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Asymptotic Behavior for Delay 2D Navier-Stokes Equations on Unbounded Channel-Like Domains

Yang Bai[1,*], **Xiaobin Yao**[1]

1. School of Mathematics and Statistics, Qinghai Minzu University, Xining, Qinghai, 810007, P.R. China

*Corresponding author

Abstract

This paper studies the global attractors of 2D Navier-Stokes equations with delay defined in unbounded Channel-like domains. We establish the uniform tail-ends estimates of the solutions by establishing all the solutions are uniformly small for overcome the non-compactness of the solutions.

Keywords: Navier-Stokes equations; global attractors; uniform tail-ends estimates
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1 Introduction

This paper considers the delay 2D Navier-Stokes equations defined on $\mathcal{O} = \mathbb{R} \times (0, d)$:

$$\begin{cases} u_t - \Delta u + \alpha u + (u \cdot \nabla)u = g(u_t) + \nabla p + f(x), \ \ divu = 0, \\ u(t,x) = 0, \ \ t > 0, x \in \partial\mathcal{O}, \\ u(0,x) = u_0(x), \ \ x \in \mathcal{O}, \\ u(x,t) = \phi(t-\tau, x), \ \ t \in (\tau - h, \tau), x \in \mathcal{O}, \end{cases} \tag{1}$$

where u and p are the velocity and the pressure of the fluid, $\alpha \geq 0$, $f \in L^2(\mathcal{O})$ is given, g is a Lipschitz nonlinear function with delay, $r \geq 0$ is constants,

$$u_t(\theta) = u(t+\theta), \ \forall\, \theta \in (-r, 0). \tag{2}$$

When $g(u_t) = 0$, then (2) is the standard Navier-Stokes equations. When (2) does not contain the delay term, Wang has investigated the existence of the global attractors of the 2D Navier-Stokes equations on the Channel-like unbounded domains in [1]. Caraballo has studied the attractors of 2D Navier-Stokes equations with delay on bounded domains in reference [2],

Contact: Yang Bai 1476812351@qq.com; Xiaobin Yao yaoxiaobin@qhmu.edu.cn;

and then proved the existence of pull-back attractors for non-autonomous 2D Navier-Stokes equations with delay on unbounded domains. This paper mainly studied the asymptotic compactness of the solutions of (2) defined in unbounded Channel-like domains $\mathcal{O}$. It is well known that such compactness is crucial for understanding the long term asymptotic behavior of the solutions, including the existence, stability and convergence of the global attractors. In bounded domains, the asymptotic compactness of Navier-Stokes equations follows the compactness of the standard Sobolev embeddings (see reference[3]). In unbounded domains, the main difficulty is non-compactness of Sobolev embeddings.

In fact, we can be overcome difficulty by Rosa's idea of the Ball energy equations proposed (in reference [4],[5]), which is the only method to establish asymptotic compactness of the Navier-Stokes equations solutions when $\mathcal{O}$ is unbounded in the phase space $L^2(\mathcal{O},\mathbb{R}^2)$ (see references [6] - [11]. In order to overcome the non-compactness of Sobolev embeddings in unbounded domains, we used uniform tail-ends estimates methods for the solutions is proposed in reference[12] to proved the asymptotic compactness of the solution to the reaction-diffusion equations on $\mathbb{R}^n$(see references [13],[14]).

In this paper, we will derive the uniform tail-ends estimates of solutions of the 2D Navier-Stokes equations under the suppose conditions $p = 0$ and $divu = 0$. To that end, we will employ the scalar stream function for the Navier-Stokes equations (see [6],[13]), and first derive the uniform tail-ends estimates of the stream functions (see lemma3.1 and lemma3.2). Based on the uniform tail-ends estimates, we immediately obtain the asymptotic compactness of solutions of (2) defined in $\mathcal{O}$ (see lemma3.1) and hence the existence of global attractors in H (see lemma3.2).

This paper is organized as following, we will review some basic concepts and results in Section 2. In Section 3, we will discuss the uniform estimates of the solutions in H and V. At last,we will devoted to the uniform tail-ends estimates of solutions of (2) as well as the asymptotic compactness and the existence of global attractors in H.

In the sequel, we will denote the norm and the inner product of the $L^2(\mathcal{O},\mathbb{R}^2)$ by $\|\cdot\|$ and $(\cdot,\cdot)$. The norm of V is defined by $\|u\|_V = \|\nabla u\|$ for $u \in V$. The duality between V and V^* is written as $\langle\cdot,\cdot\rangle$. we will denote the norm and the inner product of the $C_H = (-r, 0; H)$ by $(\cdot,\cdot)_{C_H}$ and $\|\cdot\|_{C_H}$.

2 Preliminaries

In this section, we recall some basic results and knowledges.

