hence system (1) is bounded.

4 Existence of equilibria and stability analysis

Throughout this article, we assume that $r_1 > m$. Evidently, system (1) has seven non-negative equilibrium points. The population free equilibrium point is $E_0 = (0,0,0)$. The second prey and predator prey free equilibrium point $E_1 = (\frac{r_1-m}{a_{11}}, 0, 0)$ as $r_1 > m$. The first prey and predator free equilibrium point $E_2 = (0, \frac{r_2}{a_{22}}, 0)$. There exists a predator free equilibrium point $E_{12} = (\bar{x_1}, \bar{x_2}, 0)$, where $\bar{x_1} = \frac{r_1 a_{22} - r_2 a_{12}}{a_{11}a_{22} - a_{12}a_{21}}$ and $\bar{x_2} = \frac{r_2 a_{11} - r_1 a_{21}}{a_{11}a_{22} - a_{12}a_{21}}$ provided the following conditions hold:

$$\frac{a_{11}}{a_{21}} > \frac{r_1 - m}{r_2} > \frac{a_{12}}{a_{21}} \text{ or } \frac{a_{11}}{a_{21}} < \frac{r_1 - m}{r_2} < \frac{a_{12}}{a_{21}}. \tag{4}$$

If $a_{11}d + mc_1p_1 < r_1c_1p_1$, then there exits a unique second prey free equilibrium point $E_{13} = (\tilde{x_1}, 0, \tilde{y})$ where

$$\tilde{x_1} = \frac{d + h\tilde{y}}{c_1 p_1}$$

and

$$\tilde{y} = \frac{-\{a_{11}(kd+h) + c_1p_1^2 + mkc_1p_1\} \pm \sqrt{\{a_{11}(kd+h) + c_1p_1^2 + mkc_1p_1\}^2 - 4k(a_{11}h + c_1p_1^2)(a_{11}d - r_1c_1p_1 + mc_1p_1)}}{2k(a_{11}h + c_1p_1^2)}.$$

If $d_2a_{22} + r_2\alpha < r_2c_2p_2$, then there exists a unique first prey free equilibrium point $E_{23} = (0, \hat{x_2}, \hat{y})$ where $\hat{x_2} = \frac{r_2 - p_2\hat{y}}{a_{22}}$, and $\hat{y}$ is given by

$$\hat{y} = \frac{-\{d\beta a_{22} - c_2p_2\beta r_2 + p_2(c_2p_2 - \alpha) + a_{22}h\} + \sqrt{\{d\beta a_{22} - c_2p_2\beta r_2 + p_2(c_2p_2 - \alpha) + a_{22}h\}^2 - 4\beta(c_2p_2^2 + a_{22}h)(a_{22}d - r_2(c_2p_2 - \alpha))}}{2\beta(c_2p_2^2 + a_{22}h)}$$

provided $r_2 > p_2\hat{y}$.

To locate the positive equilibrium point $E^* = (x_1^*, x_2^*, y^*)$ of system (1), we apply isocline method. x_1^*, x_2^* and y^* are the positive solutions of the following system of equations:

$$\frac{r_1}{1+ky} - m - a_{11}x_1 - a_{12}x_2 - p_1y = 0, \tag{5}$$

$$r_2 - a_{21}x_1 - a_{22}x_2 - p_2y = 0, \tag{6}$$

$$-d + c_1p_1x_1 + c_2p_2x_2 - \frac{\alpha x_2}{1+\beta y} - hy = 0. \tag{7}$$

From equation (6), we get $y = \frac{r_2 - a_{21}x_1 - a_{22}x_2}{p_2} = y_e$ (say).
For positivity of $y, r_2 > a_{21}x_1 + a_{22}x_2$. Now we substitute the value of y in (5) and (7) and obtain

$$f_1(x_1, x_2) = \frac{r_1}{1+ky_e} - m - a_{11}x_1 - a_{12}x_2 - p_1y_e = 0, \tag{8}$$

$$f_2(x_1, x_2) = -d + c_1p_1x_1 + c_2p_2x_2 - \frac{\alpha x_2}{1+\beta y_e} - hy_e = 0 \tag{9}$$

Now, we have

$$\frac{dx_1}{dx_2} = -\frac{\frac{\partial f_1}{\partial x_2}}{\frac{\partial f_1}{\partial x_1}} = -\frac{M_1}{N_1}$$

where $M_1 = \frac{r_1ka_{22}}{p_2(1+ky_e)^2} - a_{12} + \frac{p_1a_{22}}{p_2}, N_1 = -a_{11} + \frac{r_1ka_{21}}{p_2(1+ky_e)^2} + \frac{p_1a_{21}}{p_2}$. It is evident that $\frac{dx_1}{dx_2} > 0$ if either (i) $M_1 > 0$ and $N_1 < 0$ or (ii) $M_1 < 0$ and $N_1 > 0$ hold.

Also we get

$$\frac{dx_1}{dx_2} = -\frac{\partial f_2}{\partial x_2} / \frac{\partial f_2}{\partial x_1} = -\frac{M_2}{N_2}$$

where $M_2 = c_2p_2 - \frac{\alpha\{p_2(1+\beta y_e)+a_{222}\beta x_2\}}{p_2(1+\beta y_e)^2} + \frac{ha_{22}}{p_2}, N_2 = c_1p_1 - \frac{\alpha x_2\beta a_{21}}{p_2(1+\beta y_e)^2} + \frac{ha_{21}}{p_2}$. We note that $\frac{dx_1}{dx_2} < 0$ if either (i) $M_2 > 0$ and $N_2 > 0$ or (ii) $M_2 < 0$ and $N_2 < 0$.
From the above analysis, we conclude that the two isoclines (8) and (9) intersect at the point (x_1^*, x_2^*) under certain restrictions. Throughout this paper we assume that E^* exists.

