EJAM
www.ejamjournal.com
Electronic Journal of Applied Mathematics
Research article
Mersin University Publishing
DOI: 10.61383/ejam.20242383
2024, Vol. 2, No. 3, 51–62

Study of Left Rank-Preserving Complex Matrices

Zehui Zhao[1]

1. School of Mathematics and Computer Sciences, Gannan Normal University, Ganzhou 431000, P.R. China

Abstract

In this paper, we introduce the notions of left rank-preserving and right rank-preserving matrices, and establish three equivalent conditions of a left rank-preserving complex matrix. Furthermore, we obtain a kind of classification for all $m \times n$ left rank-preserving complex matrices if $m \leq 3$ or $n \leq 3$.

Keywords: rank of complex matrices, left (right) rank-preserving matrix, Binet-Cauchy's formula
2020 Mathematics Subject Classification: 15A03, 15A04
Article history: Received 10 Feb 2024; Accepted 15 Aug 2024; Online 13 Sep 2024

1 Introduction

Matrix preserver problems mainly concern the characterization of maps that preserve specific invariants (e.g. rank functions, matrix relations, subset structures, etc.) between distinct matrix spaces V_1 and V_2 (e.g. see [1, 2, 3, 4, 5, 6, 7, 12, 24]). Given its broad practical applications in fields such as differential equations, system control, quantum mechanics, and mathematical statistics, this problem has emerged as a prominent research topic in matrix theory. In 1897, Frobenius [8] characterized linear transformations preserving the determinant. Since that time, much research has been published on linear preserver problems. See [9, 10, 11, 13, 14, 15, 16, 17, 18, 19, 20, 21, 22, 23] for an excellent survey.

Let $M_{m,n}(\mathbb{F})$ denote the set of all $m \times n$ matrices over the algebraically closed field $\mathbb{F}$, and let $\mathrm{rank}(A)$ denote the rank of A. Define the maps $L_{\mathbb{F}} : M_{m,n}(\mathbb{F}) \to M_{n,n}(\mathbb{F})$ and $R_{\mathbb{F}} : M_{m,n}(\mathbb{F}) \to M_{m,m}(\mathbb{F})$ by

$$L_{\mathbb{F}}(A) = A^\top A, \;\; R_{\mathbb{F}}(A) = AA^\top \;\text{ for all }\; A \in M_{m,n}(\mathbb{F}),$$

where $A^\top$ denotes the transposed matrix of A. The map $L_{\mathbb{F}} : M_{m,n}(F) \to M_{n,n}(\mathbb{F})$ is left rank-preserving if $\mathrm{rank}(L_{\mathbb{F}}(A)) = \mathrm{rank}(A)$, i.e., $\mathrm{rank}(A^\top A) = \mathrm{rank}(A)$, so, the matrix A is also called *left rank-preserving matrix* over field $\mathbb{F}$. And the map $R_{\mathbb{F}} : M_{m,n}(\mathbb{F}) \to M_{m,m}(\mathbb{F})$ is right rank-preserving if $\mathrm{rank}(R_{\mathbb{F}}(A)) = \mathrm{rank}(A)$, i.e., $\mathrm{rank}(AA^\top) = \mathrm{rank}(A)$, so, the matrix A is also called *right rank-preserving matrix* over field $\mathbb{F}$. A matrix A over a field $\mathbb{F}$ is called *rank-preserving* if it is both left rank-preserving and right rank-preserving. When the field $\mathbb{F}$ is the real

Contact: Zehui Zhao zehuizhao2025@126.com

field $\mathbb{R}$, it is evident that the maps $L_{\mathbb{R}} : M_{m,n}(\mathbb{R}) \to M_{n,n}(\mathbb{R})$ and $R_{\mathbb{R}} : M_{m,n}(\mathbb{R}) \to M_{m,m}(\mathbb{R})$ are left rank-preserving and right rank-preserving, respectively, namely, (e.g. see [25])

$$\operatorname{rank}(A^\top A) = \operatorname{rank}(A) = \operatorname{rank}(AA^\top) \text{ for all } A \in M_{m,n}(\mathbb{R}).$$

While the field $\mathbb{F}$ is the complex field $\mathbb{C}$, this assertion is not true. For example, let $A = \begin{pmatrix} 1 & i \\ i & -1 \end{pmatrix}$, we have $L_{\mathbb{C}}(A) = A^\top A$ is zero matrix $\mathbf{0}_{2\times 2}$. Obviously, we have $\operatorname{rank}(A^\top A) = 0$, but $\operatorname{rank}(A) = 1 \neq 0$. Hence, a quite relevant question arises: for any complex matrix $A \in M_{m,n}(\mathbb{C})$, what is the condition that satisfies the following equation

$$\operatorname{rank}(A^\top A) = \operatorname{rank}(A)?$$

After searching the literature, we found no references to this issue. The main purpose of this article is to discuss the map $L_{\mathbb{F}}$ rank-preserving problem. In this paper, we obtain three equivalent conditions of a left rank-preserving complex matrix. Moreover, we give a kind of classification for all $m \times n$ left rank-preserving complex matrices if $m \leq 3$ or $n \leq 3$.

The paper is organized as follows. Section 2 gives a brief preliminary discussion. In section 3, we obtain three equivalent conditions of a left rank-preserving complex matrix and a kind of classification for all $m \times n$ left rank-preserving complex matrices if $m \leq 3$ or $n \leq 3$. Some remarks and comments are also given in this section.

