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Regularization of ill-posed problem for evolution equation with nonlocal operator

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Abstract

The paper deals with the ill-posedness of the backward problem for fractional evolution equation. The main contribution of our paper is to construct the regularized solution using the Fourier truncation method. We also derive estimates between the regularized solution and the sought solution. Error estimates are obtained in L^2 and Hilbert space scales $\mathbb{H}^\mu$. Our main analysis is based on the estimation of the Mittag-Leffler functions. To the best of the author's knowledge, there are not any results for focusing the regularization of backward problem for elliptic equations with nonlocal operator.

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1 Introduction

Let Ω be a bounded domain in $\mathbb{R}^N$ $(N \geq 1)$ with sufficiently smooth boundary $\partial\Omega, T$ be a positive number. In this paper, we examine the following model

$$\begin{cases} {}_C D_t^\alpha u + \Delta u = F(x,t), & \text{in} \quad \Omega \times (0,T], \\ u|_{\partial\Omega} = 0, & \text{in} \quad \Omega, \\ u_t(x,0) = 0, & \text{in} \quad \Omega, \end{cases} \tag{1}$$

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with the terminal condition

$$u(x,T) = g(x), \qquad x \in \Omega. \tag{2}$$

Here g is the function defined later. The symbol ${}_C D_t^\alpha z$ denotes the Caputo derivative, which is defined in [12, 17]. This model mentioned above is an approximation of the elliptic equation. Indeed, if $\alpha \to 2^-$ then the solution of Problem (1) converges to the solution of the elliptic equation. It is a fact that Caputo derivative models have attracted the attention of many mathematicians around the world. This is a derivative containing singular kernels that has many applications in memory models, heat transfer, and past data recovery. The Cauchy problem for elliptic equations plays important role in electrical impedance tomography, optical tomography, plasma physics and transient phenomena in time-like variables, and also describes steady- state processes in physical fields [1, 4, 20]. There are several results on Cauchy elliptic equations with classical derivatives such as [11, 26] and references therein. In some phenomena related to memory effects, fractional derivative models are often used instead of the classical derivative. It explains a number of phenomena related to past distribution or viscosity models. In [6], Binh-Thang-Phuong considered the elliptic equation under the Caputo derivative on the plane

$$\begin{cases} {}_C D_t^\alpha u + \Delta u = F(x,t), & \text{in} \quad \mathbb{R}^2 \times (0,T], \\ u(x,0) = 0, & \text{in} \quad \mathbb{R}^2, \\ u_t(x,0) = 0, & \text{in} \quad \mathbb{R}^2. \end{cases} \tag{3}$$

They obtained several regularity results for the mild solution based on various assumptions of the input data. The principal techniques of the analysis is based on the bound of the Mittag-Leffler functions, combined with analysis in Hilbert scales space. In [7], the authors studied an initial value problem for a class of 2D time-fractional diffusion evolution equations with Riemann–Liouville fractional derivative. They studied the following model

$$\begin{cases} {}_C D_t^\alpha u + \Delta u = F(x,t,u(x,t)), & \text{in} \quad \Omega \times (0,T], \\ u|_{\partial\Omega} = 0, & \text{in} \quad \Omega, \\ u(x,y,0) = f(x,y), & \text{in} \quad \Omega, \end{cases} \tag{4}$$

where $0 < \alpha < 1$ and f is the initial datum. They focused on the existence and ill-posedness result (in the sense of Hadamard) in some cases of the source terms. By Fourier truncation method, the authors founded the regularized problems, and shown the error estimate between the exact solution and the approximate solution. Next, in [23], the authors considered a Cauchy problem for a semilinear fractional elliptic equation. They used the truncation method to give an approximation solution and provided the error estimate which is of logarithmic order.

This type of backward problem in the fractional diffusion equation has an important application foundation and is receiving more and more attention. The final value problem is an inverse problem that requires redefining the distribution at the initial time when the distribution at the past time is known. Backward problem appear in many applications, such as image deblurring and inpainting, see [24, 25, 26].

In the paper [21], the authors studied the terminal value problem for elliptic equation as follows

$$\begin{cases} u_{tt} + \Delta u = F(x,t), & \text{in} \quad \Omega \times (0,T], \\ u|_{\partial\Omega} = 0, & \text{in} \quad \Omega, \\ u_t(x,T) = f(x), & \text{in} \quad \Omega, \end{cases} \tag{5}$$

where $\Omega = (0, \pi) \times (0, \pi)$. They used Fourier truncation method to provide an approximate solution and focused on the error estimate between the regularized solution and the exact solution. This method has been shown to be effective in many inverse models and was recently detailed in a paper of N.M. Hai and his co-authors [13] that addressed the problem of regularizing the memory heat problem.

To the best of author's knowledge, there are not any results dealing with regularization of backward problems for fractional elliptic equations, such as (1)-(2). Notice the reader that the model (1)-(2) was also recently investigated by the authors [15]. However, there is no topic about regularization results on the paper [15]. Motivated by this reason, in this paper, we are interested to study the regularization of the problem. The main contribution of our paper is the study of ill-posedness and regularization for the backward problem. Our analysis is of using Fourier truncation method to construct a regularized solution and provide the error between the regularized solution and the sought solution. Research directions on regularization for diffusion equation with nonlocal operators can be found in the papers of Luc et al [9, 10].

There are a few tricky points when dealing with elliptic equations with the Caputo derivative. which are described as follows

- We adopted some interesting techniques from the article [25]. However, the paper [25] used the properties of some Mittag-Lefler functions on negative number axis. For fractional elliptic equation, we need the information of some Mittag-Leffler functions $E_{\alpha,1}(z)$ and $E_{\alpha,\alpha}(z)$ for any $z > 0$. This is the biggest difference compared to [25].
- We have difficulties dealing with singular integrals. To ensure the convergence of these integrals, we need clever techniques to handle them. Some evaluations and estimations have been learned and referenced by us from articles [2, 7].

This paper is organized as follows. In section 2, we introduce some premilinaries which is useful for main results. In section 3, we construct a regularized solution using Fourier truncation method. We provide the error estimate between the regularized solution and the exact solution.

2 Preliminaries

Definition 2.1. The spectral problem

$$\begin{cases} \Delta \psi_n(x) = -\lambda_n \psi_n(x), & x \in \Omega, \\ \psi_n(x) = 0, & x \in \partial\Omega, \end{cases}$$

admits the eigenvalues $0 < \lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_n \leq \ldots$, and corresponding eigenfunctions $e_n \in H_0^1(\Omega)$.