4.1 Stability analysis

The local stability properties of the steady states can be determined through the Jacobian matrix around the each steady state. Clearly, E_0 is always unstable.
E_1 is locally stable if

$$r_2a_{11} < a_{21}(r_1 - m) \text{ and } da_{11} > c_1p_1(r_1 - m).$$

E_2 is locally stable if

$$(r_1 - m)a_{22} < a_{12}r_2 \text{ and } da_{22} + \alpha r_2 > c_2p_2a_{22}.$$

E_{12} is locally stable if

$$a_{11}a_{22} > a_{12}a_{21} \text{ and } d > c_1p_1\bar{x_1} + (c_2p_2 - \alpha)\bar{x_2}.$$

E_{13} is locally stable if

$$r_2 < a_{21}\tilde{x_1} + p_2\tilde{y}.$$

E_{23} is locally stable if

$$r_1 < (1 + k\hat{y})(m + a_{12}\hat{x_2} + p_1\hat{y}), c_2p_2(1 + \beta\hat{y}) > \alpha \text{ and } h(1 + \beta\hat{y})^2 > \alpha\hat{x_2}\beta.$$

To examine the local stability of positive steady state E^*, we find out the characteristic equation around E^* which is given by:

$$\lambda^3 + a_1\lambda^2 + a_2\lambda + a_3 = 0 \tag{10}$$

where $a_1 = a_{11}x_1^* + a_{22}x_2^* - a_{33}, a_2 = -a_{22}a_{33}x_2^* + p_2x_2^*y^*(c_2p_2 - \frac{\alpha}{1+\beta y^*)}) + a_{11}x_1^*(a_{22}x_2^* - a_{33}) - a_{12}a_{21}x_1^*x_2^* + c_1p_1y^*x_1^*(\frac{kr_1}{(1+ky^*)^2} + p_1), a_3 = a_{11}x_1^*x_2^*y^*p_2(c_2p_2 - \frac{\alpha}{1+\beta y^*}) - a_{33}x_1^*x_2^*(a_{11}a_{22} - a_{12}a_{21}) - a_{12}p_1p_2c_1x_1^*x_2^*y^* - x_1^*x_2^*y^*\{\frac{kr_1}{(1+ky^*)^2} + p_1\}\{a_{21}(c_2p_2 - \frac{\alpha}{1+\beta y^*}) - c_1p_1a_{22}\}, a_{33} = \frac{\alpha x_2^*\beta}{(1+\beta y^*)^2} - h.$
From the Routh-Hurwitz criterion, we can say that E^* is locally asymptotically stable if the following conditions are fulfilled:

$$a_1 > 0, a_3 > 0 \text{ and } a_1a_2 > a_3. \tag{11}$$

It is quite complex to determine the coordinates of the positive steady state of the system. The existence of the positive steady state of the system point can be derived by showing uniform persistence of the system which in turn implies the existence [22, 23].

4.2 Uniform persistence

If there exists a compact set $E \subset D = \{(x_1, x_2, y) : x_1 > 0, x_2 > 0, y > 0\}$ where all solutions of system (1) ultimately lie, the system is said to be uniformly persistent.
The following theorem establishes uniform persistence of system (1).

Theorem 1. Suppose E_{12}, E_{13} and E_{23} exist. If there are no limit cycles in the $x_2 - y$ plane. If $\frac{a_{11}}{a_{21}} > \frac{r_1 - m}{r_2} > \frac{a_{12}}{a_{22}}, -d + c_1 p_1 \bar{x_1} + \bar{x_2}(c_2 p_2 - \alpha) > 0, r_2 > a_{21}\tilde{x_1} + p_2\tilde{y}$ and $\frac{r_1}{1+k\hat{y}} > m + a_{22}\hat{x_2} + p_1\hat{y}$ then system (1) is uniformly persistent.

Proof. We now establish the result by the concept of average Lyapunov function [22].
Define the average Lyapunov function of the form: $\rho(x) = x_1^{k_1} x_2^{k_2} y^{k_3}$ where each $k_i, i = 1,2,3$ is assumed positive. In the interior of $\mathbb{R}^3_+$, one has

$$
\begin{aligned}
\frac{1}{\rho(x)}\frac{d\rho(x)}{dt} &= \Psi(x) \\
&= \frac{k_1}{x_1}\frac{dx_1}{dt} + \frac{k_2}{x_2}\frac{dx_2}{dt} + \frac{k_3}{y}\frac{dy}{dt} \\
&= k_1\left(\frac{r_1}{1+ky} - m - a_{11}x_1 - a_{12}x_2 - p_1 y\right) + k_2(r_2 - a_{21}x_1 - a_{22}x_2 - p_2 y) \\
&+ k_3\left(-d + c_1 p_1 x_1 + c_2 p_2 x_2 - \frac{\alpha x_2}{1+\beta y} - hy\right).
\end{aligned}
$$

We have to show $\Psi(x) > 0$ for all $x \in bd\mathbb{R}^3_+$, for a suitable choice of $k_i > 0, i = 1,2,3$ to prove uniform persistence of system (1). That is one has to satisfy the following conditions corresponding to the boundary equilibria $E_0, E_1, E_2, E_{12}, E_{13}$ and E_{23} :

$$E_0 : k_1(r_1 - m) + k_2 r_2 - k_3 d > 0, \tag{12}$$
$$E_1 : k_2\left(r_2 - \frac{a_{21}(r_1 - m)}{a_{11}}\right) + k_3\left(-d + \frac{c_1 p_1 (r_1 - m)}{a_{11}}\right) > 0, \tag{13}$$
$$E_2 : k_1\left(r_1 - m - \frac{a_{12} r_2}{a_{22}}\right) + k_3\left(-d + \frac{c_2 p_2 r_2}{a_{22}}\right) > 0, \tag{14}$$
$$E_{12} : k_3(-d + c_1 p_1 \bar{x_1} + \bar{x_2}(c_2 p_2 - \alpha)) > 0, \tag{15}$$
$$E_{13} : k_2(r_2 - a_{21}\tilde{x_1} - p_2\tilde{y}) > 0, \tag{16}$$
$$E_{23} : k_1\left(\frac{r_1}{1+k\hat{y}} - m - a_{22}\hat{x_2} - p_1\hat{y}\right) > 0. \tag{17}$$