We first give the Poincare inequality: there exists $\lambda > 0$ such that

$$\|\nabla u\|^2 \geq \lambda\|u\|^2, \ \forall u \in H_0^1(\mathcal{O},\mathbb{R}^2). \tag{3}$$

Let $u, v, w \in V$, we denoted

$$b(u,v,w) = \sum_{i,j=1}^{2} \int_{\mathcal{O}} u_i \frac{\partial v_j}{\partial x_i} w_j dx. \tag{4}$$

Then we have for all $u, v, w \in V$,

$$b(u,v,w) = -b(u,w,v), \ b(u,v,v) = 0$$

and

$$|b(u,v,w)| \leq \|u\|_{L^4(\mathcal{O})}\|\nabla v\|\|w\|_{L^4(\mathcal{O})} \leq c\|u\|_V\|v\|_V\|w\|_V, \tag{5}$$

for every $c > 0$. By(4), we define a bilinear operator $B : V \times V \to V^*$ given by, for every $u, v, w \in V$,

$$\langle B(u,v), w\langle= b(u,v,w),$$

which along with (5) implies that for all $u, v \in V$,

$$\|B(u,v)\|_{V^*} \le c\|u\|_V \|v\|_V. \tag{6}$$

For $u \in V \cap H^2(\mathcal{O}, \mathbb{R}^2)$, we also have

$$\|B(u,u)\| \le c\|u\|^{\frac{1}{2}} \|\nabla u\| \|\Delta u\|^{\frac{1}{2}}. \tag{7}$$

We suppose

$$\alpha = \lambda - L_g.$$

We will make appropriate suppose for the delay term. Let $g : (-r, 0; H) \to L^2(\mathcal{O}, \mathbb{R}^2)$ satisfy the following conditions: (1) $g(0) = 0$; (2) there exists $L_g > 0$, such that $\forall\, \xi, \eta \in (-r, 0; H)$

$$|g(\xi) - g(\eta)| \le L_g \|\xi - \eta\|_{C_H}. \tag{8}$$

where $\|u\|_{C_H} = \sup_{-r \le s \le 0} |u(t+s, x)|$.

3 Uniform estimates of solutions

In this section, we will derive the uniform estimates of solutions of (2) in H and V.

Lemma 3.1. *If (2.6) holds, then for every $u_0 \in H$, the solution u of (2) satisfies for all $t \ge 0$,*

$$\|u(t)\|^2 + 2\int_0^t e^{(2\alpha-2)(s-t)} \|u(s)\|_V^2 ds \le e^{(2-2\alpha)t} \|u_0\|^2 + C_1,$$

and for all $t \ge r \ge 0$,

$$2\int_0^t e^{(2\alpha-2)(s-t)} \|u(s)\|_V^2 ds \le e^{(2-2\alpha)r} \|u_0\|^2 + C_1(1 + t - r),$$

where $C_1 > 0$ is a number depending only on λ, L_F and f, but not on t, r or u_0.

Proof. For $t > 0$,

$$\frac{d}{dt}\|u(t)\|^2 + 2\|\nabla u(t)\|^2 + 2\alpha\|u(t)\|^2 = 2(g(u_t), u(t)) + 2(f, u(t)). \tag{9}$$

By (8) we have

$$2(g(u_t), u(t)) \le 2|g(u_t)||u(t)| \le L_g \|u_t\|_{C_H}^2 + \|u\|^2. \tag{10}$$

By Young's inequality we have

$$2(f, u(t)) \le \|f\|^2 + \|u(t)\|^2. \tag{11}$$

By (3) and (9)-(11), we have for $t > 0$,

$$\frac{d}{dt}\|u(t)\|^2 + 2\|\nabla u(t)\|^2 + 2\alpha\|u(t)\|^2 \le 2\|u(t)\|^2 + L_g\|u_t\|^2 + \|f\|^2. \tag{12}$$

Solve (12) to obtain for all $t \ge 0$,

$$\|u(t)\|^2 + 2\int_0^t e^{(2\alpha-2)(s-t)} \|\nabla u(s)\|^2 ds \le e^{(-2\alpha+2)t}\|u_0\|^2 + \int_0^t e^{(2\alpha-2)(s-t)} \|u_t\|_{C_H} ds + e^{2-2\alpha}\|f\|^2. \tag{13}$$

Integrating (12) on (r, t) with $0 \le r \le t$, by (13) we have

$$2\int_r^t \|\nabla u(s)\|^2 ds \le \|u(r)\|^2 + \|f\|^2(t-r) \le e^{(2-2\alpha)r}\|u_0\|^2 + e^{2-2\alpha}\|f\|^2 + e^{2-2\alpha}\|f\|^2(t-r). \tag{14}$$

Then (13) and (14) conclude the proof. □

Next, we prove the uniform estimates of the solutions in V for $t > 0$ with initial data in H.

Lemma 3.2. *If (8) holds, then for every $u_0 \in H$, the solutions u of (2) satisfies for all $t \geq 1$,*

$$\|\nabla u(t)\|^2 + \int_t^{t+1} \|\Delta u(s)\|^2 ds + \int_t^{t+1} \|\partial_t u(s)\|^2 ds \leq C_2,$$

where $C_2 > 0$ is a number depending only on λ, L_g and u_0, but not on t.

