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A Hybrid Approach to Approximate and Exact Solutions for Linear and Nonlinear Fractional-Order Schrödinger Equations with Conformable Fractional Derivatives

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Abstract

Fractional-order Schrödinger differential equations extend the classical Schrödinger equation by incorporating fractional calculus to describe more complex physical phenomena. The Schrödinger equations are solved using fractional derivatives expressed through the Caputo derivative. However, there is limited research on exact and approximate solutions involving conformable fractional derivatives. This study aims to address this gap by employing a hybrid approach that combines the Elzaki transform with the decomposition technique to solve the Schrödinger equation with conformable fractional derivatives, considering both zero and nonzero trapping potentials. The efficiency of this approach is evaluated through the analysis of relative and absolute errors, confirming its accuracy. Moreover, the obtained results are compared with other techniques, including the homotopy analysis method (HAM) and the residual power series method (RPSM). The comparison demonstrates strong consistency with these methods, suggesting that our approach is a viable alternative to Caputo derivative-based methods for solving time-fractional Schrödinger equations. Furthermore, we conclude that the conformable fractional derivative is a suitable substitute for the Caputo derivative in modeling Schrödinger equations.

Key words: approximate solutions, Elzaki transform, exact solutions, Adomian decomposition method, Schrödinger differential equations
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1 Introduction to this Template

Fractional calculus (FC) generalizes traditional calculus to non-integer orders, offering powerful tools for modeling and analyzing systems with memory and hereditary properties. Instead of dealing exclusively with integer-order derivatives and integrals, FC involves operations of differentiation and integration of fractional orders.

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The history of FC dates back to the 17th century, with the concept of fractional derivatives and integrals emerging alongside the development of traditional calculus. Here is a brief overview of the history of FC [1, 2, 3]:

i. Beginnings (17th Century)
   - Leibniz and L'Hôpital: Leibniz introduced the idea of non-integer order derivatives in correspondence with L'Hôpital, pondering the concept of a half-order derivative.

ii. Exploration and Early Development (18th and 19th Centuries)
   - Euler: Leonhard Euler made early contributions to the theory of FC, investigating the properties of the gamma function and its connection to FC.
   - Liouville: Joseph Liouville expanded upon Euler's work, developing integral representations for fractional derivatives and integrals.
   - Grünwald and Letnikov: Independently, Grünwald and Letnikov formulated discrete versions of fractional derivatives, laying down the groundwork for numerical methods in FC.
   - Riemann-Liouville FC: Bernhard Riemann and Joseph Liouville developed the Riemann-Liouville approach to FC, providing a rigorous mathematical framework for fractional derivatives and integrals.

iii. Consolidation and Formalization (20th Century)
   - Caputo FC: Michele Caputo introduced a modification to the Riemann-Liouville approach known as the Caputo fractional derivative, which is widely used due to its compatibility with initial value problems.
   - Modern Developments: Throughout the 20th century, mathematicians continued to explore and develop the theory of FC, advancing its applications in various scientific disciplines.

iv. Resurgence and Application (Late 20th Century to Present):
   - Resurgence of Interest: FC experienced a resurgence of interest in the late 20th century and continues to be an active area of research, driven by its applications in physics, engineering, biology, finance, and signal processing.
   - Applications: FC finds applications in modeling anomalous transport phenomena, viscoelastic materials, biological processes, financial markets, signal processing, and many other fields.

FC offers several advantages over traditional integer-order calculus, particularly when dealing with complex, real-world systems. Here are some of the key benefits [4, 5, 6, 7, 8, 9]:

i. Modeling Complex Phenomena: FC provides a powerful framework for modeling and analyzing complex phenomena that exhibit non-local interactions, memory effects, and fractal properties. Systems with long-range dependencies, anomalous diffusion, and hereditary behaviors can be accurately described using fractional differential equations.

ii. Memory and Non-local Effects: FC captures memory effects and non-local interactions that are not adequately addressed by classical calculus. Fractional derivatives and integrals allow for the incorporation of memory kernels and long-range dependencies, leading to more realistic models of dynamic systems.

iii. Flexibility in Representation: FC offers a flexible and versatile approach to representing and analyzing dynamic systems. The fractional order provides a continuous spectrum of differentiation and integration, allowing for a finer-grained characterization of system dynamics compared to integer-order models.

iv. Fractional Dynamics: FC enables the study of fractional-order dynamical systems, which exhibit rich and diverse behaviors. Fractional dynamics encompasses phenomena such as fractional-order chaos, fractal patterns, and anomalous transport processes, providing insights into the underlying mechanisms governing complex systems.

v. Improved Accuracy: In many cases, FC provides more accurate descriptions of real-world phenomena compared to classical integer-order models. By capturing memory effects and long-range dependencies, FC can better match experimental observations and empirical data, leading to more reliable predictions and interpretations.

vi. Generalization of Classical Calculus: FC generalizes classical calculus by extending differentiation and integration to non-integer orders. This allows for a seamless transition between classical and FC, providing a unified framework for addressing a wide range of problems in mathematics, science, and engineering.

vii. Applications in Various Fields: FC has diverse applications across numerous fields, including physics, engineering, biology, finance, and signal processing. It is used to model and analyze phenomena such as diffusion processes, viscoelasticity, population dynamics, financial time series, and medical imaging, among others.

viii. Novel Mathematical Properties: FC possesses unique mathematical properties that distinguish it from classical calculus. It introduces fractional-order operators, fractional Taylor series, and fractional boundary value problems, enriching the theoretical toolkit of mathematicians and researchers.

Fractional derivatives are the cornerstone of FC, offering a powerful tool for describing and analyzing systems with non-local or memory-dependent behavior. Unlike integer-order derivatives, which only consider instantaneous rates of change at a given point, fractional derivatives take into account the function's behavior over a certain interval, incorporating memory into the system's dynamics. Fractional-order derivatives come in various forms, each with its own definition and properties. Here are some of the commonly used types of fractional-order derivatives: Riemann-Liouville, Caputo, Grünwald-Letnikov, Caputo-Fabrizio, and conformable fractional operators [10, 11, 12, 13, 14, 15]. Each definition has its own unique properties and advantages, and the selection of a definition depends on the specific requirements of the problem and the desired mathematical characteristics of the derivative.

Many researchers have recently been actively engaged in studying various types of fractional-order derivatives. For example, the authors [16, 17] established very important results in the sense of Caputo-Fabrizio. The authors [18] introduced a new approach to solving fractional-order differential equations (FODEs) using conformable fractional operators. Zhang et al. [19] discovered solutions to differential equations (DEs) within the framework of Caputo fractional derivatives (Cap-FD). Zhang and Xiong [20] demonstrated global exponential stability and the existence of unique solutions for the periodic solution of FODEs with semilinear impulses using the Caputo derivative framework. Syam and Al-Refai [21] established several useful results regarding solutions to FODEs using Atangana-Baleanu derivatives. Li and Wang [22] found solutions to fractional Rössler chaotic systems using Grünwald-Letnikov fractional derivatives. For further study, refer to [23, 24, 25, 26].

Conformable fractional derivatives (Con-FrD) provide a useful and intuitive way to incorporate memory effects into DEs. Con-FrD retains many properties of classical derivatives, making it easier to work with while offering the flexibility of FC. For this reason, Con-FrD is very useful for simulating real-world phenomena in physics, engineering, control systems, biology, and other fields.

The Con-FrD of a function $\mathrm{Y}(\mathrm{t})$ of order β is defined as [27]:

$$\mathrm{D}_{\mathrm{t}}^{\beta}\mathrm{Y}(\mathrm{t}) = \lim_{\epsilon\to 0} \frac{\mathrm{Y}^{\lceil\beta\rceil-1}(\mathrm{t}+\epsilon \mathrm{t}^{\lceil\beta\rceil-\beta}) - \mathrm{Y}^{\lceil\beta\rceil-1}(\mathrm{t})}{\epsilon}, \tag{1}$$

where the Con-FrD with regard to time is denoted by D_t^{β}, and the smallest integer that is equal to or greater than β is $\lceil\beta\rceil$, $\beta \in \mathscr{N}$, and $\beta - 1 < \beta \leq \beta$, $t > 0$.