Definition 2.2. The Hilbert scale space $\mathbb{H}^s(\Omega)$ given as follows

$$\mathbb{H}^s(\Omega) = \Big\{ f \in L^2(\Omega) \mid \sum_{n=1}^{\infty} \lambda_n^{2s} \Big(\int_\Omega f(x)\psi_n(x)\mathrm{d}x \Big)^2 < \infty \Big\}, \tag{6}$$

for any $s \geq 0$, with the the norm

$$\|f\|_{\mathbb{H}^s(\Omega)} = \left(\sum_{n=1}^{\infty} \lambda_n^{2s} \Big(\int_\Omega f(x)\psi_n(x)\mathrm{d}x \Big)^2 \right)^{1/2}, \quad f \in \mathbb{H}^s(\Omega). \tag{7}$$

Lemma 2.3 (See [19]). *Let $1 < \alpha_0 < \alpha_1 < 2$ and $\alpha \in [\alpha_0, \alpha_1]$. Then there exists a constant $\rho_1, \rho_2, \overline{\rho}_1, \overline{\rho}_2 > 0$ and $z > 0$ such that*

$$\frac{\rho_1}{\alpha} \exp(z^{\frac{1}{\alpha}}) \leq E_{\alpha,1}(z) \leq \frac{\rho_2}{\alpha} \exp(z^{\frac{1}{\alpha}}), \tag{8}$$

and

$$\frac{\overline{\rho}_1}{\alpha}\exp(z^{\frac{1}{\alpha}}) \leq z^{\frac{\alpha-1}{\alpha}}E_{\alpha,\alpha}(z) \leq \frac{\overline{\rho}_2}{\alpha}\exp(z^{\frac{1}{\alpha}}). \tag{9}$$

Lemma 2.4. *Let $z \in \mathbb{R}$. Then we have*

$$\frac{d}{dz}E_{\alpha,1}(z) = \frac{E_{\alpha,\alpha}(z)}{\alpha}, \frac{d}{dz}E_{\alpha,1}(\lambda z^\alpha) = \frac{1}{z}E_{\alpha,0}(\lambda z^\alpha). \tag{10}$$

Lemma 2.5. *Let $h, k > 0$. Then we get*

$$\frac{1}{\Gamma(k)}\int_0^t (t-r)^{k-1}E_{\alpha,\beta}(\lambda r^\alpha)r^{\beta-1}dr = t^{\beta+k-1}E_{\alpha,\beta+k}(\lambda t^\alpha). \tag{11}$$

Proof. We can see the proof in Vol. 1, pp. 269–295 [3]. □

Lemma 2.6. *Let $1 < \alpha < 2$, then we get*

$$\frac{\rho_1}{\alpha}\exp(\lambda_n^{\frac{1}{\alpha}}t) \leq E_{\alpha,1}\left(\lambda_n t^\alpha\right) \leq \frac{\rho_2}{\alpha}\exp(\lambda_n^{\frac{1}{\alpha}}t). \tag{12}$$

and

$$\frac{\overline{\rho}_1}{\alpha}\lambda_n^{\frac{1}{\alpha}-1}t^{1-\alpha}\exp(t\lambda_n^{\frac{1}{\alpha}}) \leq E_{\alpha,\alpha}(\lambda_n t^\alpha) \leq \frac{\overline{\rho}_2}{\alpha}\lambda_n^{\frac{1}{\alpha}-1}t^{1-\alpha}\exp(t\lambda_n^{\frac{1}{\alpha}}). \tag{13}$$

Proof. Using Lemma (2.3) with $z = \lambda_n^{\frac{1}{\alpha}}t$, (12) and (13) is proved. □

Lemma 2.7. *Let $1 < \alpha < 2$. Then we have*

$$\frac{\overline{\rho}_1}{\rho_2}t^{1-\alpha}\lambda_n^{\frac{1}{\alpha}-1}\exp((t-T)\lambda_n^{\frac{1}{\alpha}}) \leq \frac{E_{\alpha,\alpha}(\lambda_n t^\alpha)}{E_{\alpha,1}(\lambda_n T^\alpha)} \leq \frac{\overline{\rho}_2}{\rho_1}t^{1-\alpha}\lambda_n^{\frac{1}{\alpha}-1}\exp((t-T)\lambda_n^{\frac{1}{\alpha}}). \tag{14}$$

Proof. Using (9), for any $0 \leq t \leq T$, we have

$$E_{\alpha,\alpha}(\lambda_n t^\alpha) \leq \frac{\overline{\rho}_2}{\alpha}\exp((\lambda_n t^\alpha)^{\frac{1}{\alpha}})\left(\lambda_n t^\alpha\right)^{\frac{1-\alpha}{\alpha}} = \frac{\overline{\rho}_2}{\alpha}\lambda_n^{\frac{1}{\alpha}-1}t^{1-\alpha}\exp\left(t\lambda_n^{\frac{1}{\alpha}}\right). \tag{15}$$

This inequality together with (33) yields to

$$\frac{E_{\alpha,\alpha}(\lambda_n t^\alpha)}{E_{\alpha,1}(\lambda_n T^\alpha)} \leq \frac{\overline{\rho}_2}{\rho_1}t^{1-\alpha}\lambda_n^{\frac{1}{\alpha}-1}\exp\left((t-T)\lambda_n^{\frac{1}{\alpha}}\right). \tag{16}$$

□

3 Regularization and error estimate

Let us give now the Fourier explicit expansion of the mild solution. Since the fomula [6], one gets

$$u_n(t) = E_{\alpha,1}\left(\lambda_n t^\alpha\right)u_n(0) + \int_0^t (t-\tau)^{\alpha-1}E_{\alpha,\alpha}\left(\lambda_n(t-\tau)^\alpha\right)F_n(\tau)d\tau. \tag{17}$$

By let $t = T$, we find that

$$g_n = u_n(T) = E_{\alpha,1}\left(\lambda_n T^\alpha\right)u_n(0) + \int_0^T (T-\tau)^{\alpha-1}E_{\alpha,\alpha}\left(\lambda_n(T-\tau)^\alpha\right)F_n(\tau)d\tau. \tag{18}$$

Hence, one has the following equality

$$u_n(0) = \frac{g_n}{E_{\alpha,1}\left(\lambda_j T^\alpha\right)} - \frac{\int_0^T (T-\tau)^{\alpha-1} E_{\alpha,\alpha}\left(\lambda_j (T-\tau)^\alpha\right) F_n(\tau) d\tau}{E_{\alpha,1}\left(\lambda_n T^\alpha\right)}. \tag{19}$$

By some simple caculations, we derive that

$$\begin{aligned} u(x,t) &= \sum_{n=1}^{\infty} \frac{E_{\alpha,1}\left(\lambda_n t^\alpha\right)}{E_{\alpha,1}\left(\lambda_n T^\alpha\right)} g_n \psi_n(x) \\ &\quad - \sum_{n=1}^{\infty} \left(\frac{E_{\alpha,1}(\lambda_n t^\alpha) \int_0^T (T-\tau)^{\alpha-1} E_{\alpha,\alpha}\left(\lambda_n (T-\tau)^\alpha\right) F_n(\tau) d\tau}{E_{\alpha,1}\left(\lambda_n T^\alpha\right)} \right) \psi_n(x) \\ &\quad + \sum_{n=1}^{\infty} \left(\int_0^t (t-\tau)^{\alpha-1} E_{\alpha,\alpha}\left(\lambda_n (t-\tau)^\alpha\right) F_n(\tau) d\tau \right) \psi_n(x). \end{aligned} \tag{20}$$

In this section, we derive the regularization results for the backward problem (1)-(2). It is not difficult to observe that the mild solution (20) is instable in L^2 since the fact that the term $(t-\tau)^{\alpha-1} E_{\alpha,\alpha}\left(\lambda_n (t-\tau)^\alpha\right) \to +\infty$ when $\lambda_n \to +\infty$ and $0 < \tau < t$. Hence, we need to regularize backward problem (1)-(2) to ensure the stability of the problem. Let us assume that (g, F) is noised by the observed data $(g^\varepsilon, F^\varepsilon) \in L^2(\Omega) \times L^\infty(0,T; L^2(\Omega))$ such that

$$\left\| g^\varepsilon - g \right\|_{L^2(\Omega)} + \left\| F^\varepsilon - F \right\|_{L^\infty(0,T;L^2(\Omega))} \le \varepsilon. \tag{21}$$

In the following Theorem, we introduce a Fourier truncation method to approximate backward problem (1)-(2).