Positivity of (15)-(17) follow by the assumptions of the theorem. After suitable choice of k_1 and k_2, one can show the positivity of (12)-(14) by the assumptions of the theorem.

Remark 1. System (1) is uniformly persistent under the assumptions of the Theorem 1. Now we conclude that there exist a time T such that $x_1(t), x_2(t), y(t) > L, 0 < L < 1$, for $t > T$.

Remark 2. If there are finite number of limit cycles , then persistence condition becomes $\int_0^\tau \{\frac{r_1}{1+k\hat{\phi}(t)} - m - a_{22}\sigma\hat{\psi}(t) - p_1\hat{\phi}(t)\}dt > 0$ for each limit cycle $(\hat{\psi}(t), \hat{\phi}(t))$ in the $x_2 - y$ plane where τ is the appropriate period.

The above theorem implies uniform persistence of system (1) under certain restrictions. Further, it is shown that in [23], uniform persistence ensures the existence of a positive equilibrium point. Hence $E^* = (x_1^*, x_2^*, y^*)$ exists provided the conditions of Theorem 1 are fulfilled. Now we present global stability analysis of the positive steady state.

Theorem 2. Suppose all the conditions of Theorem 1 be satisfied. Further suppose that $h > \frac{\alpha\beta x_2^*}{1+\beta y^*}, 4c_1c_2a_{11}a_{22} > (c_1a_{12}+c_2a_{21})^2$ and $detM > 0$ where M is defined in the proof. Then E^* is globally asymptotically stable.

Proof. Consider the following positive definite function

$$V(x_1, x_2, y) = c_1(x_1 - x_1^* - x_1^* \ln\frac{x_1}{x_1^*}) + c_2(x_2 - x_2^* - x_2^* \ln\frac{x_2}{x_2^*}) + (y - y^* - y^* \ln\frac{y}{y^*}).$$

The time derivative of V along the solution of (1) is

$$\begin{aligned}\frac{dV}{dt} &= c_1(x_1 - x_1^*)(\frac{r_1}{1+ky} - m - a_{11}x_1 - a_{12}x_2 - p_1y) + c_2(x_2 - x_2^*)(r_2 - a_{21}x_1 - a_{22}x_2 - p_2y)\\ &+ (y-y^*)(-d + c_1p_1x_1 + c_2p_2x_2 - \frac{\alpha x_2}{1+\beta y} - hy).\end{aligned}$$

After some algebraic calculations, we have

$$\begin{aligned}\frac{dV}{dt} &= c_1(x_1 - x_1^*)\{\frac{r_1k(y^*-y)}{(1+ky)(1+ky^*)} - a_{11}(x_1 - x_1^*) - a_{12}(x_2 - x_2^*)\}\\ &- c_2(x_2 - x_2^*)\{a_{21}(x_1 - x_1^*) + a_{22}(x_2 - x_2^*)\} + (y-y^*)\{\alpha(\frac{x_2^*}{1+\beta y^*} - \frac{x_2}{1+\beta y}) - h(y-y^*)\}\\ &= -\frac{r_1kc_1(x_1 - x_1^*)(y-y^*)}{(1+ky)(1+ky^*)} - c_1a_{11}(x_1 - x_1^*)^2 - (c_1a_{12}+c_2a_{21})(x_1 - x_1^*)(x_2 - x_2^*)\\ &- c_2a_{22}(x_2 - x_2^*)^2 - \frac{\alpha(y-y^*)(x_2 - x_2^*)}{1+\beta y} + \frac{\alpha\beta x_2^*(y-y^*)^2}{(1+\beta y)(1+\beta y^*)} - h(y-y^*)^2\\ &\leq -[c_1a_{11}(x_1 - x_1^*)^2 - (c_1a_{12}+c_2a_{21})|x_1 - x_1^*||x_2 - x_2^*| + c_2(x_2 - x_2^*)^2\\ &- \frac{r_1kc_1|x_1 - x_1^*||y-y^*|}{1+ky^*} - \alpha|x_2 - x_2^*||y-y^*| + (h - \frac{\alpha\beta x_2^*}{1+\beta y^*})(y-y^*)^2]\\ &= -X^TMX\end{aligned}$$

where $X^T = \{|x_1 - x_1^*|, |x_2 - x_2^*|, |y - y^*|\}$ and $M = [m_{ij}]_{3\times3}$. Elements of matrix M are given by: $m_{11} = c_1a_{11}, m_{12} = m_{21} = -\frac{c_1a_{12}+c_2a_{21}}{2}, m_{13} = m_{31} = -\frac{r_1kc_1}{2(1+ky^*)}, m_{22} = c_2a_{22}, m_{33} = h - \frac{\alpha\beta x_2^*}{1+\beta y^*}, m_{23} = m_{32} = -\frac{\alpha}{2}$. Hence M is positive definite if

$$h > \frac{\alpha\beta x_2^*}{1+\beta y^*}, 4c_1c_2a_{11}a_{22} > (c_1a_{12}+c_2a_{21})^2$$

and $\det M > 0$.