In a specific case, if $0 < \beta \leq 1$, we then obtain

$$D_t^{\beta} Y(t) = \lim_{\epsilon \to 0} \frac{Y(t + \epsilon t^{1-\beta}) - Y(t)}{\epsilon}, \ t > 0. \tag{2}$$

The conformable fractional integral is defined as follows:

$$\mathfrak{I}_{\beta}^{\tilde{\alpha}}(Y)(t) = \int_{\tilde{\alpha}}^{t} \frac{Y(\wp)}{\left(\wp - \tilde{\alpha}\right)^{1-\beta}} d\wp, \ \beta \in (0,\ 1]. \tag{3}$$

Con-FrD provides an alternative framework for fractional differentiation that simplifies the computation of fractional derivatives and maintains some key properties of classical derivatives. The Con-FrD offers several advantages compared to other types of fractional derivatives. Here are some of the key benefits [28, 29, 30, 31]:

i Simplicity and Intuition: Con-FrD are designed to be a straightforward generalization of the classical derivative, making them easier to understand and apply. They retain many of the properties of integer-order derivatives, which can simplify the transition from classical to FC.

ii Consistency with Classical Calculus: Unlike some other fractional derivatives, the conformable derivative maintains a closer relationship to classical calculus. It satisfies properties like the product rule, chain rule, and power rule in a manner that is more consistent with classical derivatives, making it more intuitive for those familiar with traditional calculus.

iii Clear Physical Interpretation: Conformable derivatives offer a clearer physical interpretation compared to some other fractional derivatives, as they are often more closely related to the original physical quantities and processes. This makes them useful in modeling real-world phenomena where a straightforward interpretation is desired.

iv Ease of Application: Due to their simpler form and consistency with classical calculus rules, Con-FrD are often easier to apply in various mathematical and engineering problems. They can be used in solving DEs, modeling systems, and analyzing dynamic behaviors without requiring complex modifications to existing methods.

v Flexibility in Modeling: Conformable derivatives provide a flexible framework for modeling systems that exhibit non-integer order dynamics, such as memory effects or anomalous diffusion, without the complexity that comes with some other fractional derivatives. This makes them a practical choice for applications in physics, engineering, biology, and finance.

vi Compatibility with Numerical Methods: The simpler and more intuitive nature of conformable derivatives often leads to easier implementation in numerical methods. This can be beneficial for computational modeling and simulation, where the goal is to approximate solutions to FODEs.

The FODEs are an advanced mathematical tool that extends the concept of traditional DEs to non-integer orders of differentiation and integration. This generalization provides a more flexible and accurate framework for modeling complex systems that exhibit memory, hereditary properties, or anomalous diffusion, which are not adequately captured by traditional integer-order models. The power of FODEs lies in their ability to model processes where the rate of change is not merely dependent on the current state but also influenced by the history of the system. This is particularly relevant in fields like physics, engineering, biology, and finance, where systems often have memory effects, such as viscoelastic materials, anomalous transport phenomena, and financial markets with long-term dependencies.

In the field of FODEs, several well-known models have been developed to describe complex systems with memory and hereditary properties. Among these models, the fractional

Schrödinger differential equations (FSDEs) stands out as particularly important. This equation extends the classical Schrödinger differential equation (SDE) by incorporating FC, allowing for a more accurate description of quantum systems with anomalous diffusion and other non-standard behaviors. Typically, the fractional derivative is introduced into either the time or spatial components of the SDE, leading to modifications in the wave function's behavior, with potential applications in quantum mechanics and other fields such as statistical mechanics and condensed matter physics.

FSDEs have diverse applications in various fields of science and engineering, primarily due to their ability to model systems with memory and non-local interactions. Some of the key applications include [32, 33, 34, 35]:

i. Quantum Mechanics: FSDEs are used to model quantum systems where the standard SDEs may not be sufficient, such as systems with fractal geometries, complex potentials, or in scenarios involving anomalous diffusion. They help in understanding the quantum behavior in disordered or complex materials.
ii. Condensed Matter Physics: In systems with fractal or irregular structures, like certain types of semiconductors, polymers, or amorphous materials, FSDEs can provide more accurate descriptions of the electronic properties, enabling better predictions of material behavior.
iii. Quantum Optics: FSDEs are applied in the study of light-matter interactions, particularly in media with non-standard refractive indices or in systems exhibiting anomalous dispersion. This can lead to new insights into photon transport and light localization.
iv. Statistical Mechanics: In statistical mechanics, FSDEs describe systems with long-range interactions or memory effects. This is particularly useful in modeling complex systems like glasses, spin systems, or in understanding anomalous transport phenomena.
v. Signal Processing and Image Analysis: FSDEs have been employed in signal processing, particularly in the analysis of signals with fractal characteristics or in the context of time-series analysis. Similarly, they are used in image processing, especially for edge detection or texture analysis in images with irregular patterns.
vi. Biophysics: In biological systems where anomalous diffusion or memory effects are present, such as in the transport of molecules within cellular environments or in modeling neuron dynamics, FSDEs can provide more accurate models.
vii. Financial Mathematics: Fractional order models, including those derived from the SDEs, are used in financial mathematics to describe markets with memory or to model the dynamics of prices and options in more complex financial environments.

The solutions to FSDEs are crucial for understanding, modeling, and predicting the behavior of complex systems that exhibit memory effects, non-local interactions, anomalous diffusion, and other phenomena. In the literature, the SDE has been solved using fractional derivatives represented in the form of the Cap-FD. For example, FSDEs have been solved using various methods [36, 37, 38, 39, 40, 41, 42]. However, the algorithms of each of these methods have been applied in the sense of Caputo derivatives. No research work has yet been employed to find approximate solutions (App-Ss) and exact solutions (Ex-Ss) for FSDEs using the Adomian decomposition method (ADM) with the Elzaki transform (ET) in the sense of Con-FrD. In this research, we addressed this gap by solving linear and nonlinear FSDEs using a hybrid approach that combines ADM and ET, applied in the sense of Con-FrD. We refer to it as the Elzaki-Adomian decomposition method (EADM). Moreover, the correctness of EADM is assessed through an examination of absolute errors (Abs-E) and relative errors (Rel-E) presented in both numerical and graphical representations. It is observed that the App-S rapidly approaches the Ex-S, as evidenced by the evaluation of 2D and 3D graphs across various fractional-order values. The numerical and graphical findings confirm the notable precision and effectiveness of EADM. Furthermore, the EADM results are compared with those from alternative techniques, including RPSM [37] and HAM [41]. The comparison demonstrates a high degree of agreement with these methods, suggesting that EADM is a viable substitute for Cap-FD-based techniques in solving FSDEs.

The ET is based on the traditional Fourier integral. The ET was established by Tarig. M. Elzaki to facilitate the time-domain solution of ordinary and partial DEs. It is a very successful approach to solving fractional problems. Some of its key benefits include:

i. Simpler calculations: The ET simplifies the process of solving DEs, especially FODEs, as it avoids the need for complex convolution products that are often required in Laplace transforms.
ii. Effective for FODEs: It provides an efficient way to solve FODEs, including those involving fractional derivatives in the Caputo or conformable sense, often leading to more straightforward solutions.
iii. Handling discontinuous functions: The ET is better suited for dealing with discontinuous or piecewise functions due to its unique formulation, making it more versatile in practical applications.
iv. Faster convergence: The ET is known for its faster convergence compared to other transforms, which can lead to more accurate results in numerical computations.
v. Easier inversion process: Inversion of the ET is relatively easier, offering a clear advantage when reconstructing solutions from the transformed domain.