Theorem 3.1. *Let us give the following regularized solution*

$$\begin{aligned} u^\varepsilon(x,t) &= \sum_{\lambda_n \le N_\varepsilon} \frac{E_{\alpha,1}\left(\lambda_n t^\alpha\right)}{E_{\alpha,1}\left(\lambda_n T^\alpha\right)} g_j^\varepsilon \psi_j(x) \\ &\quad - \sum_{\lambda_n \le N_\varepsilon} \left(\frac{E_{\alpha,1}(\lambda_n t^\alpha) \int_0^T (T-\tau)^{\alpha-1} E_{\alpha,\alpha}\left(\lambda_n (T-\tau)^\alpha\right) F_n^\varepsilon(\tau) d\tau}{E_{\alpha,1}\left(\lambda_n T^\alpha\right)} \right) \psi_j(x) \\ &\quad + \sum_{\lambda_n \le N_\varepsilon} \left(\int_0^t (t-\tau)^{\alpha-1} E_{\alpha,\alpha}\left(\lambda_n (t-\tau)^\alpha\right) F_n^\varepsilon(\tau) d\tau \right) \psi_j(x). \end{aligned} \tag{22}$$

Let us assume that $u \in L^\infty(0,T; \mathbb{H}^\ell(\Omega))$ *for any* $\ell > 0$.
a) Let us choose N_ε *such that*

$$\lim_{\varepsilon \to 0} \exp\left(T N_\varepsilon^{\frac{1}{\alpha}}\right) \varepsilon = 0, \quad \lim_{\varepsilon \to 0} N_\varepsilon = +\infty. \tag{23}$$

Then we have

$$\left\| u^\varepsilon(.,t) - u(.,t) \right\|_{L^2(\Omega)} \le (N_\varepsilon)^{-\ell} \left\| u \right\|_{L^\infty(0,T;\mathbb{H}^\ell(\Omega))} + \frac{\rho_2}{\rho_1}\varepsilon + 2\sqrt{\frac{T\rho_2^3 \overline{\rho}_2}{\alpha^3 \rho_1^2 \lambda_1}} \exp\left(T N_\varepsilon^{\frac{1}{\alpha}}\right) \varepsilon. \tag{24}$$

b) Let us choose N_ε *such that*

$$\lim_{\varepsilon \to 0} N_\varepsilon^\mu \exp\left(T N_\varepsilon^{\frac{1}{\alpha}}\right) \varepsilon = 0, \quad \lim_{\varepsilon \to 0} N_\varepsilon = +\infty, \tag{25}$$

for any $\mu \geq 0$*. Then we have*

$$\left\|u^\varepsilon(.,t) - u(.,t)\right\|_{\mathbb{H}^\mu(\Omega)} \leq (N_\varepsilon)^{-\ell}\left\|u\right\|^2_{L^\infty(0,T;\mathbb{H}^{\ell+\mu}(\Omega))} + \frac{\rho_2}{\rho_1}N_\varepsilon^\mu \varepsilon + 2\sqrt{\frac{T\rho_2^3\overline{\rho}_2}{\alpha^3\rho_1^2\lambda_1}}N_\varepsilon^\mu \exp\left(TN_\varepsilon^{\frac{1}{\alpha}}\right)\varepsilon. \tag{26}$$

Remark 3.2. Let us choose N_ε as follows

$$N_\varepsilon = \left(\frac{1}{T(1-\sigma)}\right)^\alpha \log^\alpha(\frac{1}{\varepsilon}), \tag{27}$$

for $0 < \sigma < 1$. It follows from (24) that the error $\left\|u^\varepsilon(.,t) - u(.,t)\right\|_{L^2(\Omega)}$ is of order

$$\max\left(\log^{-\alpha\ell}(\frac{1}{\varepsilon}),\ \varepsilon^\sigma\right). \tag{28}$$

It follows from (26) that the error $\left\|u^\varepsilon(.,t) - u(.,t)\right\|_{\mathbb{H}^\mu(\Omega)}$ is of order

$$\max\left(\log^{-\alpha\ell}(\frac{1}{\varepsilon}),\ \varepsilon^\sigma \log^{\alpha\mu}(\frac{1}{\varepsilon})\right). \tag{29}$$

Proof. Let us denote

$$\mathbb{K}_1^{N_\varepsilon}(x,t) = \sum_{\lambda_n \leq N_\varepsilon} \frac{E_{\alpha,1}\left(\lambda_n t^\alpha\right)}{E_{\alpha,1}\left(\lambda_n T^\alpha\right)} g_j^\varepsilon \psi_n(x), \quad \mathbb{K}_1(x,t) = \sum_{\lambda_n \leq N_\varepsilon} \frac{E_{\alpha,1}\left(\lambda_n t^\alpha\right)}{E_{\alpha,1}\left(\lambda_n T^\alpha\right)} g_n \psi_n(x)$$

$$\mathbb{K}_2^{N_\varepsilon}(x,t) = -\sum_{\lambda_n \leq N_\varepsilon}\left(\frac{E_{\alpha,1}(\lambda_n t^\alpha)\int_0^T (T-\tau)^{\alpha-1}E_{\alpha,\alpha}\left(\lambda_n(T-\tau)^\alpha\right)F_n^\varepsilon(\tau)d\tau}{E_{\alpha,1}\left(\lambda_n T^\alpha\right)}\right)\psi_n(x)$$

$$\mathbb{K}_2(x,t) = -\sum_{\lambda_n \leq N_\varepsilon}\left(\frac{E_{\alpha,1}(\lambda_n t^\alpha)\int_0^T (T-\tau)^{\alpha-1}E_{\alpha,\alpha}\left(\lambda_n(T-\tau)^\alpha\right)F_n(\tau)d\tau}{E_{\alpha,1}\left(\lambda_n T^\alpha\right)}\right)\psi_n(x)$$

and

$$\mathbb{K}_3^{N_\varepsilon}(x,t) = \sum_{\lambda_n \leq N_\varepsilon}\left(\int_0^t (t-\tau)^{\alpha-1}E_{\alpha,\alpha}\left(\lambda_n(t-\tau)^\alpha\right)F_n^\varepsilon(\tau)d\tau\right)\psi_n(x).$$

$$\mathbb{K}_3(x,t) = \sum_{\lambda_n \leq N_\varepsilon}\left(\int_0^t (t-\tau)^{\alpha-1}E_{\alpha,\alpha}\left(\lambda_n(t-\tau)^\alpha\right)F_n(\tau)d\tau\right)\psi_n(x).$$

Let us set the following function

$$\begin{aligned} v^\varepsilon(x,t) &= \sum_{\lambda_n \leq N_\varepsilon} \frac{E_{\alpha,1}\left(\lambda_n t^\alpha\right)}{E_{\alpha,1}\left(\lambda_n T^\alpha\right)} g_n \psi_n(x) \\ &\quad - \sum_{\lambda_n \leq N_\varepsilon}\left(\frac{E_{\alpha,1}(\lambda_n t^\alpha)\int_0^T (T-\tau)^{\alpha-1}E_{\alpha,\alpha}\left(\lambda_n(T-\tau)^\alpha\right)F_n(\tau)d\tau}{E_{\alpha,1}\left(\lambda_n T^\alpha\right)}\right)\psi_n(x) \\ &\quad + \sum_{\lambda_n \leq N_\varepsilon}\left(\int_0^t (t-\tau)^{\alpha-1}E_{\alpha,\alpha}\left(\lambda_n(t-\tau)^\alpha\right)F_n(\tau)d\tau\right)\psi_n(x) \\ &= \mathbb{K}_1(x,t) + \mathbb{K}_2(x,t) + \mathbb{K}_3(x,t). \end{aligned} \tag{30}$$