4.3 Bifurcation study

Define $f(k) = a_1(k)a_2(k) - a_3(k)$.

Theorem 3. If there exist $k = k^*$ such that $(i) a_i(k^*) > 0, i = 1, 2, 3$ (ii) $f(k^*) = 0$, (iii) $f'(k^*) < 0$ then the positive equilibrium point E^* is unstable when $k > k^*$ but is stable when $k < k^*$ and a Hopf bifurcation of periodic solutions emerges at $k = k^*$.

Proof. Applying the method in [24], we observe that $f(k)$ is monotonic decreasing in the neighbourhood of $k = k^*$. As $a_i(k) > 0, i = 1,2,3, f(k) < 0$ when $k > k^*$ then E^* becomes unstable. Again, $f(k) > 0$ when $k < k^*$ then E^* becomes stable. We can show Hopf bifurcation by the result developed in [25].

Similarly we can show bifurcation of the system by considering the parameters α, a_{11}, a_{22} also. Bifurcation results relating to the above parameters are not stated here.

4.4 Stability of limit cycle

The character of the limit cycle arising through Hopf bifurcation can be determined by calculating the coefficient of curvature of the limit cycle [26]. The aim of this section is to study the stability of limit cycle of system (1). We now use the following transformations: $x_1 = u_1 + x_1^*, x_2 = u_2 + x_2^*, y = u_3 + y^*$.
Using the new variables in system (1), we obtain:

$$\begin{aligned}
\frac{du_1}{dt} &= (u_1 + x_1^*)\{\frac{r_1}{1+k(u_3+y^*)} - m - a_{11}(u_1+x_1^*) - a_{12}(u_2+x_2^*) - p_1(u_3+y^*)\},\\
\frac{du_2}{dt} &= -(u_2+x_2^*)(a_{21}u_1 + a_{22}u_2 + p_2u_3),\\
\frac{du_3}{dt} &= (u_3+y^*)\{-d + c_1p_1(u_1+x_1^*) + c_2p_2(u_2+x_2^*) - \frac{\alpha(u_2+x_2^*)}{1+\beta(u_3+y^*)} - h(u_3+y^*)
\end{aligned}$$

where the matrix of the non-linear part is given by:

$$Q = \begin{pmatrix} Q_1 \\ Q_2 \\ Q_3 \end{pmatrix} = \begin{pmatrix} \frac{u_1+x_1^*}{1+k(u_3+y^*)} - a_{11}u_1^2 - a_{12}u_1u_2 - p_1u_1u_3 \\ -a_{21}u_1u_2 - a_{22}u_2^2 - p_2u_2u_3 \\ c_1p_1u_1u_3 + c_2p_2u_2u_3 - \frac{\alpha(u_2+x_2^*)(u_3+y^*)}{1+\beta(u_3+y^*)} - hu_3^2 \end{pmatrix}.$$

From the nonlinear part above, we compute the characteristic quantities in the following:

$$\begin{aligned}
g_{20} &= \frac{1}{4}\left\{\frac{\partial^2 Q_1}{\partial u_1^2} - \frac{\partial^2 Q_1}{\partial u_2^2} + 2\frac{\partial^2 Q_2}{\partial u_1 \partial u_2} + i\left(\frac{\partial^2 Q_2}{\partial u_1^2} - \frac{\partial^2 Q_2}{\partial u_2^2} - 2\frac{\partial^2 Q_1}{\partial u_1 \partial u_2}\right)\right\}\\
&= -\frac{1}{2}\{a_{11} + a_{21} - i(a_{22} + a_{12})\},
\end{aligned}$$

$$g_{11} = \frac{1}{4}\left\{\frac{\partial^2 Q_1}{\partial u_1^2} + \frac{\partial^2 Q_1}{\partial u_2^2} + i\left(\frac{\partial^2 Q_2}{\partial u_1^2} + \frac{\partial^2 Q_2}{\partial u_2^2}\right)\right\} = -\frac{1}{2}(a_{11} + ia_{22}),$$

$$G_{110} = \frac{1}{2}\left\{\frac{\partial^2 Q_1}{\partial u_1 \partial u_2} + \frac{\partial^2 Q_2}{\partial u_2 \partial u_3} + i\left(\frac{\partial^2 Q_2}{\partial u_1 \partial u_3} - \frac{\partial^2 Q_1}{\partial u_2 \partial u_3}\right)\right\} = -\left\{\frac{k}{(1+k(u_3+y^*))^2} + p_1 + p_2\right\},$$

$$G_{101} = \frac{1}{2}\left\{\frac{\partial^2 Q_1}{\partial u_1 \partial u_2} - \frac{\partial^2 Q_2}{\partial u_2 \partial u_3} + i\left(\frac{\partial^2 Q_2}{\partial u_1 \partial u_3} + \frac{\partial^2 Q_1}{\partial u_2 \partial u_3}\right)\right\} = -\left\{\frac{k}{(1+k(u_3+y^*))^2} - p_1 + p_2\right\},$$

$$W_{11} = -\frac{1}{4(\lambda_3(a_1(k^*))}\left(\frac{\partial^2 Q_3}{\partial u_1^2} + \frac{\partial^2 Q_3}{\partial u_2^2}\right) = 0,$$

$$W_{20} = -\frac{1}{4(4i - \lambda_3(a_1(k^*)))}\left(\frac{\partial^2 Q_3}{\partial u_1^2} - \frac{\partial^2 Q_3}{\partial u_2^2} - 2i\frac{\partial^2 Q_3}{\partial u_1 \partial u_2}\right) = 0,$$

$$G_{21} = \frac{1}{8}\left\{\frac{\partial^3 Q_1}{\partial u_1^3} + \frac{\partial^3 Q_1}{\partial u_1 \partial u_2^2} + \frac{\partial^3 Q_2}{\partial u_2^3} + \frac{\partial^3 Q_2}{\partial u_1^2 \partial u_2} + i\left(\frac{\partial^3 Q_2}{\partial u_1^3} + \frac{\partial^3 Q_2}{\partial u_1 \partial u_2^2} - \frac{\partial^3 Q_1}{\partial u_1^2 \partial u_2} - \frac{\partial^3 Q_1}{\partial u_2^3}\right)\right\} = 0.$$