The ET is defined for exponentially ordered functions. We examine functions defined within the set $\mathbb{Q}$.

$$\mathbb{Q} = \{\mathrm{Y}(\mathrm{t}) | \exists \mathbb{P}, \mathbb{H}_1, \mathbb{H}_2 > 0, |\mathrm{Y}(\mathrm{t}) < \mathbb{P} \exp^{|\lambda| \mathbb{H}_\ell} \ if\ \mathrm{Y} \in (-1)^\ell \times [0, \infty)\}.$$

The constant $\mathbb{P}$ for a given function in the set $\mathbb{Q}$ must be a finite value; $\mathbb{H}_1$, $\mathbb{H}_2$ can be either finite or infinite.

The ET is defined as follows [43]:

$$\mathbb{E}[\mathrm{Y}(\mathrm{t})] = \mathrm{Y}^*(\mathrm{s}) = \mathrm{s} \int_0^\infty \mathrm{Y}(\mathrm{t}) \exp^{-\frac{\mathrm{t}}{\mathrm{s}}} d\mathrm{t},\ \mathbb{H}_1 \leq \mathrm{s} \leq \mathbb{H}_2.$$

The ADM is a powerful technique for solving DEs, including FODEs. It is particularly useful for handling non-linear DEs and can be adapted to the fractional-order case. ADM involves decomposing the solution of a DE into a series of simpler functions, making it easier to solve the equation iteratively. The main idea is to break down the problem into manageable components. The ADM holds significant importance in solving FODEs due to several key advantages and features. FODEs often include non-linear terms that can be challenging to solve using traditional methods. ADM effectively decomposes these non-linear terms into simpler components using Adomian polynomials, making it easier to find solutions even when the equations are highly non-linear. ADM provides a systematic framework for solving FODEs by breaking down the problem into a series of simpler problems. This approach allows for an iterative solution where each term in the series is computed sequentially, making it easier to handle complex DEs. The method is flexible and can be applied to a wide range of FODEs, including those with complex boundary conditions and variable coefficients. This adaptability makes ADM suitable for a diverse set of problems across different scientific and engineering disciplines.

We take into account the following: nonlinear SDE with respect to a conformable operator and a non-zero trapping potential that is time-fractional.

$$\imath \mathrm{D}_{\mathrm{t}}^{\beta} \mathrm{Y}(\mathrm{x}, \mathrm{t}) + \mathrm{A} \mathrm{Y}_{\mathrm{xx}}(\mathrm{x}, \mathrm{t}) + \mathrm{B}(\mathrm{x}) \mathrm{Y}(\mathrm{x}, \mathrm{t}) + \mathrm{C} |\mathrm{Y}(\mathrm{x}, \mathrm{t})|^2 \mathrm{Y}(\mathrm{x}, \mathrm{t}) = 0, \tag{4}$$

with the initial condition (I-C):

$$Y(x,0) = Y_0(x), \tag{5}$$

here $A, C \in \mathbb{R}$, $0 < \beta \leq 1$, $\mathfrak{i}^2 = -1$; D_t^{β} is Con-FrD; $Y(x,t)$ complex-valued function, $x \in \mathbb{R}$; $t \geq 0$; $B(x)$ represents trapping potential; $|Y(x,t)|$ is the modulus of $Y(x,t)$.

The subsequent sections are structured as follows: Section 2 presents the main characteristics and results that form the foundation of our study. Section 3 provides the algorithm of the EADM. In Section 4, we solve three types of FSDEs using the EADM. In Section 5, we discuss the results obtained in Section 4 through graphical and numerical outcomes and analyze the correctness of our approach. Finally, Section 6 presents the conclusion.

2 Preliminaries

In this section, we go over some useful characteristics of Con-FrD and ET that will be useful in this paper.

Theorem 2.1. *[44] Let $0 < \beta \leq 1$, $Y_1(x,t)$ and $Y_2(x,t)$ be β-differentiable at a point $t > 0$. Then,*

i. $D_t^{\beta}(c_1 Y_1(x,t) + c_2 Y_2(x,t)) = c_1 D_t^{\beta} Y_1(x,t) + c_2 D_t^{\beta} Y_2(x,t),\ \forall\, c_1,\ c_2 \in \mathbb{R}$.

ii. $D_t^{\beta}(t^c) = c t^{c-\beta},\ \forall c \in \mathbb{R}$.

iii. $D_t^{\beta}(c) = 0,\ c \in \mathbb{R}$.

iv. $D_t^{\beta} Y_1(x,t) Y_2(x,t)) = Y_1(x,t) D_t^{\beta} Y_2(x,t) + Y_2(x,t) D_t^{\beta} Y_1(x,t)$.

v. $D_t^{\beta} \dfrac{Y_1(x,t)}{Y_2(x,t)} = \dfrac{Y_2(x,t) D_t^{\beta} Y_1(x,t) - Y_1(x,t) D_t^{\beta} Y_2(t)}{Y_2^2(t)}$.

In addition, if Y is differentiable, then $D_t^{\beta}(Y)(t) = t^{1-\beta} \dfrac{dY}{dt}(t)$.

Lemma 2.2. *[45] Assume that $Y_1(x,t)$ and $Y_2(x,t)$ satisfies the axioms of existence of ET, $\mathbb{E}_{\beta}[Y_1(x,t)] = Y_1^*(x,s)$, $\mathbb{E}_{\beta}[Y_2(x,t)] = Y_2^*(x,s)$ and $\mathfrak{B}_1, \mathfrak{B}_2$ are constants. Then, the following axioms hold:*

i. $\mathbb{E}_{\beta}[\mathfrak{B}_1 Y_1(x,t) + \mathfrak{B}_2 Y_2(x,t)] = \mathfrak{B}_1 Y_1^*(x,s) + \mathfrak{B}_2 Y_2^*(x,s)$.

ii. $\mathbb{E}_{\beta}^{-1}[\mathfrak{B}_1 Y_1^*(x,s) + \mathfrak{B}_2 Y_2^*(x,s)] = \mathfrak{B}_1 Y_1(x,t) + \mathfrak{B}_2 Y_2(x,t)$.

iii. $\mathbb{E}_{\beta}[D_t^{\beta} Y(x,t)] = s^{-\beta} Y^*(x,s) - \sum_{\kappa=0}^{\mathfrak{P}-1} s^{(2+\kappa-\beta)} D_t^{(\kappa)}(x,0),\ \mathfrak{P} - 1 < \beta \leq \mathfrak{P}$.

iv. $\mathbb{E}_{\beta}[D_t^{\mathfrak{i}\beta} Y(x,t)] = s^{-\mathfrak{i}\beta} Y^*(x,s) - \sum_{\kappa=0}^{\mathfrak{i}-1} s^{(2+(\kappa-\mathfrak{i})\beta)} D_t^{(\kappa)}(x,0),\ 0 < \beta \leq 1$.