2 Preliminaries

Given an $m \times n$ matrix $A = (a_{ij})_{m\times n}$ with entries a_{ij}, a *minor* of A is the determinant of a smaller matrix formed from its entries by selecting only some of the rows and columns. Let $K = \{k_1, k_2 \cdots, k_p\}$ and $L = \{l_1, l_2, \cdots, l_p\}$ be subsets of $\{1, 2, \cdots, m\}$ and $\{1, 2, \cdots, n\}$, respectively. The indices are chosen such that $k_1 < k_2 < \cdots < k_p$ and $l_1 < l_2 < \cdots < l_p$. The p-th order minor defined by K and L is the following determinant

$$A\begin{pmatrix} k_1 & k_2 & \cdots & k_p \\ l_1 & l_2 & \cdots & l_p \end{pmatrix} = \begin{vmatrix} a_{k_1 l_1} & a_{k_1 l_2} & \cdots & a_{k_1 l_p} \\ a_{k_2 l_1} & a_{k_2 l_2} & \cdots & a_{k_2 l_p} \\ \vdots & \vdots & \ddots & \vdots \\ a_{k_p l_1} & a_{k_p l_2} & \cdots & a_{k_p l_p} \end{vmatrix}.$$

If p exceeds either m or n, then the minor is automatically zero. When $p = m = n$, the minor is simply the determinant of the matrix. If $K = L$, then the minor is called principal.

Lemma 2.1. *(Cauchy-Binet formula) (eg. see [25]) Let A be an $m \times n$ matrix and B an $n \times m$ matrix. Then the determinant of their product $C = BA$ can be written as a sum of products of minors of A and B:*

$$|C| = \sum_{1\leq i_1 < i_2 < \cdots < i_m \leq n} B\begin{pmatrix} 1 & 2 & \cdots & n \\ i_1 & i_2 & \cdots & i_n \end{pmatrix} A\begin{pmatrix} i_1 & i_2 & \cdots & i_n \\ 1 & 2 & \cdots & n \end{pmatrix}.$$

Basically, the sum is over the maximal (n-th order) minors of A and B. If $n > m$, then neither A nor B have minors of rank n, so $|C| = 0$. If $m = n$, this formula reduces to the usual multiplicativity of determinants $|C| = |AB| = |A||B|$.

Suppose $B = A^\top$, then the determinant $B\begin{pmatrix} 1 & 2 & \cdots & n \\ i_1 & i_2 & \cdots & i_n \end{pmatrix}$ is the transposed determinant of $A\begin{pmatrix} i_1 & i_2 & \cdots & i_n \\ 1 & 2 & \cdots & n \end{pmatrix}$. Since every determinant must be equal to its transposed determinant, and if $n \leq m$, we have

$$|A^\top A| = \sum_{1\leq k_1<k_2<\cdots<k_m\leq n} \left(A \begin{pmatrix} 1 & 2 & \cdots & n \\ i_1 & i_2 & \cdots & i_n \end{pmatrix} \right)^2 .$$

Remark 2.2. For any complex matrix $A \in M_{m,n}(\mathbb{F})$, it holds over $\mathbb{C}$ that A is left rank-preserving if and only if its transpose $A^\top$ is right rank-preserving. The introductory example in this article demonstrates the existence of complex matrices that are not left rank-preserving. Interestingly, a matrix that is left rank-preserving over $\mathbb{C}$ need not be right rank-preserving over $\mathbb{C}$, and conversely, a right rank-preserving matrix may fail to be left rank-preserving. For instance, there exists a matrix $A = (1, i)$ that is left rank-preserving but not right rank-preserving; conversely, another matrix $B = \begin{pmatrix} 1 \\ i \end{pmatrix}$ may be right rank-preserving without being left rank-preserving.

3 Left rank-preserving complex matrices

In this section, we obtain three equivalent conditions of a left rank-preserving complex matrix and give a kind of classification for all $m \times n$ left rank-preserving complex matrices if $m \leq 3$ or $n \leq 3$.

For simplicity, let

$$\mathcal{LA}_{m,n} = \{A \in M_{m,n}(\mathbb{C}) : \operatorname{rank}(A^\top A) = \operatorname{rank}(A)\}$$

and

$$\mathcal{RA}_{m,n} = \{A \in M_{m,n}(\mathbb{C}) : \operatorname{rank}(AA^\top) = \operatorname{rank}(A)\}.$$

Theorem 3.1. *Let A be an $m \times n$ nonzero matrix of rank r over the field $\mathbb{C}$. Then, A is left rank-preserving if and only if there exists a submatrix A_1 fromed by selecting r columns from A, such that $|A_1^\top A_1| \neq 0$.*

Proof. Since $\operatorname{rank}(A) = r > 0$, we show that there are an inverse matrix $P \in M_{n,n}(\mathbb{C})$ and a matrix $A_1 \in M_{m,r}(\mathbb{C})$, such that

$$AP = (A_1, \mathbf{0}), \text{ and } \operatorname{rank}(A_1) = r.$$

Obviously, $P^\top$ is also invertible, and

$$P^\top(A^\top A)P = (AP)^\top(AP) = \begin{pmatrix} A_1^\top \\ \mathbf{0} \end{pmatrix} (A_1, \mathbf{0}) = \begin{pmatrix} A_1^\top A_1 & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{pmatrix}. \tag{1}$$

Moreover, we have

$$\operatorname{rank}(AP) = \operatorname{rank}(A), \text{ and } \operatorname{rank}(P^\top A^\top AP) = \operatorname{rank}(A^\top A).$$

On the other hand, from $AP = (A_1, \mathbf{0})$ and Eq.(1), it is easy to see that

$$\operatorname{rank}(AP) = \operatorname{rank}(A_1), \text{ and } \operatorname{rank}(P^\top A^\top AP) = \operatorname{rank}(A_1^\top A_1).$$

Then we get $\operatorname{rank}(A) = \operatorname{rank}(A_1)$ and $\operatorname{rank}(A^\top A) = \operatorname{rank}(A_1^\top A_1)$. Therefore,

$$\operatorname{rank}(A^\top A) = \operatorname{rank}(A) \Leftrightarrow \operatorname{rank}(A_1^\top A_1) = \operatorname{rank}(A_1).$$