Then the coefficient of curvature of limit cycle of system (1) is

$$\sigma_1 = \mathrm{Re}\left\{\frac{g_{20}g_{11}}{4}i + G_{110}W_{11} + \frac{G_{21} + G_{101}W_{20}}{2}\right\} = \frac{1}{16}(a_{11}a_{12} - a_{22}a_{21}).$$

Thus, we note that if the coefficient of curvature $\sigma_1 < 0$ then the limit cycle of system (1) is stable otherwise unstable.

5 Numerical observations

In this section, we demonstrate our finding through numerical simulations. The reason for such inquiry is to find out the effect of varying the parameter values and validate our obtained results. The parameter values are hypothetically chosen which do not necessarily have biological significance or apply to specific population.

We mainly examine the effect of fear, anti-predator behaviour and intra-specific competition rate among the prey species on the population dynamics. For this purpose, we select the parameters as follows:

$$r_1 = 4,\ r_2 = 2,\ a_{11} = 0.1,\ a_{12} = 0.2,\ a_{21} = 0.01,\ a_{22} = 0.1,\ p_1 = 2,\ p_2 = 2,\ d = 1, \tag{18}$$

$$c_1 = 1,\ c_2 = 1.5,\ \alpha = 4,\ \beta = 1,\ h = 0.25,\ m = 0.2. \tag{19}$$

Now we state numerical results. Fig. 1 describes the phase portrait as well as time series plots of system (1) for $k = 0.76$. It shows stable behaviour and the solutions of system (1) converges to the equilibrium point $E^*(0.6118, 0.0261, 1.0042)$. Choosing $k = 0.9$, we observe that the system induces oscillatory behaviour (see Fig. 2).

Keeping $k = 0.9$, we change the value of a_{22} in (18) and (19) from 0.1 to 1 to find out the effect of intra-specific competition coefficient of prey x_2. In this case, the oscillatory behaviour observed in Fig. 2 disappears and the system becomes stable (see Fig. 3). Now we check the effect of birth rate of first prey species. For this, we select $r_1 = 3,\ a_{22} = 1,\ k = 0.9$ where other parameter values are given in (18) and (19). Here we again observe oscillatory behaviour of the system (see Fig. 4).

To obtain bifurcation diagram, we select $r_1 = 3,\ k = 0.638,\ a_{22} = 1$ and the other parameter values are same as in (18) and (19), it is easy to see that system (1) has an equilibrium point $E^* = (0.5397, 0.3883, 0.6058)$ and all the conditions of Theorem 3 are satisfied.

Thus, Hopf bifurcation occurs at $k = 0.638$. The corresponding bifurcation diagram with respect to the parameter k is shown in Fig. 5. Now we choose $k = 0.9$ and $\alpha = 0.336$ and other parameter values are same as in Fig.5, we observe Hopf bifurcation around the equilibrium point $E^* = (0.1664, 0.3352, 0.7358)$ at $\alpha = 0.336$. The corresponding bifurcation diagram with respect to the parameter α is displayed in Fig.6.

6 Discussion

In this paper, we have studied the dynamical behaviour of two competing prey-one predator model where the birth rate of one of the competing prey (weak competitor designated as x_1) is affected by the fear of predators while the other prey (strong competitor) shows anti-predator behaviour. The model also considers intra-specific competition among the predator population which exists in nature due to limited resources. Predation process follows Holling type I functional response.

For the biological validity of the model, we have shown positivity and boundedness of solutions. The existence criteria of all viable steady states are derived. By isocline method, we have indicated the existence conditions of positive steady state. It is shown that the system

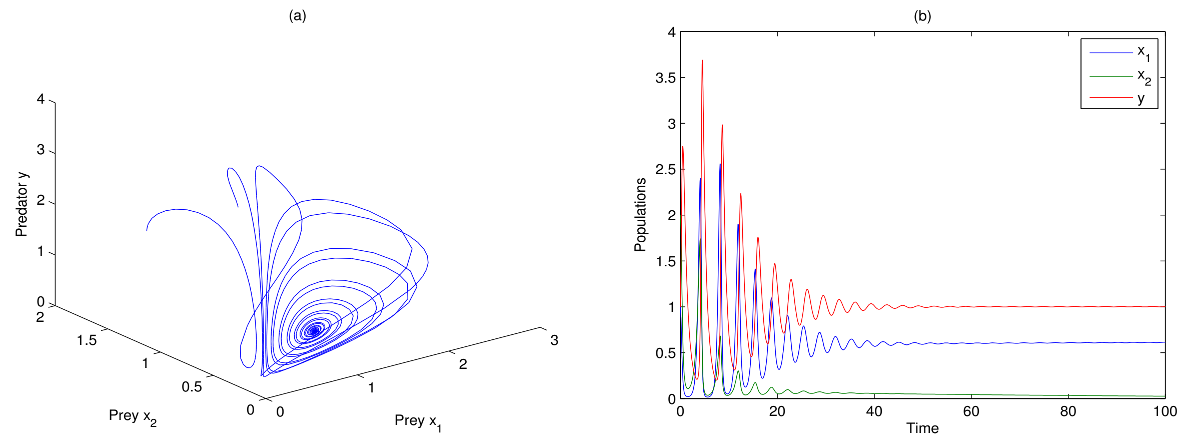

Figure 1: Phase portrait along time series plots of system (1) with parameter values $r_1 = 4, r_2 = 2, a_{11} = 0.1, a_{12} = 0.2, a_{21} = 0.01, a_{22} = 0.1, p_1 = 2, p_2 = 2, k = 0.76, d = 1, c_1 = 1, c_2 = 1.5, \alpha = 4, \beta = 1, h = 0.25, m = 0.2$ and initial point (2, 1, 2) .