3 Analysis of the EADM

This section presents the main steps for solving SFDEs using ET and ADM. We apply these steps to solve SFDEs in the standard structure. For this, take $Y(x,t) = Y_1(x,t) + \mathfrak{i} Y_2(x,t)$ in Eq. (4) and we get:

$$\begin{aligned} D_t^{\beta} Y_1(x,t) =& - A\mathbb{D}xx Y_2(x,t) - B(x) Y_2(x,t) - C\big(Y_1^2(x,t) Y_2(x,t) + Y_2^3(x,t)\big), \\ D_t^{\beta} Y_2(x,t) =& A\mathbb{D}xx Y_1(x,t) + B(x) Y_1(x,t) + C\big(Y_2^2(x,t) Y_1(x,t) + Y_1^3(x,t)\big), \end{aligned} \tag{6}$$

with I-Cs:

$$Y_1(x,0) = Y_{1,0}(x),\ Y_2(x,0) = Y_{2,0}(x), \tag{7}$$

here, $Y(x,0) = Y_{1,0}(x) + \mathfrak{i}Y_{2,0}(x)$.
By applying $\mathbb{E}_\beta$ on the above system.

$$\begin{aligned}\mathbb{E}_\beta[D_t^\beta Y_1(x,t)] =& -\mathbb{E}_\beta\big[A\mathbb{D}xxY_2(x,t) + B(x)Y_2(x,t)+\\ & C\big(Y_1^2(x,t)Y_2(x,t) + Y_2^3(x,t)\big)\big],\\ \mathbb{E}_\beta[D_t^\beta Y_2(x,t)] =& \mathbb{E}_\beta\big[A\mathbb{D}xxY_1(x,t) + B(x)Y_1(x,t)+\\ & C\big(Y_2^2(x,t)Y_1(x,t) + Y_1^3(x,t)\big)\big].\end{aligned} \tag{8}$$

Through Lemma 2.2(iii) and performing various computations, we acquire the following:

$$\begin{aligned}\mathbb{E}_\beta[Y_1(x,t)] =& \frac{1}{s}Y_1(0) - \frac{1}{s}\mathbb{E}_\beta\big[A\mathbb{D}xxY_2(x,t)\big] - \frac{1}{s}\mathbb{E}_\beta\big[B(x)Y_2(x,t)\big]\\ & - \frac{1}{s}\mathbb{E}_\beta\big[C\big(Y_1^2(x,t)Y_2(x,t) + Y_2^3(x,t)\big)\big],\\ \mathbb{E}_\beta[Y_2(x,t)] =& \frac{1}{s}Y_2(0) + \frac{1}{s}\mathbb{E}_\beta\big[A\mathbb{D}xxY_1(x,t)\big] + \frac{1}{s}\mathbb{E}_\beta\big[B(x)Y_1(x,t)\big]\\ & + \frac{1}{s}\mathbb{E}_\beta\big[C\big(Y_2^2(x,t)Y_1(x,t) + Y_1^3(x,t)\big)\big].\end{aligned} \tag{9}$$

Implementing the $\mathbb{E}_\beta^{-1}$ on the system mentioned above:

$$\begin{aligned}Y_1(x,t) =& \mathbb{E}_\beta^{-1}\big[\frac{1}{s}Y_1(0)\big] - \mathbb{E}_\beta^{-1}\big[\frac{1}{s}\mathbb{E}_\beta\big[A\mathbb{D}xxY_2(x,t)\big]\big] - \mathbb{E}_\beta^{-1}\big[\frac{1}{s}\mathbb{E}_\beta\big[B(x)Y_2(x,t)\big]\big]\\ & - \mathbb{E}_\beta^{-1}\big[\frac{1}{s}\mathbb{E}_\beta\big[C\big(Y_1^2(x,t)Y_2(x,t) + Y_2^3(x,t)\big)\big]\big],\\ Y_2(x,t) =& \mathbb{E}_\beta^{-1}\big[\frac{1}{s}Y_2(0)\big] + \mathbb{E}_\beta^{-1}\big[\frac{1}{s}\mathbb{E}_\beta\big[A\mathbb{D}xxY_1(x,t)\big]\big] + \mathbb{E}_\beta^{-1}\big[\frac{1}{s}\mathbb{E}_\beta\big[B(x)Y_1(x,t)\big]\big]\\ & + \mathbb{E}_\beta^{-1}\big[\frac{1}{s}\mathbb{E}_\beta\big[C\big(Y_2^2(x,t)Y_1(x,t) + Y_1^3(x,t)\big)\big)\big].\end{aligned} \tag{10}$$

Consider the solution of Eq. (6) in the following fractional power series (FPS) using the idea of ADM.

$$\begin{aligned}Y_1(x,t) &= \sum_{\jmath=0}^{\infty} Y_{1,\jmath}(x,t),\\ Y_2(x,t) &= \sum_{\jmath=0}^{\infty} Y_{2,\jmath}(x,t).\end{aligned} \tag{11}$$

The nonlinear terms $\mathscr{Z}_1(Y_1,Y_2)$ and $\mathscr{Z}_2(Y_1,Y_2)$ represented as

$$\begin{aligned}\mathscr{Z}_1(Y_1,Y_2) &= \sum_{\jmath=0}^{\infty} \mathfrak{A}_{1,\jmath}(Y_1,Y_2),\\ \mathscr{Z}_2(Y_1,Y_2) &= \sum_{\jmath=0}^{\infty} \mathfrak{A}_{2,\jmath}(Y_1,Y_2),\end{aligned} \tag{12}$$

here, $\mathfrak{A}_{1,\jmath}(Y_1,Y_2)$ and $\mathfrak{A}_{2,\jmath}(Y_1,Y_2)$ are the Adomian polynomials (A-Ps) for the nonlinear term.

$$\begin{aligned}\mathfrak{A}_{1,\jmath}(Y_1,Y_2) =& \frac{1}{\Gamma(\jmath+1)}\frac{\partial^\jmath}{\partial\chi^\jmath}\big[\mathscr{Z}_1\big(\sum_{\mathfrak{r}=0}^{\jmath}\chi^{\mathfrak{r}}Y_{1,\mathfrak{r}}(x,t)\big)\big]_{\chi=0},\ \jmath = 0,\ 1,\ 2,\ \ldots,\\ \mathfrak{A}_{2,\jmath}(Y_1,Y_2) =& \frac{1}{\Gamma(\jmath+1)}\frac{\partial^\jmath}{\partial\chi^\jmath}\big[\mathscr{Z}_2\big(\sum_{\mathfrak{r}=0}^{\jmath}\chi^{\mathfrak{r}}Y_{2,\mathfrak{r}}(x,t)\big)\big]_{\chi=0},\ \jmath = 0,\ 1,\ 2,\ \ldots.\end{aligned} \tag{13}$$

From Eqs. (11), (12), and (10).

$$\sum_{\jmath=0}^{\infty} Y_{1,\jmath}(x,t) = \mathbb{E}_{\beta}^{-1}\Big[\frac{1}{s}Y_1(0)\Big] - \mathbb{E}_{\beta}^{-1}\Big[\frac{1}{s}\mathbb{E}_{\beta}\Big[A\mathbb{D}xx\sum_{\jmath=0}^{\infty} Y_{2,\jmath}(x,t)\Big]\Big] - \mathbb{E}_{\beta}^{-1}\Big[\frac{1}{s}C_{\beta}\Big[B(x)\sum_{\jmath=0}^{\infty} Y_{2,\jmath}(x,t)\Big]\Big] - \mathbb{E}_{\beta}^{-1}\Big[\frac{1}{s}\mathbb{E}_{\beta}\Big[C\sum_{\jmath=0}^{\infty}\mathfrak{A}_{1,\jmath}(Y_1,Y_2)\Big],$$
$$\sum_{\jmath=0}^{\infty} Y_{2,\jmath}(x,t) = \mathbb{E}_{\beta}^{-1}\Big[\frac{1}{s}Y_2(0)\Big] + \mathbb{E}_{\beta}^{-1}\Big[\frac{1}{s}\mathbb{E}_{\beta}\Big[A\mathbb{D}xx\sum_{\jmath=0}^{\infty} Y_{1,\jmath}(x,t)\Big]\Big] + \mathbb{E}_{\beta}^{-1}\Big[\frac{1}{s}\mathbb{E}_{\beta}\Big[B(x)\sum_{\jmath=0}^{\infty} Y_{1,\jmath}(x,t)\Big]\Big] + \mathbb{E}_{\beta}^{-1}\Big[\frac{1}{s}\mathbb{E}_{\beta}\Big[C\sum_{\jmath=0}^{\infty}\mathfrak{A}_{2,\jmath}(Y_1,Y_2)\Big]. \tag{14}$$

From Eq. (14), we have as

$$Y_{1,0}(x,t) = \mathbb{E}_{\beta}^{-1}\Big[\frac{1}{s}Y_1(0)\Big],$$
$$Y_{2,0}(x,t) = \mathbb{E}_{\beta}^{-1}\Big[\frac{1}{s}Y_2(0)\Big]. \tag{15}$$