We now treat the error $\left\|v^\varepsilon(.,t)-u(.,t)\right\|_{L^2(\Omega)}$. It is obvious to see that

$$\left\|v^\varepsilon(.,t)-u(.,t)\right\|^2_{L^2(\Omega)} = \sum_{\lambda_n>N_\varepsilon} u_j^2(t) = \sum_{\lambda_n>N_\varepsilon} \lambda_n^{-2\ell}\lambda_n^{2\ell}u_j^2(t) \le (N_\varepsilon)^{-2\ell}\sum_{\lambda_n>N_\varepsilon}\lambda_n^{2\ell}u_n^2(t)$$
$$\le (N_\varepsilon)^{-2\ell}\left\|u(.,t)\right\|^2_{\mathbb{H}^\ell(\Omega)} \le (N_\varepsilon)^{-2\ell}\left\|u\right\|^2_{L^\infty(0,T;\mathbb{H}^\ell(\Omega))}. \tag{31}$$

We continue to study the term $\left\|v^\varepsilon(.,t)-u^\varepsilon(.,t)\right\|_{L^2(\Omega)}$. From (20) and (30), one gets

$$\left\|v^\varepsilon(.,t)-u^\varepsilon(.,t)\right\|_{L^2(\Omega)} \le \sum_{j=1}^{3}\left\|\mathbb{K}_j^{N_\varepsilon}(.,t)-\mathbb{K}_1(.,t)\right\|_{L^2(\Omega)}. \tag{32}$$

Using Lemma (2.3), we derive that

$$\frac{\rho_1}{\alpha}\exp(\lambda_n^{\frac{1}{\alpha}}t) \le E_{\alpha,1}\left(\lambda_n t^\alpha\right) \le \frac{\rho_2}{\alpha}\exp(\lambda_n^{\frac{1}{\alpha}}t). \tag{33}$$

Since (33), one has

$$\frac{\rho_1}{\rho_2}\exp(\lambda_n^{\frac{1}{\alpha}}(t-T)) \le \frac{E_{\alpha,1}\left(\lambda_n t^\alpha\right)}{E_{\alpha,1}\left(\lambda_n T^\alpha\right)} \le \frac{\rho_2}{\rho_1}\exp(\lambda_n^{\frac{1}{\alpha}}(t-T)). \tag{34}$$

Using this inequality, we find that

$$\left\|\mathbb{K}_1^{N_\varepsilon}(.,t)-\mathbb{K}_1(.,t)\right\|^2_{L^2(\Omega)} = \sum_{\lambda_n\le N_\varepsilon}\left(\frac{E_{\alpha,1}\left(\lambda_n t^\alpha\right)}{E_{\alpha,1}\left(\lambda_n T^\alpha\right)}\right)^2\left(g_n^\varepsilon-g_n\right)^2 \le \frac{\rho_2^2}{\rho_1^2}\left\|g^\varepsilon-g\right\|^2_{L^2(\Omega)}. \tag{35}$$

This implies that

$$\left\|\mathbb{K}_1^{N_\varepsilon}(.,t)-\mathbb{K}_1(.,t)\right\|_{L^2(\Omega)} \le \frac{\rho_2}{\rho_1}\varepsilon. \tag{36}$$

Using Parseval's equality, we obtain

$$\left\|\mathbb{K}_2^{N_\varepsilon}(.,t)-\mathbb{K}_2(.,t)\right\|^2_{L^2(\Omega)}$$
$$= \sum_{\lambda_n\le N_\varepsilon}\left(\frac{E_{\alpha,1}\left(\lambda_n t^\alpha\right)}{E_{\alpha,1}\left(\lambda_n T^\alpha\right)}\right)^2\left(\int_0^T (T-\tau)^{\alpha-1}E_{\alpha,\alpha}\left(\lambda_n(T-\tau)^\alpha\right)\left(F_n^\varepsilon(\tau)-F_n(\tau)\right)d\tau\right)^2$$
$$\le \frac{\rho_2^2}{\rho_1^2}\sum_{\lambda_n\le N_\varepsilon}\left(\int_0^T (T-\tau)^{\alpha-1}E_{\alpha,\alpha}\left(\lambda_n(T-\tau)^\alpha\right)\left(F_n^\varepsilon(\tau)-F_n(\tau)\right)d\tau\right)^2. \tag{37}$$

In view of Hölder inequality, we arrive at

$$\sum_{\lambda_n\le N_\varepsilon}\left(\int_0^T (T-\tau)^{\alpha-1}E_{\alpha,\alpha}\left(\lambda_n(T-\tau)^\alpha\right)\left(F_n^\varepsilon(\tau)-F_n(\tau)\right)d\tau\right)^2$$
$$\le \sum_{\lambda_n\le N_\varepsilon}\left(\int_0^T (T-\tau)^{\alpha-1}E_{\alpha,\alpha}\left(\lambda_n(T-\tau)^\alpha\right)d\tau\right)$$
$$\left(\int_0^T (T-\tau)^{\alpha-1}E_{\alpha,\alpha}\left(\lambda_n(T-\tau)^\alpha\right)\left(F_n^\varepsilon(\tau)-F_n(\tau)\right)^2d\tau\right)$$
$$\le \sum_{\lambda_n\le N_\varepsilon}\frac{E_{\alpha,1}(\lambda_n T^\alpha)-1}{\lambda_n}\left(\int_0^T (T-\tau)^{\alpha-1}E_{\alpha,\alpha}\left(\lambda_n(T-\tau)^\alpha\right)\left(F_n^\varepsilon(\tau)-F_n(\tau)\right)^2d\tau\right). \tag{38}$$

Since $\lambda_n \leq N_\varepsilon$ and using Lemma (2.3), one obtains

$$\frac{E_{\alpha,1}(\lambda_n T^\alpha) - 1}{\lambda_n} \leq \frac{\rho_2}{\alpha\lambda_1} \exp\left(T\lambda_n^{\frac{1}{\alpha}}\right) \leq \frac{\rho_2}{\alpha\lambda_1} \exp\left(TN_\varepsilon^{\frac{1}{\alpha}}\right) \tag{39}$$

and

$$(T-\tau)^{\alpha-1} E_{\alpha,\alpha}\left(\lambda_n (T-\tau)^\alpha\right) \leq \frac{\overline{\rho}_2}{\alpha} \exp\left((T-\tau)\lambda_n^{\frac{1}{\alpha}}\right) \leq \frac{\overline{\rho}_2}{\alpha} \exp\left(TN_\varepsilon^{\frac{1}{\alpha}}\right) \tag{40}$$

where we note that $\lambda_n \geq \lambda_1$. Since some previous observations, we get

$$\begin{aligned}
&\sum_{\lambda_n \leq N_\varepsilon} \left(\int_0^T (T-\tau)^{\alpha-1} E_{\alpha,\alpha}\left(\lambda_n (T-\tau)^\alpha\right) \left(F_n^\varepsilon(\tau) - F_n(\tau)\right) d\tau \right)^2 \\
&\leq \frac{\rho_2 \overline{\rho}_2}{\alpha^2 \lambda_1} \exp\left(2TN_\varepsilon^{\frac{1}{\alpha}}\right) \int_0^T (T-\tau)^{\alpha-1} \left\| F^\varepsilon(.,\tau) - F(.,\tau) \right\|^2_{L^2(\Omega)} d\tau \\
&\leq \frac{T\rho_2 \overline{\rho}_2}{\alpha^3 \lambda_1} \exp\left(2TN_\varepsilon^{\frac{1}{\alpha}}\right) \left\| F^\varepsilon - F \right\|^2_{L^\infty(0,T;L^2(\Omega))} \leq \frac{T\rho_2 \overline{\rho}_2}{\alpha^3 \lambda_1} \exp\left(2TN_\varepsilon^{\frac{1}{\alpha}}\right) \varepsilon^2.
\end{aligned} \tag{41}$$