Note that $A_1^\top A_1$ is a square matrix of order r, and the matrix A_1 has full column rank, i.e., $\operatorname{rank}(A_1) = r$. Then we get

$$\operatorname{rank}(A_1^\top A_1) = \operatorname{rank}(A_1) \Leftrightarrow |A_1^\top A_1| \neq 0.$$

The proof of Theorem 3.1 is completed. □

Base on Theorem 3.1, we show that

$$\operatorname{rank}(A^\top A) = \operatorname{rank}(A) \ \Leftrightarrow\ |A_1^\top A_1| \neq 0.$$

Let $A = (a_{ij})$, and let A_1 be the submatrix formed by the j_1-th, j_2-th, $\ldots, j_r$-th columns of A. Then

$$A_1^\top A_1 = \begin{pmatrix} a_{1j_1} & a_{2j_1} & \cdots & a_{mj_1} \\ a_{1j_2} & a_{2j_2} & \cdots & a_{mj_2} \\ \vdots & \vdots & \ddots & \vdots \\ a_{1j_r} & a_{2j_r} & \cdots & a_{mj_r} \end{pmatrix} \begin{pmatrix} a_{1j_1} & a_{1j_2} & \cdots & a_{1j_r} \\ a_{2j_1} & a_{2j_2} & \cdots & a_{2j_r} \\ \vdots & \vdots & \ddots & \vdots \\ a_{mj_1} & a_{mj_2} & \cdots & a_{mj_r} \end{pmatrix},$$

namely

$$A_1^\top A_1 = \begin{pmatrix} \sum\limits_{i=1}^m a_{ij_1}^2 & \sum\limits_{i=1}^m a_{ij_1}a_{ij_2} & \cdots & \sum\limits_{i=1}^m a_{ij_1}a_{ij_r} \\ \sum\limits_{i=1}^m a_{ij_2}a_{ij_1} & \sum\limits_{i=1}^m a_{ij_2}^2 & \cdots & \sum\limits_{i=1}^m a_{ij_2}a_{ij_r} \\ \vdots & \vdots & \ddots & \vdots \\ \sum\limits_{i=1}^m a_{ij_r}a_{ij_1} & \sum\limits_{i=1}^m a_{ij_r}a_{ij_2} & \cdots & \sum\limits_{i=1}^m a_{ij_r}^2 \end{pmatrix}.$$

Thus, when $r = 1$, we get

$$|A_1^\top A_1| = a_{1j_1}^2 + a_{2j_1}^2 + \cdots + a_{mj_1}^2.$$

And when $r = 2$, we obtain

$$|A_1' A_1| = \left(\sum_{i=1}^m a_{ij_1}^2\right)\left(\sum_{i=1}^m a_{ij_2}^2\right) - \left(\sum_{i=1}^m a_{ij_1}a_{ij_2}\right)^2.$$

From theorem 3.1, we get the following Proposition.

Proposition 3.2. *Let $A = (a_{ij})$ be an $m \times n$ matrix over the complex field $\mathbb{C}$.*

(1) Let A be an $m \times n$ matrix with $\operatorname{rank}(A) = 1$. Then A is left rank-preserving if and only if there exists a column index j ($1 \leq j \leq n$) such that the squared norm of the j-th column is non-zero, i.e.,

$$a_{1j}^2 + a_{2j}^2 + \cdots + a_{mj}^2 \neq 0.$$

(2) Let A be an $m \times n$ matrix with $\operatorname{rank}(A) = 2$. Then A is left rank-preserving if and only if there exist two distinct column indices j_1 and j_2 ($1 \leq j_1, j_2 \leq n$) such that the corresponding columns of A satisfy the following equation:

$$\left(\sum_{i=1}^m a_{ij_1}^2\right)\left(\sum_{i=1}^m a_{ij_2}^2\right) - \left(\sum_{i=1}^m a_{ij_1}a_{ij_2}\right)^2 \neq 0.$$

Let A_1 be an $m \times (m-1)$ column full-rank matrix over the complex field. Then there exist $m-1$ row vectors in A_1 that are linearly independent, while the remaining 1 row vector is a linear combination of these $m-1$ row vectors. Let $\alpha_1, \alpha_2, \ldots, \alpha_m$ denote the set of row vectors of A_1. Without loss of generality, assume the first $m-1$ row vectors are linearly independent. Then there exist $m-1$ complex numbers $k_1, k_2, \ldots, k_{m-1}$ such that

$$\alpha_m = k_1\alpha_1 + k_2\alpha_2 + \cdots + k_{m-1}\alpha_{m-1}. \tag{2}$$

Suppose $\alpha_i = (a_{i1}, a_{i2}, \ldots, a_{i\,m-1}),\ i = 1, 2, \ldots, m,$ and let M_0 be the minor of order $m-1$ formed by the first r rows of A_1, then

$$M_0 = \begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1\,m-1} \\ a_{21} & a_{22} & \cdots & a_{2\,m-1} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m-1\,1} & a_{m-1\,2} & \cdots & a_{m-1\,m-1} \end{vmatrix} \neq 0.$$

Let M_i (for $i = 1, 2, \ldots, m-1$) be the minor of order $m-1$ formed by deleting the i-th row of A_1 and retaining the remaining $m-1$ rows, i.e.,

$$M_i = \begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1\,m-1} \\ a_{21} & a_{22} & \cdots & a_{2\,m-1} \\ \vdots & \vdots & \ddots & \vdots \\ a_{i-1\,1} & a_{i-1\,2} & \cdots & a_{i-1\,m-1} \\ a_{i+1\,1} & a_{i+1\,2} & \cdots & a_{i+1\,m-1} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{m\,m-1} \end{vmatrix}, i = 1, 2, \ldots, m-1.$$