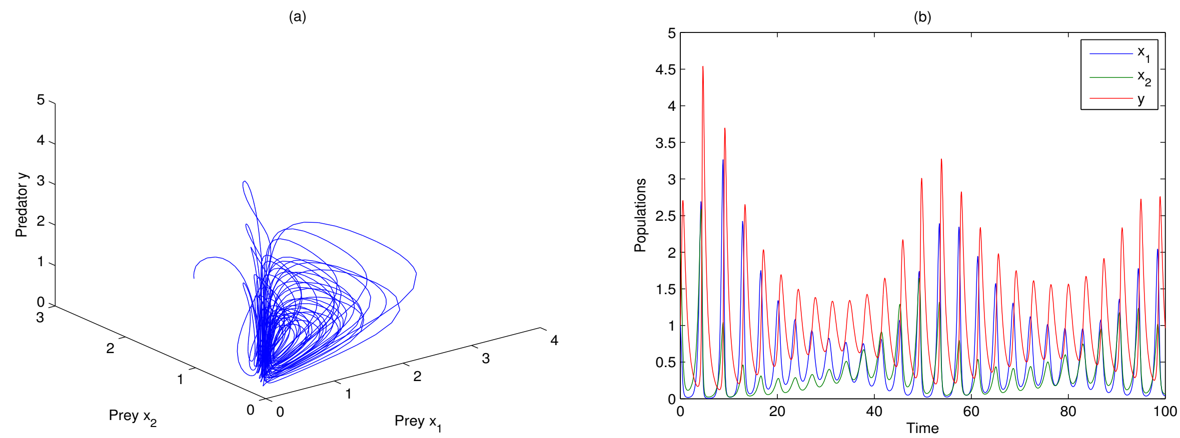

Figure 2: Phase portrait along time series plots of system (1) with parameter values $r_1 = 4, r_2 = 2, a_{11} = 0.1, a_{12} = 0.2, a_{21} = 0.01, a_{22} = 0.1, p_1 = 2, p_2 = 2, k = 0.9, d = 1, c_1 = 1, c_2 = 1.5, \alpha = 4, \beta = 1, h = 0.25, m = 0.2$ and initial point (2, 1, 2) .

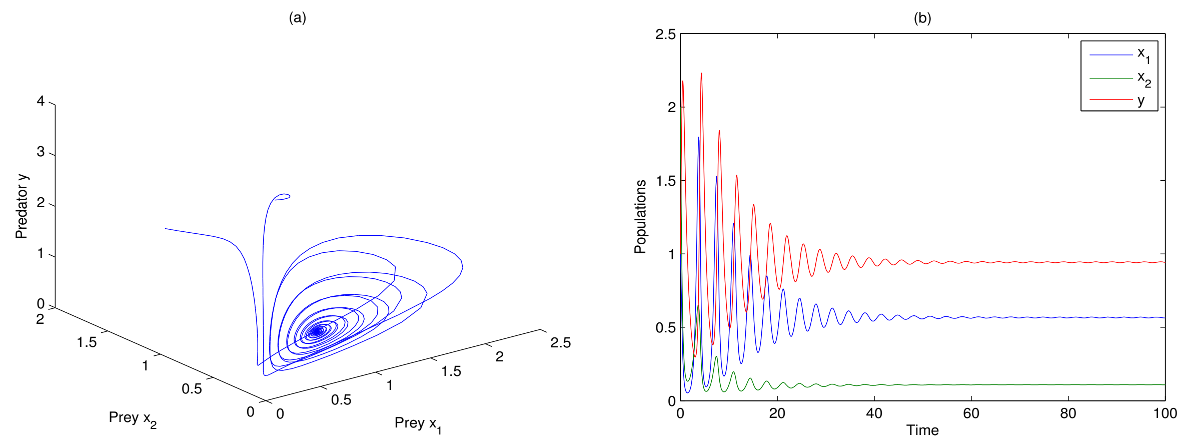

Figure 3: Phase portrait along time series plots of system (1) with parameter values $r_1 = 4, r_2 = 2, a_{11} = 0.1, a_{12} = 0.2, a_{21} = 0.01, a_{22} = 1, p_1 = 2, p_2 = 2, k = 0.9, d = 1, c_1 = 1, c_2 = 1.5, \alpha = 2, \beta = 1, h = 0.25, m = 0.2$ and initial point (2, 1, 2) .