Combining (37) and (41), one obtains

$$\left\| \mathbb{K}_2^{N_\varepsilon}(.,t) - \mathbb{K}_2(.,t) \right\|_{L^2(\Omega)} \leq \sqrt{\frac{T\rho_2^3 \overline{\rho}_2}{\alpha^3 \rho_1^2 \lambda_1}} \exp\left(TN_\varepsilon^{\frac{1}{\alpha}}\right) \varepsilon. \tag{42}$$

By a similar way, we also obtain that

$$\left\| \mathbb{K}_3^{N_\varepsilon}(.,t) - \mathbb{K}_3(.,t) \right\|_{L^2(\Omega)} \leq \sqrt{\frac{T\rho_2^3 \overline{\rho}_2}{\alpha^3 \rho_1^2 \lambda_1}} \exp\left(TN_\varepsilon^{\frac{1}{\alpha}}\right) \varepsilon. \tag{43}$$

Combining (60), (32), (36), (42) and (43), we infer that

$$\begin{aligned}
\left\| u^\varepsilon(.,t) - u(.,t) \right\|_{L^2(\Omega)} &\leq \left\| v^\varepsilon(.,t) - u(.,t) \right\|_{L^2(\Omega)} + \left\| v^\varepsilon(.,t) - u^\varepsilon(.,t) \right\|_{L^2(\Omega)} \\
&\leq (N_\varepsilon)^{-\ell} \left\| u \right\|_{L^\infty(0,T;\mathbb{H}^\ell(\Omega))} + \frac{\rho_2}{\rho_1}\varepsilon + 2\sqrt{\frac{T\rho_2^3 \overline{\rho}_2}{\alpha^3 \rho_1^2 \lambda_1}} \exp\left(TN_\varepsilon^{\frac{1}{\alpha}}\right) \varepsilon.
\end{aligned} \tag{44}$$

Let us now to prove that the estimate in $\mathbb{H}^\mu$ norm. First, we modified the proof of (60) as follows

$$\begin{aligned}
\left\| v^\varepsilon(.,t) - u(.,t) \right\|^2_{\mathbb{H}^\mu(\Omega)} &= \sum_{\lambda_n > N_\varepsilon} \lambda_n^{-2\ell} \lambda_n^{2\ell+2\mu} u_n^2(t) \leq (N_\varepsilon)^{-2\ell} \sum_{\lambda_n > N_\varepsilon} \lambda_n^{2\ell+2\mu} u_n^2(t) \\
&\leq (N_\varepsilon)^{-2\ell} \left\| u(.,t) \right\|^2_{\mathbb{H}^{\ell+\mu}(\Omega)} \leq (N_\varepsilon)^{-2\ell} \left\| u \right\|^2_{L^\infty(0,T;\mathbb{H}^{\ell+\mu}(\Omega))}.
\end{aligned} \tag{45}$$

We edit the proof of (35) as follows

$$\begin{aligned}
\left\| \mathbb{K}_1^{N_\varepsilon}(.,t) - \mathbb{K}_1(.,t) \right\|^2_{\mathbb{H}^\mu(\Omega)} &= \sum_{\lambda_n \leq N_\varepsilon} \lambda_n^{2\mu} \left(\frac{E_{\alpha,1}\left(\lambda_n t^\alpha\right)}{E_{\alpha,1}\left(\lambda_n T^\alpha\right)} \right)^2 \left(g_n^\varepsilon - g_n \right)^2 \\
&\leq \frac{\rho_2^2}{\rho_1^2} N_\varepsilon^{2\mu} \left\| g^\varepsilon - g \right\|^2_{L^2(\Omega)} \leq \frac{\rho_2^2}{\rho_1^2} N_\varepsilon^{2\mu} \varepsilon^2.
\end{aligned} \tag{46}$$

This implies that

$$\left\|\mathbb{K}_1^{N_\varepsilon}(.,t)-\mathbb{K}_1(.,t)\right\|_{\mathbb{H}^\mu(\Omega)}\le \frac{\rho_2}{\rho_1}N_\varepsilon^\mu\varepsilon. \tag{47}$$

We revise the proof of (37) in the following

$$\begin{aligned}&\left\|\mathbb{K}_2^{N_\varepsilon}(.,t)-\mathbb{K}_2(.,t)\right\|^2_{\mathbb{H}^\mu(\Omega)}\\&=\sum_{\lambda_n\le N_\varepsilon}\lambda_n^{2\mu}\left(\frac{E_{\alpha,1}(\lambda_n t^\alpha)}{E_{\alpha,1}(\lambda_n T^\alpha)}\right)^2\left(\int_0^T (T-\tau)^{\alpha-1}E_{\alpha,\alpha}\left(\lambda_n(T-\tau)^\alpha\right)\left(F_n^\varepsilon(\tau)-F_n(\tau)\right)d\tau\right)^2\\&\le\frac{\rho_2^2}{\rho_1^2}(N_\varepsilon)^{2\mu}\sum_{\lambda_n\le N_\varepsilon}\left(\int_0^T (T-\tau)^{\alpha-1}E_{\alpha,\alpha}\left(\lambda_n(T-\tau)^\alpha\right)\left(F_n^\varepsilon(\tau)-F_n(\tau)\right)d\tau\right)^2.\end{aligned} \tag{48}$$

This together with (41) allows us to deduce that

$$\left\|\mathbb{K}_2^{N_\varepsilon}(.,t)-\mathbb{K}_2(.,t)\right\|^2_{\mathbb{H}^\mu(\Omega)}\le\frac{\rho_2^2}{\rho_1^2}(N_\varepsilon)^{2\mu}\frac{T\rho_2\overline{\rho}_2}{\alpha^3\lambda_1}\exp\left(2TN_\varepsilon^{\frac{1}{\alpha}}\right)\varepsilon^2. \tag{49}$$

Therefore, one obtains

$$\left\|\mathbb{K}_2^{N_\varepsilon}(.,t)-\mathbb{K}_2(.,t)\right\|_{\mathbb{H}^\mu(\Omega)}\le\sqrt{\frac{T\rho_2^3\overline{\rho}_2}{\alpha^3\rho_1^2\lambda_1}}(N_\varepsilon)^{\mu}\exp\left(TN_\varepsilon^{\frac{1}{\alpha}}\right)\varepsilon. \tag{50}$$