From Eq.(2), apply elementary row operations to the matrix

$$M_1 = \begin{vmatrix} a_{21} & a_{22} & \cdots & a_{2\,m-1} \\ a_{31} & a_{32} & \cdots & a_{3\,m-1} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{m\,m-1} \end{vmatrix},$$

i.e., subtract k_{i+1} times the i-th row from the last row of M_1 (for $i = 1, 2, \ldots, m-1$). The resulting determinant has every element in its last row equal to k_1 times the corresponding elements of the first row of A_1. Factor out the common factor k_1, then successively swap the last row with the row above it until it reaches the first position. This row swapping process is performed $m-2$ times, ultimately yielding the determinant M_0, we have

$$M_1 = (-1)^{m-2} k_1 M_0.$$

Similarly, we obtain

$$M_2 = (-1)^{m-3} k_2 M_0,\ M_3 = (-1)^{m-4} k_3 M_0,\ \ldots, M_{m-1} = (-1)^0 k_{m-1} M_0.$$

According to the lemma 2.1, we have

$$|A_1^\top A_1| = M_0^2 + M_1^2 + M_2^2 + \cdots + M_{m-1}^2,$$

and

$$|A_1^\top A_1| = (1 + k_1^2 + k_2^2 + \cdots + k_{m-1}^2)\, M_0^2.$$

Notice that $M_0 \neq 0$, then we have

$$|A_1^\top A_1| \neq 0 \iff 1 + k_1^2 + k_2^2 + \cdots + k_{m-1}^2 \neq 0.$$

Let A_1 be an $m \times m$ matrix of full rank, then both A_1 and $A_1^\top$ are invertible. Since the product of invertible matrices remains invertible, the matrix $A_1^\top A_1$ is consequently invertible with $|A_1^\top A_1| \neq 0$. By Theorem 3.1, we obtain the following result.

Proposition 3.3. *Let $A = (a_{ij})$ be an $m \times n$ matrix over $\mathbb{C}$.*

(1) Suppose A is an $m \times n$ matrix where $m-1$ of its row vectors $\alpha_1, ..., \alpha_{m-1}$ are linearly independent, and the remaining one row vector can be expressed as $\sum_{i=1}^{m-1} k_i\alpha_i$ with coefficients $k_1, \ldots, k_{m-1}$. Then A is left rank-preserving if and only if

$$1 + k_1^2 + k_2^2 + \cdots + k_m^2 \neq 0.$$

(2) If A has full row rank ($rank(A) = m$), then it is left rank-preserving.

By applying Proposition 3.2 and Proposition 3.3, we can establish a kind of classification of all $m \times n$ left rank-preserving complex matrices when either $m \leq 3$ or $n \leq 3$. In fact, when $m = 1$:

- If A is the zero matrix, it is trivially left rank-preserving.
- Otherwise, A has full row rank. By Proposition 3.3, such matrices are also left rank-preserving.

When $n = 1$, Proposition 3.2 implies that a left rank-preserving matrix A must be either:

- The $m \times 1$ zero matrix $\mathbf{O}_{m\times 1}$, or
- A non-zero $m \times 1$ vector $(a_1, a_2, \cdots, a_m)^\top$ satisfying $\sum_{i=1}^{m} |a_i|^2 \neq 0$.

This characterization directly yields the following corollary.

Corollary 3.4. *(1) Every $1 \times n$ complex matrix with full row rank is left rank-preserving, i.e, $\mathcal{LA}_{1,n} = M_{1,n}(\mathbb{C})$.*

(2) The set of $m \times 1$ left rank-preserving complex matrices can be divided into two types, one kind is a zero matrix $\mathbf{O_{m\times 1}}$, another kind is the form such as the complex matrices $(a_1, a_2, \ldots, a_m)^\top$, where $a_1^2 + a_2^2 + \cdots + a_m^2 \neq 0$, namely,

$$\mathcal{LA}_{m,1} = \mathbf{O_{m\times 1}} \cup \left\{ (a_1, a_2, \ldots, a_m)^\top \in M_{m,1}(\mathbb{C}) : a_1^2 + a_2^2 + \cdots + a_m^2 \neq 0 \right\}.$$

A similar discussion leads to the following four corollaries.

Corollary 3.5. *The set of $2 \times n$ left rank-preserving complex matrices can be divided into four types, they are*

(1) $2 \times n$ zero matrix $\mathbf{O_{2\times n}}$;

(2) $\mathcal{N}_1 = \left\{ \begin{pmatrix} 0 & 0 & \cdots & 0 \\ a_1 & a_2 & \ldots & a_n \end{pmatrix} \in M_{2,n}(\mathbb{C}) : a_1^2 + a_2^2 + \cdots + a_n^2 \neq 0 \right\}$;

(3) $\mathcal{N}_2 = \left\{ \begin{pmatrix} a_1 & a_2 & \cdots & a_n \\ ka_1 & ka_2 & \ldots & ka_n \end{pmatrix} \in M_{2,n}(\mathbb{C}) : 1 + k^2 \neq 0,\ a_1^2 + a_2^2 + \cdots + a_n^2 \neq 0 \right\}$;

(4) $\mathcal{N}_3 = \{ A \in M_{2,n}(\mathbb{C}) : rank(A) = 2 \}$, *namely:*

$$\mathcal{LA}_{2,n} = \mathbf{O_{2\times n}} \cup \mathcal{N}_1 \cup \mathcal{N}_2 \cup \mathcal{N}_3.$$

Corollary 3.6. *The set of $m \times 2$ left rank-preserving complex matrices can be divided into four types, they are*

(1) $m \times 2$ zero matrix $\mathbf{O_{m\times 2}}$;

(2) $\mathcal{M}_1 = \left\{ \begin{pmatrix} 0 & a_1 \\ 0 & a_2 \\ \vdots & \vdots \\ 0 & a_m \end{pmatrix} \in M_{m,2}(\mathbb{C}) : a_1^2 + a_2^2 + \cdots + a_m^2 \neq 0 \right\}$;

(3) $\mathcal{M}_2 = \left\{ \begin{pmatrix} a_1 & ka_1 \\ a_2 & ka_2 \\ \vdots & \vdots \\ a_m & ka_m \end{pmatrix} \in M_{m,2}(\mathbb{C}) : 1 + k^2 \neq 0, \, a_1^2 + a_2^2 + \cdots + a_m^2 \neq 0 \right\}$;