Proof. By a limiting process, we will derive the uniform estimates. We have for $t \geq 0$,

$$\begin{aligned}\frac{d}{dt}\|\nabla u(t)\|^2 + 2\|\Delta u(t)\|^2 + 2\alpha\|\nabla u(t)\|^2 =& 2(B(u(t), u(t)), \Delta u(t)) \\ & - 2(g(u_t(t)), \Delta u(t)) - 2(f, \Delta u(t)).\end{aligned} \tag{15}$$

For the first term on the right-hand said of (15) we have

$$\begin{aligned}2(B(u(t), u(t)), \Delta u(t)) &\leq \|u\|_{L^4(\mathcal{O},\mathbb{R}^2)} \|\nabla u\|_{L^4(\mathcal{O},\mathbb{R}^4)} \|\Delta u\| \\ &\leq c_1\|u\|^{\frac{1}{2}}\|\nabla u\|\|\Delta u\|^{\frac{3}{2}} \leq \frac{1}{4}\|\Delta u\|^2 + c_2\|u\|^2\|\nabla u\|^4,\end{aligned} \tag{16}$$

where $c_2 > 0$ depends only on $\mathcal{O}$. By (8) and Young's inequality, we have

$$-2(g(u_t(t)), \delta u(t)) - 2(f, \Delta u(t)) \leq \frac{1}{4}\|\Delta u\|^2 + 8\|f\|^2 + 8L_g^2\|u_t\|_{C_H}^2 \tag{17}$$

It follows from (15)-(17) that for $t > 0$,

$$\frac{d}{dt}\|\nabla u(t)\|^2 + \frac{3}{2}\|\Delta u(t)\|^2 \leq c_2\|u\|^2\|\nabla u\|^4 + 8\|f\|^2 + 8L_g^2\|u_t\|^2. \tag{18}$$

By Lemma 3.1 we get that for all $t \geq 0$,

$$\|u(t)\|^2 + \int_t^{t+1} \|u(s)\|_V^2 ds \leq c_3,$$

where $c_3 > 0$ depends on λ, L_g, f and u_0, which implies that for all $t \geq 0$,

$$\int_t^{t+1} \|\nabla u(s)\|^2 ds \leq c_3, \quad \int_t^{t+1} \|u(s)\|^2\|\nabla u(s)\|^2 ds \leq c_3^2,$$

and by Gronwall Lemma we get for all $t \geq 1$,

$$\|\nabla u(t)\|^2 \leq c_4, \tag{19}$$

where $c_4 > 0$ depends on λ, L_g, f and u_0. Integrate (18) on $(t, t+1)$ to get

$$\int_t^{t+1} \|\Delta u(s)\|^2 ds \leq c_5,$$

where $c_5 > 0$ depends on λ, L_g, f and u_0. This (7) and (19) are proved. □

From Lemma 3.2, we have that the following uniform estimates for large time are also proved.

Lemma 3.3. *For every $R > 0$, there exists $T_0 = T_0(R) > 0$ such that for all $t \geq T_0$ and $u_0 \in H$ with $\|u_0\| \leq R$, the solutions u of (2) satisfies*

$$\|\nabla u(t)\|^2 + \int_t^{t+1} \|\Delta u(s)\|^2 ds + \int_t^{t+1} \|\partial_t u(s)\|^2 \leq C_3,$$

where $M_3 > 0$ is a number depending only on λ, L_g and f, but not on t or u_0.

4 Uniform tail-end estimates

In this section, we will prove the uniform estimates on the tail of the solutions of (2), which will be used to prove the asymptotic compactness of the solutions and the existence of global attractors in H.

Let $\phi : \mathbb{R} \to \mathbb{R}$ be a smooth function such that $0 \le \phi \le 1$ and

$$\phi(s) = 0, |s| \le \frac{1}{2}; \ \ \phi(s) = 1, |s| \le 1.$$

Note that there exists $C > 0$ such that $|\phi'(s)| + |\phi''(s)| \le C$ for all $s \in \mathbb{R}$. Given $k \in \mathbb{N}$ and $x = (x_1, x_2) \in \mathcal{O}$, let $\phi_k(x) = \phi(\frac{x_1}{k})$.

We first give the following inequalities.