With the same procedure as above, we also obtain

$$\left\|\mathbb{K}_3^{N_\varepsilon}(.,t)-\mathbb{K}_3(.,t)\right\|_{\mathbb{H}^\mu(\Omega)}\le\sqrt{\frac{T\rho_2^3\overline{\rho}_2}{\alpha^3\rho_1^2\lambda_1}}(N_\varepsilon)^{\mu}\exp\left(TN_\varepsilon^{\frac{1}{\alpha}}\right)\varepsilon. \tag{51}$$

Combining (45), (47), (50) and (51), we confirm that

$$\begin{aligned}\left\|u^\varepsilon(.,t)-u(.,t)\right\|_{\mathbb{H}^\mu(\Omega)}&\le\left\|v^\varepsilon(.,t)-u(.,t)\right\|_{\mathbb{H}^\mu(\Omega)}+\left\|v^\varepsilon(.,t)-u^\varepsilon(.,t)\right\|_{\mathbb{H}^\mu(\Omega)}\\&\le (N_\varepsilon)^{-\ell}\left\|u\right\|^2_{L^\infty(0,T;\mathbb{H}^{\ell+\mu}(\Omega))}+\frac{\rho_2}{\rho_1}N_\varepsilon^\mu\varepsilon+2\sqrt{\frac{T\rho_2^3\overline{\rho}_2}{\alpha^3\rho_1^2\lambda_1}}N_\varepsilon^\mu\exp\left(TN_\varepsilon^{\frac{1}{\alpha}}\right)\varepsilon.\end{aligned} \tag{52}$$

The proof is completed. □

In the next Theorem, we give the regularized solution which approximate the first derivative of u with respect to variable t.

Theorem 3.3. *Let us assume that F has a separable form $F(x,t)=\varphi(t)f(x)$. Let us assume that the functions g,φ,f are noised by $g^\varepsilon,\varphi^\varepsilon,f^\varepsilon$ such that*

$$\left\|g^\varepsilon-g\right\|_{L^2(\Omega)}+\left\|\varphi^\varepsilon-\varphi\right\|_{L^\infty(0,T)}+\left\|f^\varepsilon-f\right\|_{L^2(\Omega)}\le\varepsilon. \tag{53}$$

Let us give the following regularized solution

$$\begin{aligned}w^\varepsilon(x,t)=&\sum_{\lambda_n\le N_\varepsilon}\frac{\lambda_n t^{\alpha-1}E_{\alpha,\alpha}(\lambda_n t^\alpha)}{E_{\alpha,1}(\lambda_n T^\alpha)}g_n^\varepsilon\psi_n(x)\\&-\sum_{\lambda_n\le N_\varepsilon}\left(\frac{\lambda_n t^{\alpha-1}E_{\alpha,\alpha}(\lambda_n t^\alpha)\int_0^T (T-\tau)^{\alpha-1}E_{\alpha,\alpha}\left(\lambda_n(T-\tau)^\alpha\right)F_n^\varepsilon(\tau)d\tau}{E_{\alpha,1}(\lambda_n T^\alpha)}\right)\psi_n(x)\\&+\sum_{\lambda_n\le N_\varepsilon}\left(\int_0^t (t-\tau)^{\alpha-2}E_{\alpha,\alpha-1}\left(\lambda_n(t-\tau)^\alpha\right)F_n^\varepsilon(\tau)d\tau\right)\psi_n(x),\end{aligned} \tag{54}$$

where $F_n^\varepsilon(t) = \varphi^\varepsilon(t)F_n^\varepsilon$, $F_n^\varepsilon = \int_\Omega f(x)\psi_n(x)dx$. Let us assume that backward problem (1)-(2) has a unique solution u such that $u_t \in L^\infty(0,T;\mathbb{H}^\ell(\Omega))$ for any $\ell > 0$. Then we get the following estimate

$$\begin{aligned}\left\|W^\varepsilon(.,t) - u_t(.,t)\right\|_{L^2(\Omega)} &\le (N_\varepsilon)^{-\ell}\left\|u_t\right\|_{L^\infty(0,T;\mathbb{H}^\ell(\Omega))} + \frac{\rho_2 T^{\alpha-1}N_\varepsilon}{\rho_1}\varepsilon \\ &+ C(\rho_2,\alpha,\overline{\rho}_2,\lambda_1,T)\left(\sqrt{N_\varepsilon}+1\right)\exp\left(TN_\varepsilon^{\frac{1}{\alpha}}\right)\varepsilon\left(\left\|\varphi^\varepsilon\right\|_{L^\infty(0,T)} + \left\|f\right\|_{L^2(\Omega)}\right). \end{aligned} \tag{55}$$

Here N_ε satisfies that

$$\lim_{\varepsilon\to 0}\left(\sqrt{N_\varepsilon}+1\right)\exp\left(TN_\varepsilon^{\frac{1}{\alpha}}\right)\varepsilon = \lim_{\varepsilon\to 0} N_\varepsilon\varepsilon = 0, \quad \lim_{\varepsilon\to 0} N_\varepsilon = +\infty. \tag{56}$$

Remark 3.4. Let us choose N_ε as follows

$$N_\varepsilon = \left(\frac{1}{T(1-\sigma)}\right)^\alpha \log^\alpha(\frac{1}{\varepsilon}), \tag{57}$$

for $0 < \sigma < 1$. It follows from (55) that the error $\left\|W^\varepsilon(.,t) - u_t(.,t)\right\|_{L^2(\Omega)}$ is of order

$$\max\left(\log^{-\alpha\ell}(\frac{1}{\varepsilon}),\ \varepsilon^\sigma\sqrt{\log(\frac{1}{\varepsilon})}\right). \tag{58}$$

Proof. Our analysis technique of proving this theorem is somewhat similar to the proof of the theorem (3.1), but there are many other adjustments. We will provide more details for readers. Let us denote

$$\mathbb{Q}_1^{N_\varepsilon}(x,t) = \sum_{\lambda_n\le N_\varepsilon}\frac{\lambda_n t^{\alpha-1}E_{\alpha,\alpha}\left(\lambda_n t^\alpha\right)}{E_{\alpha,1}\left(\lambda_n T^\alpha\right)} g_n^\varepsilon \psi_n(x)$$

$$\mathbb{Q}_2^{N_\varepsilon}(x,t) = -\sum_{\lambda_n\le N_\varepsilon}\left(\frac{\lambda_n t^{\alpha-1}E_{\alpha,\alpha}(\lambda_n t^\alpha)\int_0^T (T-\tau)^{\alpha-1}E_{\alpha,\alpha}\left(\lambda_n(T-\tau)^\alpha\right)F_n^\varepsilon(\tau)d\tau}{E_{\alpha,1}\left(\lambda_n T^\alpha\right)}\right)\psi_n(x)$$

and

$$\mathbb{Q}_3^{N_\varepsilon}(x,t) = \sum_{\lambda_n\le N_\varepsilon}\left(\int_0^t (t-\tau)^{\alpha-2}E_{\alpha,\alpha-1}\left(\lambda_n(t-\tau)^\alpha\right)F_n^\varepsilon(\tau)d\tau\right)\psi_n(x).$$