(4) $\mathcal{M}_3 = \left\{ \begin{pmatrix} a_1 & b_1 \\ a_2 & b_2 \\ \vdots & \vdots \\ a_m & b_m \end{pmatrix} \in M_{m,2}(\mathbb{C}) : \left(\sum_{i=1}^m a_i^2\right)\left(\sum_{i=1}^m b_i^2\right) - \left(\sum_{i=1}^m a_i b_i\right)^2 \neq 0 \right\}$, *namely:*

$$\mathcal{LA}_{m,2} = \mathbf{O_{m\times 2}} \cup \mathcal{M}_1 \cup \mathcal{M}_2 \cup \mathcal{M}_3.$$

Corollary 3.7. *The set of $3 \times n$ left rank-preserving complex matrices can be divided into four types, they are*

(1) $3 \times n$ *zero matrix* $\mathbf{O_{3\times n}}$;

(2) $\mathcal{P}_1 = \left\{ \begin{pmatrix} 0 & 0 & \dots & 0 \\ 0 & 0 & \dots & 0 \\ a_1 & a_2 & \dots & a_n \end{pmatrix} \in M_{3,n}(\mathbb{C}) : a_1^2 + a_2^2 + \cdots + a_n^2 \neq 0 \right\}$;

(3) $\mathcal{P}_2 = \left\{ \begin{pmatrix} 0 & 0 & \dots & 0 \\ a_1 & a_2 & \dots & a_n \\ ka_1 & ka_2 & \dots & ka_n \end{pmatrix} \in M_{3,n}(\mathbb{C}) : 1 + k^2 \neq 0, \, a_1^2 + a_2^2 + \cdots + a_n^2 \neq 0 \right\}$;

(4) $\mathcal{P}_3 = \left\{ \begin{pmatrix} a_1 & a_2 & \dots & a_n \\ ka_1 & k_2a_2 & \dots & ka_n \\ la_1 & l_2a_2 & \dots & la_n \end{pmatrix} \in M_{3,n}(\mathbb{C}) : 1 + k^2 + l^2 \neq 0, \, a_1^2 + a_2^2 + \cdots + a_n^2 \neq 0 \right\}$;

(5) $\mathcal{P}_4 = \left\{ \begin{pmatrix} 0 & 0 & \dots & 0 \\ a_1 & a_2 & \dots & a_n \\ b_1 & b_2 & \dots & b_n \end{pmatrix} \in M_{3,n}(\mathbb{C}) : rank \begin{pmatrix} a_1 & b_1 \\ a_2 & b_2 \\ \vdots & \vdots \\ a_m & b_m \end{pmatrix} = 2 \right\}$;

(6) $\mathcal{P}_5 = \left\{ \begin{pmatrix} a_1 & a_2 & \dots & a_n \\ ka_1 & ka_2 & \dots & ka_n \\ b_1 & b_2 & \dots & b_n \end{pmatrix} \in M_{3,n}(\mathbb{C}) : 1 + k^2 \neq 0, \; rank \begin{pmatrix} a_1 & b_1 \\ a_2 & b_2 \\ \vdots & \vdots \\ a_m & b_m \end{pmatrix} = 2 \right\}$;

(7) $\mathcal{P}_6 = \left\{ \begin{pmatrix} a_1 & a_2 & \dots & a_n \\ b_1 & b_2 & \dots & b_n \\ ka_1 + lb_1 & ka_2 + lb_2 & \dots & ka_n + lb_n \end{pmatrix} \in M_{3,n}(\mathbb{C}) : 1 + k^2 + l^2 \neq 0, \; rank \begin{pmatrix} a_1 & b_1 \\ a_2 & b_2 \\ \vdots & \vdots \\ a_m & b_m \end{pmatrix} = 2 \right\}$;

(8) $\mathcal{P}_7 = \{A \in M_{3,n}(\mathbb{C}) : rank(A) = 3\}$, *namely:*

$$\mathcal{LA}_{3,n} = \mathbf{O_{3\times n}} \cup \bigcup_{j=1}^{j=7} \mathcal{P}_j.$$

Corollary 3.8. *The set of $m \times 3$ left rank-preserving complex matrices can be divided into four types, they are*

(1) $3 \times n$ *zero matrix* $\mathbf{O_{m\times 3}}$;

(2) $\mathcal{Q}_1 = \left\{ \begin{pmatrix} 0 & 0 & a_1 \\ 0 & 0 & a_2 \\ \vdots & \vdots & \vdots \\ 0 & 0 & a_m \end{pmatrix} \in M_{m,3}(\mathbb{C}) : a_1^2 + a_2^2 + \cdots + a_n^2 \neq 0 \right\}$;

(3) $\mathcal{Q}_2 = \left\{ \begin{pmatrix} 0 & a_1 & ka_1 \\ 0 & a_2 & ka_2 \\ \vdots & \vdots & \vdots \\ 0 & a_m & ka_m \end{pmatrix} \in M_{m,3}(\mathbb{C}) : 1+k^2 \neq 0,\, a_1^2 + a_2^2 + \cdots + a_m^2 \neq 0 \right\}$;

(4) $\mathcal{Q}_3 = \left\{ \begin{pmatrix} a_1 & ka_1 & la_1 \\ a_2 & ka_2 & la_2 \\ \vdots & \vdots & \vdots \\ a_m & ka_m & la_m \end{pmatrix} \in M_{m,3}(\mathbb{C}) : 1+k^2+l^2 \neq 0,\, a_1^2 + a_2^2 + \cdots + a_m^2 \neq 0 \right\}$;