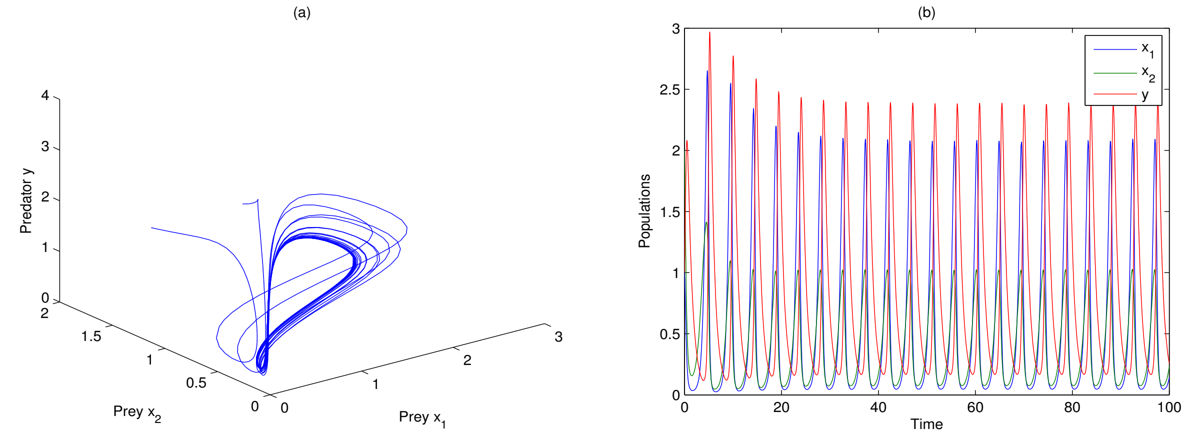

Figure 4: Phase portrait along time series plots of system (1) with parameter values $r_1 = 3, r_2 = 2, a_{11} = 0.1, a_{12} = 0.2, a_{21} = 0.01, a_{22} = 1, p_1 = 2, p_2 = 2, k = 0.9, d = 1, c_1 = 1, c_2 = 1.5, \alpha = 4, \beta = 1, h = 0.25, m = 0.2$ and initial point (2, 1, 2) .

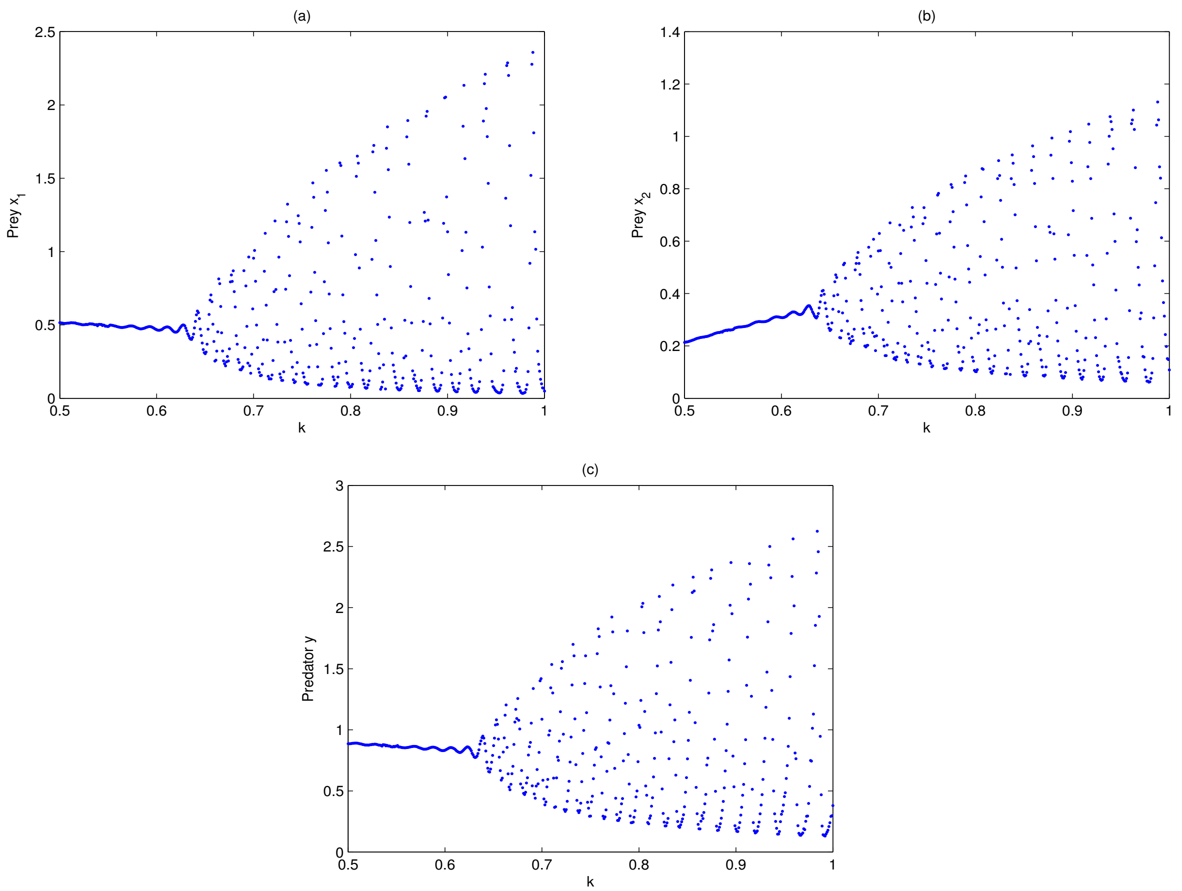

Figure 5: Bifurcation diagram for x_1, x_2 and y respectively with respect to the parameter k for the system (1) with parameter values $r_1 = 3, r_2 = 2, a_{11} = 0.1, a_{12} = 0.2, a_{21} = 0.01, a_{22} = 1, p_1 = 2, p_2 = 2, d = 1, c_1 = 1, c_2 = 1.5, \alpha = 4, \beta = 1, h = 0.25, m = 0.2$ and initial point (.5, .5, .5) .