Let us set the following function

$$\begin{aligned} W^\varepsilon(x,t) &= \sum_{\lambda_n\le N_\varepsilon}\frac{\lambda_n t^{\alpha-1}E_{\alpha,\alpha}\left(\lambda_n t^\alpha\right)}{E_{\alpha,1}\left(\lambda_n T^\alpha\right)} g_n\psi_n(x) \\ &\quad - \sum_{\lambda_n\le N_\varepsilon}\left(\frac{\lambda_n t^{\alpha-1}E_{\alpha,\alpha}(\lambda_n t^\alpha)\int_0^T (T-\tau)^{\alpha-1}E_{\alpha,\alpha}\left(\lambda_n(T-\tau)^\alpha\right)F_n(\tau)d\tau}{E_{\alpha,1}\left(\lambda_n T^\alpha\right)}\right)\psi_n(x) \\ &\quad + \sum_{\lambda_n\le N_\varepsilon}\left(\int_0^t (t-\tau)^{\alpha-2}E_{\alpha,\alpha-1}\left(\lambda_n(t-\tau)^\alpha\right)F_n(\tau)d\tau\right)\psi_n(x) \\ &= \mathbb{Q}_1(x,t) + \mathbb{Q}_2(x,t) + \mathbb{Q}_3(x,t). \end{aligned} \tag{59}$$

Let us now treat the error $\left\|W^\varepsilon(.,t)-u_t(.,t)\right\|_{L^2(\Omega)}$. It is obvious to claim that

$$\begin{aligned}\left\|W^\varepsilon(.,t)-u_t(.,t)\right\|^2_{L^2(\Omega)} &= \sum_{\lambda_n>N_\varepsilon}\Big(\int_\Omega u_t(x,t)\psi_n(x)dx\Big)^2\\ &= \sum_{\lambda_n>N_\varepsilon}\lambda_n^{-2\ell}\lambda_n^{2\ell}\Big(\int_\Omega u_t(x,t)\psi_n(x)dx\Big)^2\\ &\le (N_\varepsilon)^{-2\ell}\sum_{\lambda_n>N_\varepsilon}\lambda_n^{2\ell}\Big(\int_\Omega u_t(x,t)\psi_n(x)dx\Big)^2\\ &\le (N_\varepsilon)^{-2\ell}\left\|u_t(.,t)\right\|^2_{\mathbb{H}^\ell(\Omega)} \le (N_\varepsilon)^{-2\ell}\left\|u_t\right\|^2_{L^\infty(0,T;\mathbb{H}^\ell(\Omega))}. \end{aligned} \tag{60}$$

Our next aim is to prove the error $\left\|W^\varepsilon(.,t)-w^\varepsilon(.,t)\right\|_{L^2(\Omega)}$. It is clear to observe that

$$\left\|W^\varepsilon(.,t)-w^\varepsilon(.,t)\right\|_{L^2(\Omega)} \le \sum_{j=1}^{3}\left\|\mathbb{Q}_j^{N_\varepsilon}(.,t)-\mathbb{Q}_j(.,t)\right\|_{L^2(\Omega)}. \tag{61}$$

Using the bound (34), we give the following bound

$$\begin{aligned}\left\|\mathbb{Q}_1^{N_\varepsilon}(.,t)-\mathbb{Q}_1(.,t)\right\|^2_{L^2(\Omega)} &= \sum_{\lambda_n\le N_\varepsilon}\lambda_n^2 t^{2\alpha-2}\Big(\frac{E_{\alpha,1}(\lambda_n t^\alpha)}{E_{\alpha,1}(\lambda_n T^\alpha)}\Big)^2\Big(g_n^\varepsilon-g_n\Big)^2\\ &\le \frac{\rho_2^2T^{2\alpha-2}N_\varepsilon^2}{\rho_1^2}\left\|g^\varepsilon-g\right\|^2_{L^2(\Omega)} \le \frac{\rho_2^2T^{2\alpha-2}N_\varepsilon^2}{\rho_1^2}\varepsilon^2. \end{aligned} \tag{62}$$

In view of Parseval's equality, we obtain

$$\begin{aligned}&\left\|\mathbb{Q}_2^{N_\varepsilon}(.,t)-\mathbb{Q}_2(.,t)\right\|^2_{L^2(\Omega)}\\ &= \sum_{\lambda_n\le N_\varepsilon}\lambda_n^2 t^{2\alpha-2}\Big(\frac{E_{\alpha,1}(\lambda_n t^\alpha)}{E_{\alpha,1}(\lambda_n T^\alpha)}\Big)^2\left(\int_0^T (T-\tau)^{\alpha-1}E_{\alpha,\alpha}\left(\lambda_n(T-\tau)^\alpha\right)\left(F_n^\varepsilon(.,\tau)-F_n(.,\tau)\right)d\tau\right)^2\\ &\le \frac{\rho_2^2T^{2\alpha-2}}{\rho_1^2}\sum_{\lambda_n\le N_\varepsilon}\lambda_n^2\left(\int_0^T (T-\tau)^{\alpha-1}E_{\alpha,\alpha}\left(\lambda_n(T-\tau)^\alpha\right)\left(F_n^\varepsilon(.,\tau)-F_n(.,\tau)\right)d\tau\right)^2. \end{aligned} \tag{63}$$

In view of Hölder inequality, we arrive at

$$\begin{aligned}&\sum_{\lambda_n\le N_\varepsilon}\lambda_n^2\left(\int_0^T (T-\tau)^{\alpha-1}E_{\alpha,\alpha}\left(\lambda_n(T-\tau)^\alpha\right)\left(F_n^\varepsilon(\tau)-F_n(\tau)\right)d\tau\right)^2\\ &\le \sum_{\lambda_n\le N_\varepsilon}\lambda_n^2\left(\int_0^T (T-\tau)^{\alpha-1}E_{\alpha,\alpha}\left(\lambda_n(T-\tau)^\alpha\right)d\tau\right)\\ &\qquad\qquad\left(\int_0^T (T-\tau)^{\alpha-1}E_{\alpha,\alpha}\left(\lambda_n(T-\tau)^\alpha\right)\left(F_n^\varepsilon(\tau)-F_n(\tau)\right)^2d\tau\right)\\ &\le \sum_{\lambda_n\le N_\varepsilon}\lambda_n E_{\alpha,1}(\lambda_n T^\alpha)\left(\int_0^T (T-\tau)^{\alpha-1}E_{\alpha,\alpha}\left(\lambda_n(T-\tau)^\alpha\right)\left(F_n^\varepsilon(\tau)-F_n(\tau)\right)^2d\tau\right). \end{aligned} \tag{64}$$

Using (41) and the fact that $\lambda_n E_{\alpha,1}(\lambda_n T^\alpha) \leq \frac{\rho_2 N_\varepsilon}{\alpha} \exp\left(T N_\varepsilon^{\frac{1}{\alpha}}\right)$, $\lambda_n \leq N_\varepsilon$, we find that the following bound

$$\begin{aligned}
&\sum_{\lambda_n \leq N_\varepsilon} \lambda_n^2 \left(\int_0^T (T-\tau)^{\alpha-1} E_{\alpha,\alpha}\left(\lambda_n (T-\tau)^\alpha\right) \left(F_n^\varepsilon(\tau) - F_n(\tau)\right) d\tau \right)^2 \\
&\leq \frac{\rho_2 N_\varepsilon}{\alpha} \exp\left(T N_\varepsilon^{\frac{1}{\alpha}}\right) \sum_{\lambda_n \leq N_\varepsilon} \left(\int_0^T (T-\tau)^{\alpha-1} E_{\alpha,\alpha}\left(\lambda_n (T-\tau)^\alpha\right) \left(F_n^\varepsilon(\tau) - F_n(\tau)\right)^2 d\tau \right) \\
&\leq \frac{\rho_2}{\alpha} \frac{\rho_2 \overline{\rho}_2}{\alpha^2 \lambda_1} N_\varepsilon \exp\left(2T N_\varepsilon^{\frac{1}{\alpha}}\right) \int_0^T (T-\tau)^{\alpha-1} \left\| F^\varepsilon(.,\tau) - F(.,\tau) \right\|_{L^2(\Omega)}^2 d\tau. \qquad (65)
\end{aligned}$$