(5) $\mathcal{Q}_4 = \left\{ \begin{pmatrix} 0 & a_1 & b_1 \\ 0 & a_2 & b_2 \\ \vdots & \vdots & \vdots \\ 0 & a_m & b_m \end{pmatrix} \in M_{m,3}(\mathbb{C}) : rank \begin{pmatrix} a_1 & b_1 \\ a_2 & b_2 \\ \vdots & \vdots \\ a_m & b_m \end{pmatrix} = 2 \right\}$;

(6) $\mathcal{Q}_5 = \left\{ \begin{pmatrix} a_1 & ka_1 & b_1 \\ a_2 & ka_2 & b_2 \\ \vdots & \vdots & \vdots \\ a_m & ka_m & b_m \end{pmatrix} \in M_{m,3}(\mathbb{C}) : 1+k^2 \neq 0,\ rank \begin{pmatrix} a_1 & b_1 \\ a_2 & b_2 \\ \vdots & \vdots \\ a_m & b_m \end{pmatrix} = 2 \right\}$;

(7) $\mathcal{Q}_6 = \left\{ \begin{pmatrix} a_1 & b_1 & ka_1+lb_1 \\ a_2 & b_2 & ka_2+lb_2 \\ \vdots & \vdots & \vdots \\ a_m & b_m & ka_m+lb_1 \end{pmatrix} \in M_{m,3}(\mathbb{C}) : 1+k^2+l^2 \neq 0,\ rank \begin{pmatrix} a_1 & b_1 \\ a_2 & b_2 \\ \vdots & \vdots \\ a_m & b_m \end{pmatrix} = 2 \right\}$;

(8) $\mathcal{Q}_7 = \{ A \in M_{m,3}(\mathbb{C}) : |A^\top A| \neq 0 \}$, *namely:*

$$\mathcal{LA}_{m,3} = \mathbf{O}_{\mathbf{m}\times\mathbf{3}} \cup \bigcup_{j=1}^{j=7} \mathcal{Q}_j.$$

Next, we extend the discussion to left rank-preserving complex $m \times n$ matrices A where the rank $r = rank(A)$ satisfies neither $r \leq 2$ nor $r \geq m-1$.

Let A_1 be an $m \times r$ complex matrix with $r \leq m$. By the Cauchy-Binet formula 2.1, the determinant $|A_1^\top A_1|$ equals the sum of squares of all $r \times r$ minors of A_1 (over $\binom{m}{r}$ possible choices). This result, combined with the framework established in Theorem 3.1, yields the following theorem.

Theorem 3.9. *Let A be a non-zero $m \times n$ matrix over the complex field $\mathbb{C}$ with* $\mathrm{rank}(A) = r$*. Then A is left rank-preserving if and only if there exists an $m \times r$ column submatrix A_1 (formed by selecting r columns of A) such that the sum of squares of all $r \times r$ minors of A_1 is non-vanishing, i.e.,*

$$\sum_{B \in \mathcal{M}_r(A_1)} |B|^2 \neq 0,$$

where $\mathcal{M}_r(A_1)$ denotes the set of all $r \times r$ submatrices of A_1.

The smaller matrix A_1 in the above theorem has full column rank, which implies that it contains at least one non-zero $r \times r$ minor. Let $\alpha_1, \alpha_2, \ldots, \alpha_m$ be the row vectors of A_1. Without loss of generality, we may assume that the first r row vectors of A_1 are linearly independent. For every α_i, there are r complex numbers $t_{i1}, t_{i2}, \ldots, t_{ir}$, such that $\alpha_i = t_{i1}\alpha_1 + t_{i2}\alpha_2 + \cdots + t_{ir}\alpha_r$. In particular, when $i \leq r$, we have $t_{ij} = \delta_{ij}$ $(j = 1, 2, \ldots, r)$, where δ is the Kronecker function. Thus the r-subdeterminant $M_{r\{i_1, i_2, \ldots, i_r\}}$, which consisting of the i_1-th, i_2-th, $\ldots$, i_r-th rows of A_1, can be expressed as

$$M_{r\{i_1,i_2,\ldots,i_r\}} = \begin{vmatrix} t_{i_1 1}\,\alpha_1 + t_{i_1 2}\,\alpha_2 + \cdots + t_{i_1 r}\,\alpha_r \\ t_{i_2 1}\,\alpha_1 + t_{i_2 2}\,\alpha_2 + \cdots + t_{i_2 r}\,\alpha_r \\ \vdots \\ t_{i_r 1}\,\alpha_1 + t_{i_r 2}\,\alpha_2 + \cdots + t_{i_r r}\,\alpha_r \end{vmatrix}.$$

Let $\alpha_i = (a_{i1}, a_{i2}, \ldots, a_{ir})$, $i = 1, 2, \ldots, m$. Then the above formula can be rewritten as

$$M_{r\{i_1,i_2,\ldots,i_r\}} = \left| \begin{pmatrix} t_{i_1 1} & t_{i_1 2} & \ldots & t_{i_1 r} \\ t_{i_2 1} & t_{i_2 2} & \ldots & t_{i_2 r} \\ \vdots & \vdots & \ddots & \vdots \\ t_{i_r 1} & t_{i_r 2} & \ldots & t_{i_r r} \end{pmatrix} \begin{pmatrix} a_{11} & a_{12} & \ldots & a_{1r} \\ a_{21} & a_{22} & \ldots & a_{2r} \\ \vdots & \vdots & \ddots & \vdots \\ a_{r1} & a_{r2} & \ldots & a_{rr} \end{pmatrix} \right|.$$

According to the multiplication theorem of the determinant, we obtain

$$M_{r\{i_1,i_2,\ldots,i_r\}} = \begin{vmatrix} t_{i_1 1} & t_{i_1 2} & \ldots & t_{i_1 r} \\ t_{i_2 1} & t_{i_2 2} & \ldots & t_{i_2 r} \\ \vdots & \vdots & \ddots & \vdots \\ t_{i_r 1} & t_{i_r 2} & \ldots & t_{i_r r} \end{vmatrix} \begin{vmatrix} a_{11} & a_{12} & \ldots & a_{1r} \\ a_{21} & a_{22} & \ldots & a_{2r} \\ \vdots & \vdots & \ddots & \vdots \\ a_{r1} & a_{r2} & \ldots & a_{rr} \end{vmatrix}. \tag{3}$$