The 2nd term of the FPS is as

$$Y_{1,1}(x,t) = -\mathbb{E}_{\beta}^{-1}\Big[\frac{1}{s}\mathbb{E}_{\beta}\big[A\mathbb{D}xxY_{2,0}(x,t)\big]\Big] - \mathbb{E}_{\beta}^{-1}\Big[\frac{1}{s}\mathbb{E}_{\beta}\big[B(x)Y_{2,0}(x,t)\big]\Big] - \mathbb{E}_{\beta}^{-1}\Big[\frac{1}{s}\mathbb{E}_{\beta}\big[C\mathfrak{A}_{1,0}(Y_1,Y_2)\big],$$
$$Y_{2,1}(x,t) = \mathbb{E}_{\beta}^{-1}\Big[\frac{1}{s}\mathbb{E}_{\beta}\big[A\mathbb{D}xxY_{1,0}(x,t)\big]\Big] + \mathbb{E}_{\beta}^{-1}\Big[\frac{1}{s}\mathbb{E}_{\beta}\big[B(x)Y_{1,0}(x,t)\big]\Big] + \mathbb{E}_{\beta}^{-1}\Big[\frac{1}{s}\mathbb{E}_{\beta}\big[C\mathfrak{A}_{2,0}(Y_1,Y_2)\big]. \tag{16}$$

The following are the 3rd and 4th terms of the FPS:

$$Y_{1,2}(x,t) = -\mathbb{E}_{\beta}^{-1}\Big[\frac{1}{s}\mathbb{E}_{x}\big[A\mathbb{D}xxY_{2,1}(x,t)\big]\Big] - \mathbb{E}_{\beta}^{-1}\Big[\frac{1}{s}\mathbb{E}_{\beta}\big[B(x)Y_{2,1}(x,t)\big]\Big] - \mathbb{E}_{\beta}^{-1}\Big[\frac{1}{s}\mathbb{E}_{\beta}\big[C\mathfrak{A}_{1,1}(Y_1,Y_2)\big],$$
$$Y_{2,2}(x,t) = \mathbb{E}_{\beta}^{-1}\Big[\frac{1}{s}\mathbb{E}_{\beta}\big[A\mathbb{D}xxY_{1,1}(x,t)\big]\Big] + \mathbb{E}_{\beta}^{-1}\Big[\frac{1}{s}\mathbb{E}_{\beta}\big[B(x)Y_{1,1}(x,t)\big]\Big] + \mathbb{E}_{\beta}^{-1}\Big[\frac{1}{s}\mathbb{E}_{\beta}\big[C\mathfrak{A}_{2,1}(Y_1,Y_2)\big]. \tag{17}$$

$$Y_{1,3}(x,t) = -\mathbb{E}_{\beta}^{-1}\Big[\frac{1}{s}\mathbb{E}_{\beta}\big[A\mathbb{D}xxY_{2,2}(x,t)\big]\Big] - \mathbb{E}_{\beta}^{-1}\Big[\frac{1}{s}\mathbb{E}_{\beta}\big[B(x)Y_{2,2}(x,t)\big]\Big] - \mathbb{E}_{\beta}^{-1}\Big[\frac{1}{s}\mathbb{E}_{\beta}\big[C\mathfrak{A}_{1,2}(Y_1,Y_2)\big],$$
$$Y_{2,3}(x,t) = \mathbb{E}_{\beta}^{-1}\Big[\frac{1}{s}\mathbb{E}_{\beta}\big[A\mathbb{D}xxY_{1,2}(x,t)\big]\Big] + \mathbb{E}_{\beta}^{-1}\Big[\frac{1}{s}\mathbb{E}_{\beta}\big[B(x)Y_{1,2}(x,t)\big]\Big] + \mathbb{E}_{\beta}^{-1}\Big[\frac{1}{s}\mathbb{E}_{\beta}\big[C\mathfrak{A}_{2,2}(Y_1,Y_2)\big]. \tag{18}$$

By making generalizing we get the following:

$$\begin{aligned}
Y_{1,\jmath+1}(x,t) =& -\mathbb{E}_\beta^{-1}\Big[\frac{1}{s}\mathbb{E}_\beta\big[A\mathbb{D}xxY_{2,\jmath}(x,t)\big]\Big] - \mathbb{E}_\beta^{-1}\Big[\frac{1}{s}\mathbb{E}_\beta\big[B(x)Y_{2,\jmath}(x,t)\big]\Big] - \\
& \mathbb{E}_\beta^{-1}\Big[\frac{w}{u}\mathbb{E}_\beta\big[C\mathfrak{A}_{1,\jmath}(Y_1,Y_2)\big]\Big], \\
Y_{2,\jmath+1}(x,t) =& \mathbb{E}_\beta^{-1}\Big[\frac{1}{s}\mathbb{E}_\beta\big[A\mathbb{D}xxY_{1,\jmath}(x,t)\big]\Big] + \mathbb{E}_\beta^{-1}\Big[\frac{1}{s}\mathbb{E}_\beta\big[B(x)Y_{1,\jmath}(x,t)\big]\Big] + \\
& \mathbb{E}_\beta^{-1}\Big[\frac{1}{s}\mathbb{E}_\beta\big[C\mathfrak{A}_{2,\jmath}(Y_1,Y_2)\big]\Big].
\end{aligned} \tag{19}$$

Lastly, the FPS is obtained in the following ways:

$$\begin{aligned}
Y_1(x,t) &= \lim_{\mathfrak{r}\to\infty}\sum_{\jmath=0}^{\mathfrak{r}} Y_{1,\jmath}(x,t), \\
Y_2(x,t) &= \lim_{\mathfrak{r}\to\infty}\sum_{\jmath=0}^{\mathfrak{r}} Y_{2,\jmath}(x,t).
\end{aligned} \tag{20}$$

4 Numerical Problems

Using the method outlined in the preceding section, we find the App-Ss and Ex-Ss of linear and nonlinear FSDEs in this section.

Problem 4.1. Examine the subsequent linear SFDE:

$$\imath D_t^\beta Y(x,t) - \mathbb{D}xxY(x,t) = 0,\ t \geq 0,\ 0 < \beta \leq 1, \tag{21}$$

with the I-C:

$$Y(x,0) = 1 + \cosh(2x),\ x \in \mathbb{R}.$$

By utilizing $Y(x,t) = Y_1(x,t) + \imath Y_2(x,t)$ and $Y(x,0) = Y_{1,0}(x) + \imath Y_{2,0}(x)$, we have the following:

$$\begin{aligned}
D_t^\beta Y_1(x,t) =& \mathbb{D}xxY_2(x,t), \\
D_t^\beta Y_2(x,t) =& -\mathbb{D}xxY_1(x,t),
\end{aligned} \tag{22}$$

with I-Cs:

$$\begin{aligned}
Y_{1,0}(x,0) =& 1 + \cosh(2x), \\
Y_{2,0}(x,0) =& 0.
\end{aligned} \tag{23}$$

By applying $\mathbb{E}_\beta$ to the above system and implementing some calculations, we get the subsequent:

$$\begin{aligned}
\mathbb{E}_\beta[Y_1(x,t)] &= \frac{1}{s}Y_1(0) + \frac{1}{s}\mathbb{E}_\beta\big[\mathbb{D}xxY_2(x,t)\big], \\
\mathbb{E}_\beta[Y_2(x,t)] &= \frac{1}{s}Y_2(0) - \frac{1}{s}\mathbb{E}_\beta\big[\mathbb{D}xxY_1(x,t)\big].
\end{aligned} \tag{24}$$

Taking $\mathbb{E}_\beta^{-1}$.