Lemma 4.1. *(i)If $u \in H^1(\mathcal{O})$, then*

$$|\|\phi_k \nabla u\| - \|\nabla(\phi_k u)\|| \le \frac{C_4}{k}\|u\|.$$

(ii)If $u \in H^2(\mathcal{O})$, then

$$|\|\phi_k \Delta u\| - \|\Delta(\phi_k u)\|| \le \frac{C_4}{k}\|u\|_{H^1(\mathcal{O})}.$$

(iii)If $u \in H_0^1(\mathcal{O})$, then

$$\|\phi_k \nabla u\| \ge \lambda^{\frac{1}{2}}\|\phi_k u\| - \frac{C_4}{k}\|u\|.$$

(iv)If $u \in H^2(\mathcal{O}) \cap H_0^1(\mathcal{O})$, then

$$\|\phi_k \Delta u\| \ge \lambda^{\frac{1}{2}}\|\phi_k \nabla u\| - \frac{C_4}{k}\|u\|_{H^1(\mathcal{O})},$$

where $C_4 > 0$ is a number independent of k and u.

Proof. (i) Note that $\nabla(\phi_k u) = u\nabla\phi_k + \phi_k \nabla u$,

$$|\|\phi_k \nabla u\| - \|\nabla(\phi_k u)\|| \le \|\phi_k \nabla u - \nabla(\phi_k u)\| = \|u \nabla \phi_k\| \le \frac{c}{k}\|u\|,$$

where $c > 0$ is a number independent of k and u.
(ii) Suppose $\Delta(\phi_k u) = u(\Delta\phi_k) + \phi_k(\Delta u) + 2\nabla\phi_k \cdot \nabla u$ such that

$$\|\phi_k(\Delta u) - \Delta(\phi_k u)\| \le \|u(\Delta\phi_k)\| + 2\|\nabla\phi_k \cdot \nabla u\| \le \frac{c}{k}(\|u\| + \|\nabla u\|).$$

(iii) Since $\|\nabla(\phi_k u)\| \ge \lambda^{\frac{1}{2}}\|\phi_k u\|$, by (1) we get $\|\phi_k \nabla u\| \ge \lambda^{\frac{1}{2}}\|\phi_k u\| - \frac{c}{k}\|u\|$.
(iv) Note that

$$\begin{aligned}\|\nabla(\phi_k u)\|^2 &= (\nabla(\phi_k u), \nabla(\phi_k u)) = -(\Delta(\phi_k u), \phi_k u) \\ &\le \|\Delta(\phi_k u)\|\|\phi_k u\| \le \lambda^{\frac{1}{2}}\|\Delta(\phi_k u)\|\|\nabla(\phi_k u)\|\end{aligned}$$

and $\|\nabla(\phi_k u)\| \le \lambda^{-\frac{1}{2}}\|\Delta(\phi_k u)\|$. By (i) and (ii) we have

$$\begin{aligned}\|\phi_k \nabla u\| &\le \|\nabla(\phi_k u)\| + \frac{c_2}{k}\|u\| \le \lambda^{-\frac{1}{2}}\|\Delta(\phi_k u)\| + \frac{c_2}{k}\|u\| \\ &\le \lambda^{-\frac{1}{2}}(\|\phi_k \Delta u\| + \frac{c_3}{k}\|u\|_{H^1(\mathcal{O})}) + \frac{c_2}{k}\|u\| \le \lambda^{-\frac{1}{2}}\|\phi_k \Delta u\| + \frac{c_4}{k}\|u\|_{H^1(\mathcal{O})},\end{aligned}$$

which shows that

$$\lambda^{\frac{1}{2}}\|\phi_k \nabla u\| \leq \|\phi_k \Delta u\| + \frac{c_4}{k}\lambda^{\frac{1}{2}}\|u\|_{H^1(\mathcal{O})},$$

as proved.

Next, we give a scalar stream function for (2). Given $u = (u_1, u_2) \in V$ and $x = (x_1, x_2) \in \mathcal{O}$, define

$$\widehat{u}(x) = \widehat{u}(x_1, x_2) = \int_{(0,0)}^{(x_1,x_2)} -u_2 dx_1 + u_1 dx_2. \tag{20}$$

Since $u = 1$ on $\partial\mathcal{O}$ and $\operatorname{div}(u) = 0$ in $\mathcal{O}$, integral (20) is independent of the path of the integration. Therefore, we get

$$\partial_{x_1}\widehat{u} = -u_2, \ \ \partial_{x_2}\widehat{u} = u_1, \tag{21}$$

and

$$\widehat{u}|_{\partial\mathcal{O}} = 0, \quad \nabla\widehat{u}|_{\partial\mathcal{O}} = 0. \tag{22}$$

Let $\mathcal{L}$ be the curl operator given by

$$Tu = \partial_{x_2}u_1 - \partial_{x_1}u_2, \ \forall u = (u_1, u_2). \tag{23}$$

First applying (21), and then (23), we get from (2) the following equation for $\widehat{u}$

$$\partial_t \Delta\widehat{u} = \Delta^2\widehat{u} - \alpha\Delta\widehat{u} + \widehat{B}(\widehat{u}, \widehat{u}) + \widehat{g}(u_t) + \widehat{f}, \tag{24}$$

where

$$\widehat{g}(u_t) = T(g(u_t)), \ \ \widehat{f} = Tf, \ \ \widehat{B}(\widehat{u}, \widehat{u}) = \partial_{x_2}((\partial_{x_1}\widehat{u})\Delta\widehat{u}) - \partial_{x_2}((\partial_{x_2}\widehat{u})\Delta\widehat{u}).$$