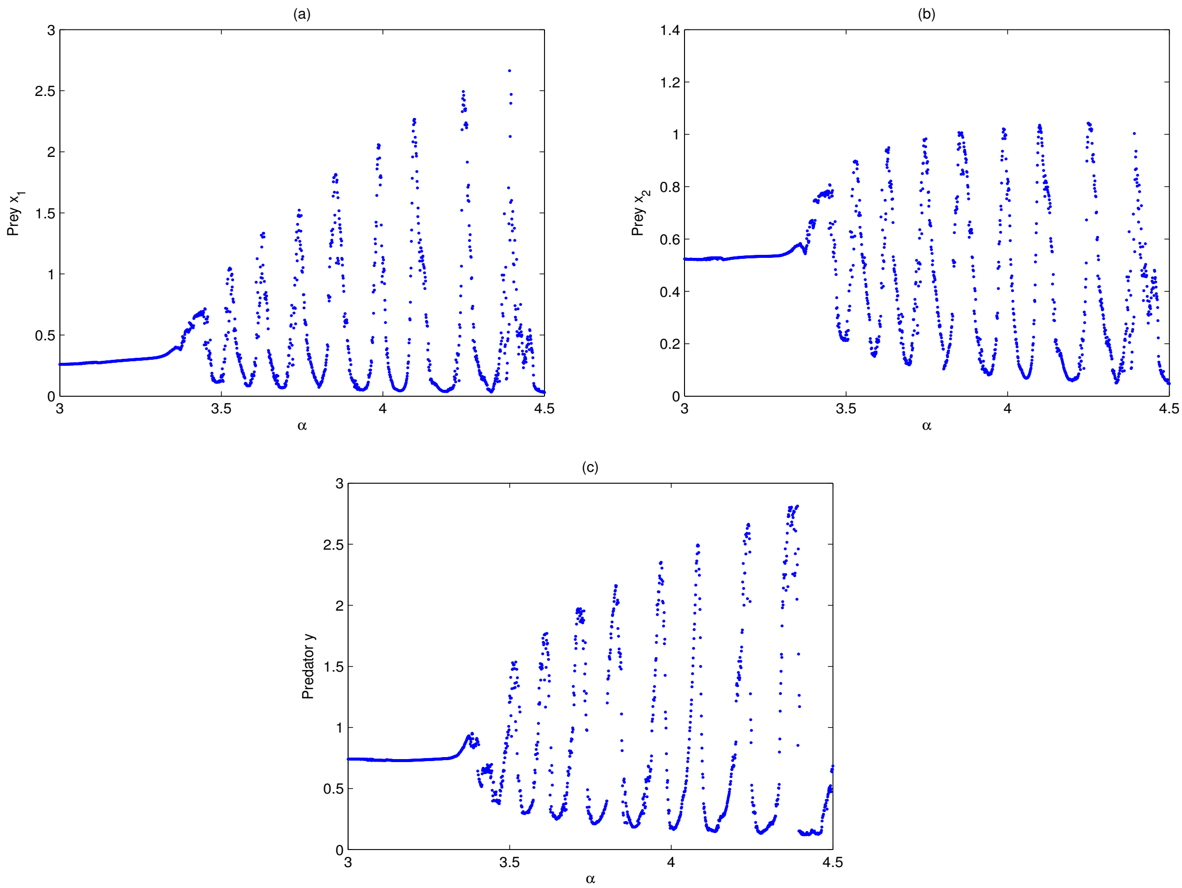

Figure 6: Bifurcation diagram for x_1, x_2 and y respectively with respect to the parameter α for the system (1) with parameter values $r_1 = 3, r_2 = 2, a_{11} = 0.1, a_{12} = 0.2, a_{21} = 0.01, a_{22} = 1, p_1 = 2, p_2 = 2, d = 1, c_1 = 1, c_2 = 1.5, k = 0.9, \beta = 1, h = 0.25, m = 0.2$ and initial point (.5, .5, .5) .

cannot collapse for any value of the parameter as the origin is always unstable. We have developed criteria for uniform persistence of the system which ensures the long term survival of all the populations. Furthermore, this criterion guarantees the existence of positive steady state. Local stability of all the boundary steady states is examined. By constructing a suitable Lyapunov function, it is possible to show the global stability of the positive steady state under certain restrictions on the system parameters, We have identified four parameters that give rise to Hopf bifurcation. In [5], the authors investigated the effect of fear in their model and remarked that oscillatory behaviour is not possible for Holling type I predator functional response. This is not true in our model.

It is observed in Fig. 1, that the effect of fear has a stabilizing effect on the system, whereas it decreases the population density of the prey x_1. The reason behind that higher level of fear, prey population becomes more vigilant and migrate to adverse regions at the cost of high death rate. There is no impact of fear on prey species x_2.

The establishment of ecological networks can promote biodiversity conservation. Protected area may be developed so that movement of species can be controlled.

The novelty of our work is the inclusion of anti-predator behaviour, intra-specific competition among the predators and competition between the prey population. In case of Holling type I predator functional response, we have observed oscillatory behaviour of the system when the fear factor $k = 0.9$ (see Fig. 2). This oscillatory behaviour can be controlled by increasing the intra-specific competition among the strong competing prey species (see Fig. 3). If we decrease the birth rate of weak prey species , then system induces a stable oscillation (see Fig. 4). Bifurcation diagram (Fig.5) shows that the system remains stable as long as $k < 0.638$. Fig. 6 depicts that the system stays stable when $\alpha < 0.336$. From bifurcation diagram one can conclude that the system emerges an oscillation when level of fear and anti-predator behaviour rate are increased after some critical value.

There are some limitations of our model. In this work, we have studied the effect of fear on the birth rate of prey species. It is justifiable to consider the fear effect on the other interaction terms such as intra-specific, inter-specific competition of the prey and predation rate etc. For mathematical simplicity, we have taken predator functional response as Holling type I. It would be better to study the other response function to compare dynamical behaviour in the present study.

7 Declarations

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Competing Interests

I declare that there are no competing interests.

Ethical Approval

Not applicable.

Authors' Contributions

Not applicable.

Availability Data and Materials

Not applicable.

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