$$\begin{aligned}
Y_1(x,t) =& \mathbb{E}_\beta^{-1}\Big[\frac{1}{s}Y_1(0)\Big] + \mathbb{E}_\beta^{-1}\Big[\frac{1}{s}\mathbb{E}_\beta\big[\mathbb{D}xxY_2(x,t)\big]\Big], \\
Y_2(x,t) =& \mathbb{E}_\beta^{-1}\Big[\frac{1}{s}Y_2(0)\Big] - \mathbb{E}_\beta^{-1}\Big[\frac{1}{s}\mathbb{E}_\beta\big[\mathbb{D}xxY_1(x,t)\big]\Big].
\end{aligned} \tag{25}$$

We derive the following result:

$$\begin{aligned}\sum_{j=0}^{\infty} \mathrm{Y}_{1,j}(\mathrm{x},\mathrm{t}) =& \mathbb{E}_{\beta}^{-1}\big[\frac{1}{\mathrm{s}}\mathrm{Y}_1(0)\big] + \mathbb{E}_{\beta}^{-1}\big[\frac{1}{\mathrm{s}}\mathbb{E}_{\beta}\big[\mathbb{D}\mathrm{xx}\sum_{j=0}^{\infty}\mathrm{Y}_{2,j}(\mathrm{x},\mathrm{t})\big]\big],\\ \sum_{j=0}^{\infty} \mathrm{Y}_{2,j}(\mathrm{x},\mathrm{t}) =& \mathbb{E}_{\beta}^{-1}\big[\frac{1}{\mathrm{s}}\mathrm{Y}_2(0)\big] - \mathbb{E}_{\beta}^{-1}\big[\frac{1}{\mathrm{s}}\mathbb{E}_{\beta}\big[\mathbb{D}\mathrm{xx}\sum_{j=0}^{\infty}\mathrm{Y}_{1,j}(\mathrm{x},\mathrm{t})\big]\big].\end{aligned} \tag{26}$$

We extract the following from the above:

$$\begin{aligned}\mathrm{Y}_{1,0}(\mathrm{x},\mathrm{t}) =& \mathbb{E}_{\beta}^{-1}\big[\frac{1}{\mathrm{s}}\mathrm{Y}_{1,0}(\mathrm{x},0)\big],\\ \mathrm{Y}_{2,0}(\mathrm{x},\mathrm{t}) =& \mathbb{E}_{\beta}^{-1}\big[\frac{1}{\mathrm{s}}\mathrm{Y}_{2,0}(\mathrm{x},0)\big].\end{aligned} \tag{27}$$

Below is the first term:

$$\begin{aligned}\mathrm{Y}_{1,0}(\mathrm{x},\mathrm{t}) =& 1+\cosh(2\mathrm{x}),\\ \mathrm{Y}_{2,0}(\mathrm{x},\mathrm{t}) =& 0.\end{aligned} \tag{28}$$

The next term expressions are as follows:

$$\begin{aligned}\mathrm{Y}_{1,1}(\mathrm{x},\mathrm{t}) =& \mathbb{E}_{\beta}^{-1}\big[\frac{1}{\mathrm{s}}\mathbb{E}_{\beta}\big[\mathbb{D}\mathrm{xx}\mathrm{Y}_{2,0}(\mathrm{x},\mathrm{t})\big]\big],\\ \mathrm{Y}_{2,1}(\mathrm{x},\mathrm{t}) =& \mathbb{E}_{\beta}^{-1}\big[\frac{1}{\mathrm{s}}\mathbb{E}_{\beta}\big[\mathbb{D}\mathrm{xx}\mathrm{Y}_{1,0}(\mathrm{x},\mathrm{t})\big]\big].\end{aligned} \tag{29}$$

From the above system, we have

$$\begin{aligned}\mathrm{Y}_{1,1}(\mathrm{x},\mathrm{t}) =& \mathbb{E}_{\beta}^{-1}\big[\frac{1}{\mathrm{s}}\mathbb{E}_{\beta}\big[\mathbb{D}\mathrm{xx}(0)\big]\big],\\ \mathrm{Y}_{2,1}(\mathrm{x},\mathrm{t}) =& \mathbb{E}_{\beta}^{-1}\big[\frac{1}{\mathrm{s}}\mathbb{E}_{\beta}\big[\mathbb{D}\mathrm{xx}(1+\cosh(2\mathrm{x}))\big]\big].\end{aligned} \tag{30}$$

As a result,

$$\begin{aligned}\mathrm{Y}_{1,1}(\mathrm{x},\mathrm{t}) =& 0,\\ \mathrm{Y}_{2,1}(\mathrm{x},\mathrm{t}) =& -\frac{4}{\Gamma(2)}\cosh(2\mathrm{x})\frac{\mathrm{t}^{\beta}}{\beta}.\end{aligned} \tag{31}$$

The third-term expressions are as follows:

$$\begin{aligned}\mathrm{Y}_{1,2}(\mathrm{x},\mathrm{t}) =& \mathbb{E}_{\beta}^{-1}\big[\frac{1}{\mathrm{s}}\mathbb{E}_{\beta}\big[\mathbb{D}\mathrm{xx}\mathrm{Y}_{2,1}(\mathrm{x},\mathrm{t})\big]\big],\\ \mathrm{Y}_{2,2}(\mathrm{x},\mathrm{t}) =& \mathbb{E}_{\beta}^{-1}\big[\frac{1}{\mathrm{s}}\mathbb{E}_{\beta}\big[\mathbb{D}\mathrm{xx}\mathrm{Y}_{1,1}(\mathrm{x},\mathrm{t})\big]\big].\end{aligned} \tag{32}$$

We have the following from above:

$$\begin{aligned}\mathrm{Y}_{1,2}(\mathrm{x},\mathrm{t}) =& \mathbb{E}_{\beta}^{-1}\big[\frac{1}{\mathrm{s}}\mathbb{E}_{\beta}\big[\mathbb{D}\mathrm{xx}\big(\frac{4}{\Gamma(2)}\cosh(2\mathrm{x})\frac{\mathrm{t}^{\beta}}{\beta}\big)\big]\big],\\ \mathrm{Y}_{2,2}(\mathrm{x},\mathrm{t}) =& \mathbb{E}_{\beta}^{-1}\big[\frac{1}{\mathrm{s}}\mathbb{E}_{\beta}\big[\mathbb{D}\mathrm{xx}(0)\big]\big].\end{aligned} \tag{33}$$

The third term is as follows:

$$\begin{aligned} Y_{1,2}(x,t) =& -\frac{8}{\Gamma(3)}\cosh(2x)\frac{t^{2\beta}}{\beta^2}, \\ Y_{2,2}(x,t) =& 0. \end{aligned} \tag{34}$$

The next terms are as follows:

$$\begin{aligned} Y_{1,3}(x,t) =& 0, \\ Y_{2,3}(x,t) =& \frac{64}{\Gamma(4)}\cosh(2x)\frac{t^{3\beta}}{\beta^3}. \end{aligned} \tag{35}$$

$$\begin{aligned} Y_{1,4}(x,t) =& \frac{256}{\Gamma(5)}\cosh(2x)\frac{t^{4\beta}}{\beta^4}, \\ Y_{2,4}(x,t) =& 0. \end{aligned} \tag{36}$$

The procedure is repeated to achieve the results for the sixth term.

$$\begin{aligned} Y_{1,5}(x,t) =& 0, \\ Y_{2,5}(x,t) =& -\frac{1024}{\Gamma(6)}\cosh(2x)\frac{t^{5\beta}}{\beta^5}. \end{aligned} \tag{37}$$

As a result,

$$Y(x,t) = 1 + \cosh(2x)\Big(\sum_{\jmath=0}^{\infty}\frac{(-1)^{\jmath}}{\Gamma(2\jmath+1)}\big(\frac{4t^{\beta}}{\beta}\big)^{2\jmath} - \mathbf{i}\sum_{\jmath=0}^{\infty}\frac{(-1)^{\jmath}}{\Gamma(2\jmath+2)}\big(\frac{4t^{\beta}}{\beta}\big)^{2\jmath+1}\Big). \tag{38}$$

The Ex-S when $\beta = 1.0$ is $1 + \cosh(2x)\exp^{-4\mathbf{i}t}$. A similar result was obtained by [37].