For convenience, for every $k \in \mathbb{N}$, denote $\mathcal{O}_k = (-k, k) \times (0, d)$. By using equation (24) for the stream function, we can prove the uniform estimates on the tail of the solutions of (2) on $\mathcal{O} \setminus \mathcal{O}_k$ as below. □

Lemma 4.2. *For every $u_0 \in H$ and $\varepsilon > 0$, there exists $\mathcal{K} = \mathcal{K}(\lambda, L_g, f, u_0, \varepsilon) \geq 1$ such that the solution u of* (2) *satisfies, for all $t \geq 1$ and $k \geq K$,*

$$\int_{\mathcal{O}\setminus\mathcal{O}_k} |u(t, x)|^2 dx \leq \varepsilon.$$

Proof. We derive the uniform estimates, but all calculations can be justified by a limiting process. By (24) we get

$$-(\partial_t\Delta\widehat{u}, \phi_k^2\widehat{u}) = -(\Delta^2\widehat{u}, \phi_k^2\widehat{u}) - \alpha(\Delta\widehat{u}, \phi_k^2\widehat{u}) - (\widehat{B}(\widehat{u}, \widehat{u}), \phi_k^2\widehat{u}) - (\widehat{g}(u_t), \phi_k^2\widehat{u}) - (\widehat{f}, \phi_k^2\widehat{u}). \tag{25}$$

For the left-hand side of (25) we have

$$\begin{aligned} -(\partial_t\Delta\widehat{u}, \phi_k^2\widehat{u}) &= \frac{1}{2}\frac{d}{dt}\int_{\mathcal{O}} \phi_k^2|\nabla\widehat{u}|^2 dx + \int_{\mathcal{O}} (\nabla\widehat{u}_t, \nabla\phi_k^2)\widehat{u} dx \\ &\geq \frac{1}{2}\frac{d}{dt}\int_{\mathcal{O}} \phi_k^2|\nabla\widehat{u}|^2 dx - \frac{c_1}{k}\|\nabla\widehat{u}_t\|\|\widehat{u}\|, \end{aligned} \tag{26}$$

where $c_1 > 0$ is independent of k. For the first term on the right-hand side of (25) we obtain

$$\begin{aligned} -(\Delta^2\widehat{u}, \phi_k^2\widehat{u}) &= -\int_{\mathcal{O}} \Delta\widehat{u}(\phi_k^2\Delta\widehat{u} + (\Delta\phi_k^2)\widehat{u} + 2\nabla\phi_k^2 \cdot \nabla\widehat{u}) dx \\ &\leq -\|\phi_k\Delta\widehat{u}\|^2 + \frac{c_2}{k^2}\|\widehat{u}\|\|\Delta\widehat{u}\| + \frac{c_2}{k}\|\nabla\widehat{u}\|\|\Delta\widehat{u}\|. \end{aligned} \tag{27}$$

By Lemma 4.1 we find that there exists $c_3 > 0$ independent of k such that

$$\lambda^{\frac{1}{2}}\|\phi_k\nabla\widehat{u}\| \leq \|\phi_k\Delta\widehat{u}\| + \frac{c_3}{k}\|\widehat{u}\|_{H^1(\mathcal{O})}$$

and hence by Young's inequality we have

$$\begin{aligned}\lambda\|\phi_k\nabla\widehat{u}\|^2 &\leq \|\phi_k\Delta\widehat{u}\|^2 + \frac{2c_3}{k}\|\phi_k\Delta\widehat{u}\|\|\widehat{u}\|_{H^1\mathcal{O}} + \frac{c_3^2}{k^2}\|\widehat{u}\|_{H^1\mathcal{O}}\\ &\leq \|\phi_k\Delta\widehat{u}\|^2 + \frac{\alpha}{4\lambda-\alpha}\|\phi_k\Delta\widehat{u}\|^2 + \frac{(4\lambda-\alpha)c_3^2}{\alpha k^2}\|\widehat{u}\|^2_{H^1(\mathcal{O})} + \frac{c_3^2}{k^2}\|\widehat{u}\|^2_{H^1(\mathcal{O})}\\ &= \frac{4\lambda}{4\lambda-\alpha}\|\phi_k\Delta\widehat{u}\|^2 + \frac{4\lambda c_3^2}{\alpha k^2}\|\widehat{u}\|^2_{H^1(\mathcal{O})}.\end{aligned}$$

we have

$$-\|\phi_k\Delta\widehat{u}\|^2 \leq (\frac{1}{4}\alpha-\lambda)\|\phi_k\nabla\widehat{u}\|^2 + \frac{(4\lambda-\alpha)c_3^2}{\alpha k^2}\|\widehat{u}\|^2_{H^1(\mathcal{O})}, \tag{28}$$

By (27) and (28) we have

$$-(\Delta^2\widehat{u},\phi_k^2\widehat{u}) \leq (-\frac{1}{4}\alpha-\lambda)\|\phi_k\nabla\widehat{u}\|^2 + \frac{c_4}{k}\|\Delta\widehat{u}\|^2, \tag{29}$$

where $c_4 > 0$ is independent of k.