Note that the first r row vectors of A are linearly independent, we have $M_{r\{1,2,\ldots,r\}} \neq 0$. According to lemma 2.1, we have

$$|A_1^\top A_1| = \sum_{1\le i_1<i_2<\cdots<i_r\le m} M^2_{r\{i_1,i_2,\ldots,i_r\}}.$$

Let

$$D_{i_1,i_2,\ldots,i_r} = \begin{vmatrix} t_{i_1 1} & t_{i_1 2} & \ldots & t_{i_1 r} \\ t_{i_2 1} & t_{i_2 2} & \ldots & t_{i_2 r} \\ \vdots & \vdots & \ddots & \vdots \\ t_{i_r 1} & t_{i_r 2} & \ldots & t_{i_r r} \end{vmatrix}, \tag{4}$$

Then formula (3) becomes $M_{r\{i_1,i_2,\ldots,i_r\}} = D_{i_1,i_2,\ldots,i_r}\, M_{r\{1,2,\ldots,r\}}$. Hence,

$$|A_1^\top A_1| = M^2_{r\{1,2,\ldots,r\}} \sum_{1\le i_1<i_2<\cdots<i_r\le m} D^2_{i_1,i_2,\ldots,i_r}.$$

From $M_{r\{1,2,\ldots,r\}} \neq 0$, and the above equation, we have

$$|A_1^\top A_1| \neq 0 \;\Leftrightarrow \sum_{1\le i_1<i_2<\cdots<i_r\le m} D^2_{i_1,i_2,\ldots,i_r} \neq 0.$$

Now, we consider the r determinant $D_{i_1,i_2,\ldots,i_r}$ of the form (4). Let

$$T = \begin{pmatrix} t_{11} & t_{12} & \ldots & t_{1r} \\ t_{21} & t_{22} & \ldots & t_{2r} \\ \vdots & \vdots & \ddots & \vdots \\ t_{m1} & t_{m2} & \ldots & t_{mr} \end{pmatrix} \text{ and } T_1 = \begin{pmatrix} t_{r+1,1} & t_{r+1,2} & \ldots & t_{r+1,r} \\ t_{r+2,1} & t_{r+2,2} & \ldots & t_{r+2,r} \\ \vdots & \vdots & \ddots & \vdots \\ t_{m1} & t_{m2} & \ldots & t_{mr} \end{pmatrix}.$$

Obviously, every $r \times r$ minor of T is a determinant of the form (4), and vice versa. It is easy to see that T_1 is an $(m-r) \times r$ submatrix formed by the last $m-r$ rows of T. Note that when

$i \leq r$, we have $t_{ij} = \delta_{ij}$ $(j = 1, 2, \ldots, r)$. By considering the submatrix formed by the first r rows of T, which is the identity matrix I_r, we obtain

$$D_{1,2,\ldots,r} = |I_r| = 1.$$

Secondly, suppose the $r \times r$ minor $D_{i_1,i_2,\ldots,i_r}$ of T contains some rows from the first r rows of T (e.g., the s-th row) and some from the last $m - r$ rows. We demonstrate that when expanding this minor along the s selected rows from the first r rows via Laplace expansion, its value becomes Δ or $-\Delta$, where Δ is an $(r - s)$-minor of T_1. Conversely, for any k-minor Δ of T_1 with $k < r$, there exists an r-minor $D_{i_1,i_2,\ldots,i_r}$ of T such that

$$D_{i_1,i_2,\ldots,i_r} = \pm\Delta.$$

The specific construction method is as follows: select the rows of Δ in T as the last k rows of $D_{i_1,i_2,\ldots,i_r}$, and choose $r - k$ rows from the first r rows of T as the first $r - k$ rows of $D_{i_1,i_2,\ldots,i_r}$. This ensures that the columns of T corresponding to Δ and the selected $r - k$ rows intersect at entries that do not contain the value 1.

The above analysis shows that, excluding the $r \times r$ minor $D_{1,2,\ldots,r}$, every $r \times r$ minor of T coincides with the absolute value of some $k \times k$ minor of T_1 for $1 \leq k \leq r$. Conversely, every $k \times k$ minor of T_1 (where $1 \leq k \leq r$) corresponds to an $r \times r$ minor of T through this relationship. Now, let S denote the sum of squares of all $k \times k$ minors of T_1 for each k in $1 \leq k \leq r$, then

$$\sum_{1 \leq i_1 < i_2 < \cdots < i_r \leq m} D^2_{i_1,i_2,\ldots,i_r} = 1 + S,$$

hence,

$$\sum_{1 \leq i_1 < i_2 < \cdots < i_r \leq m} = D^2_{i_1,i_2,\ldots,i_r} \neq 0 \;\Leftrightarrow\; S \neq -1.$$

From theorem 3.1, we get the next theorem.

Theorem 3.10. *Let A be an $m \times n$ nonzero complex matrix of rank r over the field $\mathbb{C}$. Suppose the row vectors $\alpha_{i_1}, \alpha_{i_2}, \ldots, \alpha_{i_r}$ are linearly independent in A. Let T_1 be an $(m - r) \times r$ nonzero complex matrix of rank r over $\mathbb{C}$, whose $m - r$ rows correspond to the coefficients in the linear combinations expressing the remaining $m - r$ row vectors of A in terms of these r linearly independent rows. Then, A is left rank-preserving if and only if the sum of the squares of all $k \times k$ minors of T_1 (for $1 \leq k \leq r$) is not equal to -1.*

Finally, we briefly introduce the properties of right rank-preserving matrices. Let $B = A^\top$, then each column of B corresponds to a row of A, and vice versa. Note that $B^\top B = AA^\top$. In the above discussion, we need to interchange rows with columns, columns with rows, and replace right multiplications with left multiplications.