Problem 4.2. Take into consideration the subsequent nonlinear FSDE:

$$\mathbf{i}D_t^{\beta}Y(x,t) + Y_{xx}(x,t) + 2|Y(x,t)|^2 Y(x,t) = 0, \tag{39}$$

with the I-C:

$$Y(x,0) = \exp^{\mathbf{i}x}, x \in \mathbb{R}. \tag{40}$$

Suppose $Y(x,t) = Y_1(x,t) + \mathbf{i}Y_2(x,t)$ so that $Y(x,0) = Y_{1,0}(x) + \mathbf{i}Y_{2,0}(x)$. As a result,

$$\begin{aligned} D_t^{\beta}Y_1(x,t) &= -\mathbb{D}xxY_2(x,t) - 2\big(Y_1^2(x,t)Y_2(x,t) + Y_2^3(x,t)\big), \\ T_t^{\beta}Y_2(x,t) &= \mathbb{D}xxY_1(x,t) + 2\big(Y_2^2(x,t)Y_1(x,t) + Y_1^3(x,t)\big). \end{aligned} \tag{41}$$

with the I.Cs:

$$\begin{aligned} Y_1(x,0) &= \cos(x), \\ Y_2(x,0) &= \sin(x), \end{aligned} \tag{42}$$

Applying ET to Eq. (41), using the linear property of ET and Lemma 2.2(iii) and making some calculations, we get

$$\begin{aligned} \mathbb{E}_{\beta}[Y_1(x,t)] =& \frac{1}{s}Y_1(0) - \frac{1}{s}\mathbb{E}_{\beta}\big[\mathbb{D}xxY_2(x,t)\big] \\ & - \frac{1}{s}\mathbb{E}_{\beta}\big[2\big(Y_1^2(x,t)Y_2(x,t) + Y_2^3(x,t)\big)\big], \\ \mathbb{E}_{\beta}[Y_2(x,t)] =& \frac{1}{s}Y_2(0) + \frac{1}{s}\mathbb{E}_{\beta}\big[\mathbb{D}xxY_1(x,t)\big] \\ & + \frac{1}{s}\mathbb{E}_{\beta}\big[2\big(Y_2^2(x,t)Y_1(x,t) + Y_1^3(x,t)\big)\big]. \end{aligned} \tag{43}$$

Through $\mathbb{E}_{\beta}^{-1}$ on the system mentioned before.

$$\begin{aligned}
Y_1(x,t) =& \mathbb{E}_{\beta}^{-1}\Big[\frac{1}{s}Y_1(0)\Big] - \mathbb{E}_{\beta}^{-1}\Big[\frac{1}{s}\mathbb{E}_{\beta}\big[\mathbb{D}xxY_2(x,t)\big]\Big] \\
&- \mathbb{E}_{\beta}^{-1}\Big[\frac{1}{s}\mathbb{E}_{\beta}\big[2\big(Y_1^2(x,t)Y_2(x,t) + Y_2^3(x,t)\big)\big]\Big], \\
Y_2(x,t) =& \mathbb{E}_{\beta}^{-1}\Big[\frac{1}{s}Y_2(0)\Big] + \mathbb{E}_{\beta}^{-1}\Big[\frac{1}{s}\mathbb{E}_{\beta}\big[\mathbb{D}xxY_1(x,t)\big]\Big] \\
&+ \mathbb{E}_{\beta}^{-1}\Big[\frac{1}{s}\mathbb{E}_{\beta}\big[2\big(Y_2^2(x,t)Y_1(x,t) + Y_1^3(x,t)\big)\big]. \qquad (44)
\end{aligned}$$

We get the following:

$$\begin{aligned}
\sum_{\jmath=0}^{\infty} Y_{1,\jmath}(x,t) =& \mathbb{E}_{\beta}^{-1}\Big[\frac{1}{s}Y_1(0)\Big] - \mathbb{E}_{\beta}^{-1}\Big[\frac{1}{s}\mathbb{E}_{\beta}\big[\mathbb{D}xx\sum_{\jmath=0}^{\infty} Y_{2,\jmath}(x,t)\big]\Big] - \\
& \mathbb{E}_{\beta}^{-1}\Big[\frac{1}{s}\mathbb{E}_{\beta}\big[2\sum_{\jmath=0}^{\infty}\mathfrak{A}_{1,\jmath}(Y_1,Y_2)\big)\big], \\
\sum_{\jmath=0}^{\infty} Y_{2,\jmath}(x,t) =& \mathbb{E}_{\beta}^{-1}\Big[\frac{1}{s}Y_2(0)\Big] + \mathbb{E}_{\beta}^{-1}\Big[\frac{1}{s}\mathbb{E}_{\beta}\big[\mathbb{D}xx\sum_{\jmath=0}^{\infty} Y_{1,\jmath}(x,t)\big]\Big] + \\
& \mathbb{E}_{\beta}^{-1}\Big[\frac{w}{u}\mathbb{E}_{\beta}\big[2\sum_{\jmath=0}^{\infty}\mathfrak{A}_{2,\jmath}(Y_1,Y_2)\big)\big]. \qquad (45)
\end{aligned}$$

From the above system, we obtain the first terms of the FPS solutions.

$$\begin{aligned}
Y_{1,0}(x,t) &= \cos(x), \\
Y_{2,0}(x,t) &= \sin(x). \qquad (46)
\end{aligned}$$

We extract the following terms of the FPS as follows:

$$\begin{aligned}
Y_{1,1}(x,t) &= -\sin(x)\frac{t^{\beta}}{\beta\Gamma(2)}, \\
Y_{2,1}(x,t) &= \cos(x)\frac{t^{\beta}}{\beta\Gamma(2)}. \qquad (47)
\end{aligned}$$

$$\begin{aligned}
Y_{1,2}(x,t) &= -\frac{t^{2\beta}}{\beta^2 2!}\cos(x), \\
Y_{2,2}(x,t) &= -\sin(x)\frac{t^{2\beta}}{\beta^2\Gamma(3)}. \qquad (48)
\end{aligned}$$

$$\begin{aligned}
Y_{1,3}(x,t) &= \sin(x)\frac{t^{3\beta}}{\beta^3\Gamma(4)}, \\
Y_{2,3}(x,t) &= -\cos(x)\frac{t^{3\beta}}{\beta^3\Gamma(4)}. \qquad (49)
\end{aligned}$$

$$\begin{aligned}
Y_{1,4}(x,t) &= \cos(x)\frac{t^{4\beta}}{\beta^4\Gamma(5)}, \\
Y_{2,4}(x,t) &= \sin(x)\frac{t^{4\beta}}{\beta^4\Gamma(5)}. \qquad (50)
\end{aligned}$$

The procedure is repeated to achieve the results for the sixth terms.

$$\begin{aligned} Y_{1,5}(x,t) &= -\sin(x)\frac{t^{5\beta}}{\beta^5\Gamma(6)}, \\ Y_{2,5}(x,t) &= \cos(x)\frac{t^{5\beta}}{\beta^5\Gamma(6)}. \end{aligned} \tag{51}$$

The FPS solution is as follows:

$$Y(x,t) = (\cos(x) + \mathfrak{i}\sin(x))\Big(\sum_{\jmath=0}^{\infty}\frac{1}{\Gamma(\jmath+1)}(\frac{\mathfrak{i}t^{\beta}}{\beta})^{\jmath}\Big). \tag{52}$$