For the second term on the right-hand side of (25) we obtain

$$\begin{aligned}-\alpha(\Delta\widehat{u},\phi_k^2\widehat{u}) &= \alpha\int_{\mathcal{O}}(\nabla\widehat{u}\cdot\nabla\phi_k^2\widehat{u})\widehat{u}dx + \alpha\int_{\mathcal{O}}\phi_k^2|\nabla u|^2dx\\ &\leq \alpha\int_{\mathcal{O}}\phi_k^2|\nabla u|^2dx + \frac{c}{k}\|\nabla\widehat{u}\|\|\widehat{u}\|.\end{aligned} \tag{30}$$

For the second term on the right-hand side of (25), by the divergence formula we obtain

$$\begin{aligned}-(\widehat{B}(\widehat{u},\widehat{u}),\phi_k^2\widehat{u}) &= -\int_{\mathcal{O}}\widehat{u}(\Delta\widehat{u})(\partial_{x_2}\widehat{u})(\partial_{x_1}\phi_k^2)dx\\ &\leq \frac{c_5}{k}\|\Delta\widehat{u}\|\|\widehat{u}\|_{L^4(\mathcal{O})}\|\partial_{x_2}\widehat{u}\|_{L^4(\mathcal{O})}\\ &\leq \frac{c_6}{k}\|\widehat{u}\|^{\frac{3}{2}}\|\nabla\widehat{u}\|^{\frac{3}{2}} \leq \frac{c_7}{k}(\|\Delta\widehat{u}\|^2 + \|\nabla\widehat{u}\|^6),\end{aligned} \tag{31}$$

where $c_7 > 0$ is independent of k.

Suppose $g(u_t) = (g_1(u_t), g_2(u_t))$ and $f = (f_1, f_2)$. Then for the last two terms on the right-hand side of (25), by (8), (21) we have

$$\begin{aligned}-(\widehat{g}(u_t),\phi_k^2\widehat{u}) - (\widehat{f},\phi_k^2\widehat{u}) &= \int_{\mathcal{O}}(-g_2(u_t),g_1(u_t))\cdot\nabla(\phi_k^2\widehat{u})dx + \int_{\mathcal{O}}(-f_2,f_1)\cdot\nabla(\phi_k^2\widehat{u})dx\\ &\leq \int_{\mathcal{O}}\phi_k^2|\nabla\widehat{u}|(|g(u_t)|_{C_H} + |f|)dx + \int_{\mathcal{O}}|\widehat{u}|(|g(u_t)| + |f|)|\nabla\phi_k^2|dx\\ &\leq L_g\int_{\mathcal{O}}\phi_k^2|\nabla\widehat{u}|^2_{C_H}dx + \int_{\mathcal{O}}\phi_k^2|\nabla\widehat{u}||f|dx + \int_{\mathcal{O}}|\widehat{u}|(L_g|\nabla\widehat{u}| + |f|)|\nabla\psi_k^2|dx\\ &\leq (L_g + \frac{1}{4}\alpha)\int_{\mathcal{O}}\phi_k^2|\nabla\widehat{u}|^2_{C_H}dx + \frac{1}{\alpha}\int_{\mathcal{O}}\phi_k^2|f|^2dx + \frac{c_8}{k}(\|\nabla\widehat{u}\|^2 + \|f\|^2),\end{aligned} \tag{32}$$

where $c_8 > 0$ is independent of k. which along with (25)-(32) we have

$$\frac{d}{dt}\|\phi_k\nabla\widehat{u}\|^2 + \alpha\|\phi_k\nabla\widehat{u}\|^2 \leq \frac{c_9}{k}(\|\nabla\widehat{u}_t\|^2_{C_H} + \|\nabla\widehat{u}\|^6 + \|\Delta\widehat{u}\|^2 + \|f\|^2) + c_9\|\phi_k f\|^2, \tag{33}$$

where $c_9 > 0$ is independent of k. By (21) and (33) we have

$$\frac{d}{dt}\|\phi_k u\|^2 + \alpha\|\phi_k u\|^2 \leq \frac{c_9}{k}(\|u_t\|^2_{C_H} + \|u\|^6 + 2\|\nabla u\|^2 + \|f\|^2) + c_9\|\phi_k f\|^2. \tag{34}$$