4 Declarations

Funding

This research received no external funding.

Competing Interests

I declare that there are no competing interests.

Ethical Approval

Not applicable.

Authors' Contributions

Not applicable.

Availability Data and Materials

Not applicable.

Acknowledgments

I would like to thank the editors and reviewers for their time, effort, and valuable commentswhich led to improvements in my manuscript.

References

[1] L. B. Beasley, *Linear transformations on matrices: The invariance of rank-k matrices*, Linear Algebra Appl., **3** (1970), no. 4, 407-427, https://doi.org/10.1016/0024-3795(70)90033-9.

[2] L. B. Beasley, *Linear transformations which preserve fixed rank*, Linear Algebra Appl., **40** (1981), 183-187, https://doi.org/10.1016/0024-3795(81)90148-8.

[3] L. B. Beasley, *Rank-k preservers and preservers of sets of ranks*, Linear Algebra Appl., **55** (1983), 11-17, https://doi.org/10.1016/0024-3795(83)90163-5.

[4] L. B. Beasley, *Linear operators on matrices: The invariance of rank-k matrices*, Linear Algebra Appl., **107** (1988), 161-167, https://doi.org/10.1016/0024-3795(88)90242-X.

[5] L. B. Beasley, and T. J. Laffey, *Linear operators on matrices: The invariance of rank-k matrices*, Linear Algebra Appl., **133** (1990), 175-184, https://doi.org/10.1016/0024-3795(90)90248-B.

[6] L. B. Beasley, and T. J. Laffey, *Linear preservers of matrices of rank-2*, Linear Multilinear Algebra, **48** (2001), No. 4, 319-331, https://doi.org/10.1080/03081080108818678.

[7] L. B. Beasley, and S. Z. Song, *Primitive symmetric matrices and their preservers*, Linear and multilinear algebra, **65** (2017), no. 1, 129-139, https://doi.org/10.1080/03081087.2016.1175414.

[8] G. Frobenius, *Uber die darstellung der entlichen gruppen durch linear substitutionen [Over the representation of finite groups by linear substitution]*, S. B. Deutsch. Akad. Wiss. Berlin. 1897, 994-1015.

[9] S. Pierce, et al, *A survey of linear preserver problems*, Linear Multilinear Algebra, **33** (1992), 1-119.

[10] C. G. Cao, amd X. Zhang, *Additive operators preserving idempotent matrices over fields and applications*, Linear Algebra and its Applications, **248** (1996), 327-338, https://doi.org/10.1016/0024-3795(96)89195-6.

[11] R. Horn, C. K. Li, and N. K. Tsing, *Linear operators preserving certain equivalence relations on matrices*, SIAM Journal on Matrix Analysis and Applications, **12** (1991), no. 2, 195-204, https://doi.org/10.1137/0612015.

[12] Z. X. Wan, *Geometry of matrices: in memory of Professor L.K.Hua (1910-1985)*, World Scientic: Singapore, 1996, https://doi.org/10.1142/3080.

[13] G. Dolinar, and P Semrl, *Determinant preserving maps on matrix algebras*, Linear Algebra and its Application, **348** (2002), no. 1-3, 189-192, https://doi.org/10.1016/S0024-3795(01)00578-X.

[14] S. W. Liu, *Linear maps preserving idempotence on matrix modules over principal ideal domains*, Linear Algebra and its Applications, **258** (1997), 219-319, https://doi.org/10.1016/S0024-3795(96)00203-0.

[15] X. M. Tang, *Linear operators preserving adjoint matrix between matrix spaces*, Linear Algebra and its Applications, **372** (2003), 287-293, https://doi.org/10.1016/S0024-3795(03)00532-9.

[16] X. M. Tang, *Additive rank-1 preservers between Hermitian matrix spaces and applications*, Linear Algebra and its Applications, **395** (2005), 333-342, https://doi.org/10.1016/j.laa.2004.08.016.

[17] X. M. Tang, and X. Zhang, *Additive adjoint preservers between matrix spaces*, Linear and Multilinear Algebra, **54** (2006), no. 4, 285-300, https://doi.org/10.1080/03081080500423858.

[18] L. P. Huang, *Adjacency Preserving bijective maps of Hermitian matrices over any division ring with an involution*, Acta Mathematica Sinica, **23** (2007), no.1, 95-102, https://doi.org/10.1007/s10114-005-0770-7.

[19] H. You, and Z. Y. Wang, *K-Potence preserving maps without the linearity and surjectivity assumpyions*, Linear Algebra and its Applications, **426** (2007), no. 1, 238-254, https://doi.org/10.1016/j.laa.2007.04.024.

[20] H. You, and X. M. Tang, *Additive preservers of rank-additivity on the space of symmetric and alternate matrices*, Linear Algebra and its Applications, **380** (2004), 185-198, https://doi.org/10.1016/j.laa.2003.10.008.

[21] M. Omladič, and P. Šemrl, *Additive mappings Preserving Operators of rank one*, Linear Algebra and its Applications, **182** (1993), 239-256, https://doi.org/10.1016/0024-3795(93)90502-F.

[22] X. Zhang, *Linear/Additive preservers of ranks 2 and 4 on alternate matrix spaces over fields*, Algebra and its Applications, **392** (2004), 25-38, https://doi.org/10.1016/j.laa.2004.06.001.

[23] X. Zhang, *Idempotence preserving maps without the linearity and surjectivity assumptions*, Linear Algebra and its Applications, **387** (2004), 162-187, https://doi.org/10.1016/j.laa.2004.02.011.

[24] X. Zhang, X. M. Tang, and C. G. Cao, *Preserver Problems on Spaces of Matrices*, Beijing Science Press: Beijing, 2007.

[25] W. S. Qiu, *Advanced algebra (second edition)*, Peking University Press: Beijing, 2003.