Since $F(x,t) = \varphi(t) f(x)$ and $F^\varepsilon(x,t) = \varphi^\varepsilon(t) f^\varepsilon(x)$, we obtain for any $0 \leq t \leq T$

$$\begin{aligned}
\left\| F^\varepsilon(.,t) - F(.,t) \right\|_{L^2(\Omega)} &= \left\| \varphi^\varepsilon(t) f^\varepsilon(.) - \varphi(t) f(.) \right\|_{L^2(\Omega)} \\
&\leq \left| \varphi^\varepsilon(t) \right| \left\| f^\varepsilon(.) - f(.) \right\|_{L^2(\Omega)} + \left\| f(.) \right\|_{L^2(\Omega)} \left| \varphi^\varepsilon(t) - \varphi(t) \right| \\
&\leq \varepsilon \left(\left\| \varphi^\varepsilon \right\|_{L^\infty(0,T)} + \left\| f \right\|_{L^2(\Omega)} \right), \qquad (66)
\end{aligned}$$

where we have used the assumption (53). By collecting many previous results (63), (64), (65), (66), we infer that

$$\left\| Q_2^{N_\varepsilon}(.,t) - Q_2(.,t) \right\|_{L^2(\Omega)}^2 \leq C(\rho_2, \alpha, \overline{\rho}_2, \lambda_1, T) N_\varepsilon \exp\left(2T N_\varepsilon^{\frac{1}{\alpha}}\right) \varepsilon^2 \left(\left\| \varphi^\varepsilon \right\|_{L^\infty(0,T)} + \left\| f \right\|_{L^2(\Omega)} \right)^2. \qquad (67)$$

Thus, we deduce that

$$\begin{aligned}
&\left\| Q_2^{N_\varepsilon}(.,t) - Q_2(.,t) \right\|_{L^2(\Omega)} \\
&\leq C(\rho_2, \alpha, \overline{\rho}_2, \lambda_1, T) \sqrt{N_\varepsilon} \exp\left(T N_\varepsilon^{\frac{1}{\alpha}}\right) \varepsilon \left(\left\| \varphi^\varepsilon \right\|_{L^\infty(0,T)} + \left\| f \right\|_{L^2(\Omega)} \right). \qquad (68)
\end{aligned}$$

Using Parseval's equality and Hölder inequality, we derive that

$$\begin{aligned}
&\left\| Q_3^{N_\varepsilon}(.,t) - Q_3(.,t) \right\|_{L^2(\Omega)}^2 \\
&= \sum_{\lambda_n \leq N_\varepsilon} \left(\int_0^t (t-\tau)^{\alpha-2} E_{\alpha,\alpha-1}\left(\lambda_n (t-\tau)^\alpha\right) \left(F_n^\varepsilon(\tau) - F_n(\tau)\right) d\tau \right)^2 \\
&\leq \sum_{\lambda_n \leq N_\varepsilon} \left(\int_0^t (t-\tau)^{\alpha-2} E_{\alpha,\alpha-1}\left(\lambda_n (t-\tau)^\alpha\right) d\tau \right) \\
&\qquad \left(\int_0^t (t-\tau)^{\alpha-2} E_{\alpha,\alpha-1}\left(\lambda_n (t-\tau)^\alpha\right) \left(F_n^\varepsilon(\tau) - F_n(\tau)\right)^2 d\tau \right). \qquad (69)
\end{aligned}$$

Using (66) and noting that

$$\varepsilon^2 \left(\left\| \varphi^\varepsilon \right\|_{L^\infty(0,T)} + \left\| f \right\|_{L^2(\Omega)} \right)^2 \geq \left\| F^\varepsilon(.,\tau) - F(.,\tau) \right\|_{L^2(\Omega)}^2 \geq \left(F_n^\varepsilon(\tau) - F_n(\tau)\right)^2,$$

we obtain that the following bound

$$\int_0^t (t-\tau)^{\alpha-2} E_{\alpha,\alpha-1}\left(\lambda_n (t-\tau)^\alpha\right) \left(F_n^\varepsilon(\tau) - F_n(\tau)\right)^2 d\tau$$
$$\le \varepsilon\left(\left\|\varphi^\varepsilon\right\|_{L^\infty(0,T)} + \left\|f\right\|_{L^2(\Omega)}\right)\left(\int_0^t (t-\tau)^{\alpha-2} E_{\alpha,\alpha-1}\left(\lambda_n (t-\tau)^\alpha\right) d\tau\right). \tag{70}$$

On other hand, if $\lambda_n \le N_\varepsilon$ then since (13) and Lemma (2.4), we find that

$$\int_0^t (t-\tau)^{\alpha-2} E_{\alpha,\alpha-1}\left(\lambda_n (t-\tau)^\alpha\right) d\tau = t^{\alpha-1} E_{\alpha,\alpha}\left(\lambda_n t^\alpha\right)$$
$$\le \frac{\overline{\rho}_2}{\alpha} \lambda_1^{\frac{1}{\alpha}-1} \exp(t\lambda_n^{\frac{1}{\alpha}}) \le \frac{\overline{\rho}_2}{\alpha} \lambda_1^{\frac{1}{\alpha}-1} \exp(tN_\varepsilon^{\frac{1}{\alpha}}). \tag{71}$$

By connecting three resusts (69), (70) and (71), we arrive at

$$\left\|\mathbb{Q}_3^{N_\varepsilon}(.,t) - \mathbb{Q}_3(.,t)\right\|_{L^2(\Omega)} \le \frac{\overline{\rho}_2}{\alpha} \lambda_1^{\frac{1}{\alpha}-1} \exp(tN_\varepsilon^{\frac{1}{\alpha}}) \varepsilon\left(\left\|\varphi^\varepsilon\right\|_{L^\infty(0,T)} + \left\|f\right\|_{L^2(\Omega)}\right). \tag{72}$$

Combining (60), (61), (62), (63) and (72), we deduce that

$$\left\|W^\varepsilon(.,t) - u_t(.,t)\right\|_{L^2(\Omega)} \le (N_\varepsilon)^{-\ell} \left\|u_t\right\|_{L^\infty(0,T;\mathbb{H}^\ell(\Omega))} + \frac{\rho_2 T^{\alpha-1} N_\varepsilon}{\rho_1}\varepsilon$$
$$+ C(\rho_2, \alpha, \overline{\rho}_2, \lambda_1, T)\left(\sqrt{N_\varepsilon} + 1\right) \exp\left(TN_\varepsilon^{\frac{1}{\alpha}}\right) \varepsilon\left(\left\|\varphi^\varepsilon\right\|_{L^\infty(0,T)} + \left\|f\right\|_{L^2(\Omega)}\right). \tag{73}$$

□

4 Declarations

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The authors declare that they have no competing interests.

Ethical Approval

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