Solve (34) on $(1,t)$ for $t \geq 1$ to have

$$\begin{aligned}\|\phi_k u(t)\|^2 &\leq e^{\alpha(1-t)}\|\phi_k u(1)\|^2 \\ &\quad + \frac{c_9}{k}\int_1^t e^{\alpha(s-t)}(\|u_t(s)\|^2_{C_H} + \|u(s)\|^6 + 2\|\nabla u(s)\|^2 + \|f\|^2)ds + c_9\alpha^{-1}\|\phi_k f\|^2. \end{aligned}\tag{35}$$

For the first term on the right-hand side of (35), we have for $t \geq 1$,

$$e^{\alpha(1-t)}\|\phi_k u(1)\|^2 \leq \int_{\{(x_1,x_2)\in\mathcal{O}:|x_1|\geq\frac{1}{2}k\}} |u(1,x)|^2 dx \to 0, \ \ k \to \infty. \tag{36}$$

For the second term on the right-hand side of (35), by Lemma 3.1 and Lemma 3.2 we have for $t \geq 1$,

$$\begin{aligned}&\frac{c_9}{k}\int_1^t e^{\alpha(s-t)}(\|u_t(s)\|^2_{C_H} + \|u(s)\|^6 + 2\|\nabla u(s)\|^2 + \|f\|^2)\\
&\leq \frac{c_9}{k}(c_{10} + \|f\|)\int_1^t e^{\alpha(s-t)}ds + \frac{c_9}{k}\int_1^t e^{\alpha(s-t)}\|u_t(s)\|^2_{C_H}ds\\
&\leq \frac{c_9}{k}(c_{10} + \|f\|) + \frac{c_9}{k}(\int_1^2 e^{\alpha(s-t)}\|u_t(s)\|^2_{C_H}ds + \cdots + \int_{[t]}^t e^{\alpha(s-t)}\|u_t(s)\|^2_{C_H}ds)\\
&\leq \frac{c_9}{k}(c_{10} + \|f\|) + \frac{c_9}{k}(e^{\alpha(2-t)}\int_1^2 \|u_t(s)\|^2_{C_H}ds + \cdots\\
&+ e^{\alpha([t]-t)}\int_{[t]-1}^{[t]} \|u_t(s)\|^2_{C_H}ds + \int_{[t]}^t \|u_t(s)\|^2_{C_H}ds)\\
&\leq \frac{c_{11}}{k} + \frac{c_{11}}{k}(e^{\alpha(2-t)} + \cdots + e^{\alpha([t]-t)}) \leq \frac{c_{11}}{k} + \frac{c_{11}}{k}e^{\alpha([t]-t)}\Sigma_{j=0}^{\infty}r^{-\alpha j}\\
&\leq \frac{c_{11}}{k} + \frac{c_{11}}{k}(1-e^{-\alpha})^{-1} \to 0, \ \ k \to \infty,\end{aligned}\tag{37}$$

where $c_{11} > 0$ depends on λ, L_g, f and u_0, but not on k.

Since $f \in H$, for the last term on the right-hand side of (35), we have

$$c_9\alpha^{-1}\|\phi_k f\|^2 = c_9\alpha^{-1}\int_{\{(x_1,x_2)\in\mathcal{O}:|x_1|\geq\frac{1}{2}k\}} |f(x)|^2 dx \to 0, \ \ k \to \infty. \tag{38}$$

It follows from (34)-(38) that for every $\varepsilon > 0$, there exists $K = K(\lambda, L_g, f, u_0, \varepsilon) \geq 1$, such that for all $t \geq 1$ and $k \geq K$,

$$\int_{\{(x_1,x_2)\in\mathcal{O}:|x_1|\geq k\}} |u(t,x)|^2 dx \leq \|\phi_k u(t)\|^2 \leq \varepsilon,$$

are proved. □

Next, we consider an improvement of Lemma4.3 for the tail-ends estimates of the solutions which are uniform with respect to bounded initial data in H.

Lemma 4.3. *For every $R > 0$ and $\varepsilon > 0$, there exists $T_1 = T_1(\lambda, L_g, f, R, \varepsilon) > 0$ and $K = K(\lambda, L_g, f, \varepsilon) \geq 1$, such that if $u_0 \in H$ with $\|u_0\| \leq R$, then the solution u of (1.1) satisfies, for all $t \geq T_1$ and $k \geq K$,*

$$\int_{\mathcal{O}\setminus\mathcal{O}_k} |u(t,x)|^2 dx \leq \varepsilon.$$

Proof. Let $u_0 \in H$ with $\|u_0\| \leq R$. Then by Lemma 3.3 we have that there exists $T_0 = T_0(R) > 0$ such that for all $t \geq T_0$,

$$\|\nabla u(t)\|^2 + \int_t^{t+1} \|\partial_t u(s)\|^2 ds \leq C_1, \tag{39}$$

where $C_1 > 0$ depends only on λ, L_g and f, but not on u_0 or R.

Solve (34) on (T_0, t) for $t \geq T_0$ to obtain

$$\begin{aligned}&\|\phi_k u(u)\|^2 \leq e^{\alpha(T_0-t)}\|\phi_k u(T_0)\|^2 \\ &\quad + \frac{c_9}{k}\int_{T_0}^t e^{\alpha(s-t)}(\|u_t(s)\|^2_{C_H} + \|u(s)\|^6 + 2\|\nabla u(s)\|^2 + \|f\|^2)ds + c_9\alpha^{-1}\|\phi_k f\|^2. \end{aligned} \tag{40}$$

For the first term on the right-hand side of (40), by Lemma 3.1 we have for $t \geq T_0$,

$$e^{\alpha(T_0-t)}\|\phi_k u(T_0)\|^2 \leq e^{\alpha(T_0-t)}\|u(T_0)\|^2 \leq e^{\alpha(T_0-t)}(e^{-\frac{1}{2}\alpha T_0}R^2 + C_2),$$

where $C_2 > 0$ depends only on λ, L_g and f, but not on u_0 or R, for every $\varepsilon > 0$, there exists $T_1 = T_1(\lambda, L_g, f, R, \varepsilon) \geq T_0$ such that for all $t \geq T_1$,

$$e^{\alpha(T_0-t)}\|\phi_k u(T_0)\|^2 \leq \frac{1}{4}\varepsilon. \tag{41}$$

For the second term on the light-hand side of (40), by (39) we have for $t \geq T_0$,

$$\begin{aligned}&\frac{c_9}{k}\int_{T_0}^t e^{\alpha(s-t)}(\|u_t(s)\|^2_{C_H} + \|u(s)\|^6 + 2\|\nabla u(s)\|^2 + \|f\|^2) \\ &\leq \frac{C_3}{k} + \frac{C_3}{k}(e^{\alpha(T_0+1-t)}\int_{T_0}^{T_0+1}\|u_t(s)\|^2_{C_H}ds + \cdots \\ &+ e^{\alpha([t]-t)}\int_{[t]-1}^{[t]}\|u_t(s)\|^2_{C_H}ds + \int_{[t]}^t \|u_t(s)\|^2_{C_H}ds) \\ &\leq \frac{C_3}{k} + \frac{C_1C_3}{k} + \frac{C_1C_3}{k}(1-e^{-\alpha})^{-1} \to 0, \quad k \to \infty, \end{aligned} \tag{42}$$

where $C_3 > 0$ depends only on λ, L_g and f, but not on k, u_0 or R.

By(39)-(42) and (38), we have that there exists $K = K(\lambda, L_g, \varepsilon) \geq 1$ such that for all $t \geq T_1$ and $k \geq \mathcal{K}$,

$$\int_{\{(x_1,x_2)\in\mathcal{O}:|x_1|\geq k\}} |u(t,x)|^2 dx \leq \|\phi_k u(t)\|^2 \leq \varepsilon.$$

By a consequence of Lemma4.3, we have the asymptotic compactness of the solutions in H. □

Lemma 4.4. *Under condition (2.6), if $\{u_{0,n}\}_{n=1}^\infty$ is bounded in H and $t_n \to \infty$, then $\{S(t_n)u_{0,n}\}_{n=1}^\infty$ is precompact in H.*

Proof. Since $\{u_{0,n}\}_{n=1}^\infty$ is bounded in H, by Lemma4.3 we have that for every $\varepsilon > 0$, there exist $N_1 = N_1(\varepsilon) \geq 1$ and $\mathcal{K} = \mathcal{K}(\varepsilon) \geq 1$ such that for all $n \geq N_1$,

$$\|S(t_n)u_{0,n}\|_{L^2(\mathcal{O}\setminus\mathcal{O}_K)} < \frac{\varepsilon}{4}. \tag{4.23}$$

By Lemma3.3 we have that there exists $N_2 = N_2(\varepsilon) \geq N_1$ such that $\{S(t_n)u_{0,n}\}_{n=N_2}^\infty$ is bounded in V. Then by the compactness of the embedding $H^1(\mathcal{O}_K) \hookrightarrow L^2(\mathcal{O}_K)$ we infer that $\{S(t_n)u_{0,n}\}_{n=1}^\infty$ has a finite cover of radius $\frac{1}{4}$ in $L^2(\mathcal{O}_K)$, which along with (42) implies that $\{S(t_n)u_{0,n}\}_{n=1}^\infty$ has a finite cover of radius ε in H.

From Lemma3.3 and 4.4, the existence of global attractors of (2) follows. □

Theorem 4.5. *If (8) holds true, then system (2) has a unique global attractors in H.*

5 Conclusion

We establish the uniform estimates of 2D Navier-Stokes equation of the solution in H and v to obtain uniform estimates of the solution. We proved the asymptotic compactness and the existence of global attractors in H by the uniform tail-end estimate.

6 Declarations

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Conceptualization, Y. Bai; methodology, Y. Bai and X. Yao; investigation, Y. Bai;writing-original draft preparation, Y. Bai; writing-review and editing, X. Yao.

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