The Ex-S when $\beta = 1$ is $Y(x,t) = \exp^{\mathfrak{i}(x+t)}$. The same solution was achieved by [41].
Problem 4.3. Consider the following nonlinear FSDE that follows:

$$\mathfrak{i}D_t^{\beta}Y(x,t) + \frac{1}{2}Y_{xx}(x,t) - \cos^2(x)Y(x,t) - |Y(x,t)|^2Y(x,t) = 0, \tag{53}$$

with the I-C:

$$Y(x,0) = \sin(x),\ x \in \mathbb{R}. \tag{54}$$

Suppose $Y(x,t) = Y_1(x,t) + \mathfrak{i}Y_2(x,t)$ then $Y(x,0) = Y_{1,0}(x) + \mathfrak{i}Y_{2,0}(x)$. As a result,

$$\begin{aligned} D_t^{\beta}Y_1(x,t) &= -\frac{1}{2}\mathbb{D}xxY_2(x,t) + \cos^2(x)Y_2(x,t) + \big(Y_1^2(x,t)Y_2(x,t) + Y_2^3(x,t)\big), \\ D_t^{\beta}Y_2(x,t) &= \frac{1}{2}\mathbb{D}xxY_1(x,t) - \cos^2(x)Y_1(x,t) - \big(Y_2^2(x,t)Y_1(x,t) + Y_1^3(x,t)\big), \end{aligned} \tag{55}$$

with the I-Cs:

$$\begin{aligned} Y_1(x,0) &= \sin(x), \\ Y_2(x,0) &= 0. \end{aligned} \tag{56}$$

After applying ET to Eq. (55), utilizing Lemma 2.2(iii) and the linear property of ET, and performing certain computations, we obtain

$$\begin{aligned} \mathbb{E}[Y_1(x,t)] =& \frac{1}{s}Y_1(0) - \frac{1}{s}\mathbb{E}_{\beta}\big[\frac{1}{2}\mathbb{D}xxY_2(x,t)\big] + \frac{1}{s}\mathbb{E}_{\beta}\big[\cos^2(x)Y_2(x,t)\big] \\ &+ \frac{1}{s}\mathbb{E}_{\beta}\big[\big(Y_1^2(x,t)Y_2(x,t) + Y_2^3(x,t)\big)\big], \\ \mathbb{E}_{\beta}[Y_2(x,t)] =& \frac{1}{u}Y_2(0) + \frac{1}{s}\mathbb{E}_{\beta}\big[\frac{1}{2}\mathbb{D}xxY_1(x,t)\big] - \frac{1}{s}\mathbb{E}_{\beta}\big[\cos^2(x)Y_1(x,t)\big] \\ &- \frac{1}{s}\mathbb{E}_{\beta}\big[\big(Y_2^2(x,t)Y_1(x,t) + Y_1^3(x,t)\big)\big]. \end{aligned} \tag{57}$$

Assess $\mathbb{E}_{\beta}^{-1}$ in the system mentioned earlier.

$$\begin{aligned} Y_1(x,t) =& \mathbb{E}_{\beta}^{-1}\big[\frac{1}{s}Y_1(0)\big] - \mathbb{E}_{\beta}^{-1}\big[\frac{1}{s}\mathbb{E}\big[\frac{1}{2}\mathbb{D}xxY_2(x,t)\big]\big] + \mathbb{E}^{-1}\big[\frac{1}{s}\mathbb{E}\big[\cos^2(x)Y_2(x,t)\big]\big] \\ &+ \mathbb{E}_{\beta}^{-1}\big[\frac{1}{s}\mathbb{E}_{\beta}\big[\big(Y_1^2(x,t)Y_2(x,t) + Y_2^3(x,t)\big)\big]\big], \\ Y_2(x,t) =& \mathbb{E}_{\beta}^{-1}\big[\frac{1}{s}Y_2(0)\big] + \mathbb{E}_{\beta}^{-1}\big[\frac{1}{s}\mathbb{E}_{\beta}\big[\frac{1}{2}\mathbb{D}xxY_1(x,t)\big]\big] - \mathbb{E}_{\beta}^{-1}\big[\frac{1}{s}\mathbb{E}_{\beta}\big[\cos^2(x)Y_1(x,t)\big]\big] \\ &- \mathbb{E}_{\beta}^{-1}\big[\frac{1}{s}\mathbb{E}\big[\big(Y_2^2(x,t)Y_1(x,t) + Y_1^3(x,t)\big)\big]\big]. \end{aligned} \tag{58}$$

Using the method outlined in Section 3, we derive the following result from Eq. (58):

$$
\begin{aligned}
\sum_{\jmath=0}^{\infty} Y_{1,\jmath}(x,t) =& \mathbb{E}_{\beta}^{-1}\Big[\frac{1}{s}Y_1(0)\Big] - \mathbb{E}_{\beta}^{-1}\Big[\frac{1}{s}\mathbb{E}_{\beta}\Big[\frac{1}{2}\mathbb{D}xx\sum_{\jmath=0}^{\infty} Y_{2,\jmath}(x,t)\Big]\Big]+ \\
& \mathbb{E}_{\beta}^{-1}\Big[\frac{1}{s}\mathbb{E}_{\beta}\Big[\cos^2(x)\sum_{\jmath=0}^{\infty} Y_{2,\jmath}(x,t)\Big]\Big] + \mathbb{E}_{\beta}^{-1}\Big[\frac{1}{s}\mathbb{E}_{\beta}\Big[\sum_{\jmath=0}^{\infty}\mathfrak{A}_{1,\jmath}(Y_1,Y_2)\Big],\\
\sum_{\jmath=0}^{\infty} Y_{2,\jmath}(x,t) =& \mathbb{E}_{\beta}^{-1}\Big[\frac{1}{s}Y_2(0)\Big] + \mathbb{E}_{\beta}^{-1}\Big[\frac{1}{s}\mathbb{E}\Big[\frac{1}{2}\mathbb{D}xx\sum_{\jmath=0}^{\infty} Y_{1,\jmath}(x,t)\Big]\Big]- \\
& \mathbb{E}_{\beta}^{-1}\Big[\frac{1}{s}\mathbb{E}_{\beta}\Big[\cos^2(x)\sum_{\jmath=0}^{\infty} Y_{1,\jmath}(x,t)\Big]\Big] - \mathbb{E}_{\beta}^{-1}\Big[\frac{1}{s}\mathbb{E}_{\beta}\Big[\sum_{\jmath=0}^{\infty}\mathfrak{A}_{2,\jmath}(Y_1,Y_2)\Big]. \qquad (59)
\end{aligned}
$$

We extracted the first term of the FPS solution for the Eq. (55) via the equivalent on the two ends of Eq. (59).

$$
\begin{aligned}
Y_{1,0}(x,t) &= \sin(x),\\
Y_{2,0}(x,t) &= 0. \qquad (60)
\end{aligned}
$$

We extract the following second term of the FPS.

$$
\begin{aligned}
Y_{1,1}(x,t) &= 0,\\
Y_{2,1}(x,t) &= -\frac{3}{2}\sin(x)\frac{t^{\beta}}{\beta\Gamma(2)}. \qquad (61)
\end{aligned}
$$

Similarly, we found the third, fourth, and fifth terms.

$$
\begin{aligned}
Y_{1,2}(x,t) &= -\frac{9}{4}\sin(x)\frac{t^{2\beta}}{\beta^2\Gamma(3)},\\
Y_{2,2}(x,t) &= 0. \qquad (62)
\end{aligned}
$$

$$
\begin{aligned}
Y_{1,3}(x,t) &= 0,\\
Y_{2,3}(x,t) &= \frac{27}{8}\sin(x)\frac{t^{3\beta}}{\beta^3\Gamma(4)}. \qquad (63)
\end{aligned}
$$

$$
\begin{aligned}
Y_{1,4}(x,t) &= \frac{81}{16}\sin(x)\frac{t^{4\beta}}{\beta^4\Gamma(5)},\\
Y_{2,4}(x,t) &= 0. \qquad (64)
\end{aligned}
$$

The procedure is repeated to achieve the results for the sixth term.

$$
\begin{aligned}
Y_{1,5}(x,t) &= 0,\\
Y_{2,5}(x,t) &= -\frac{243}{32}\sin(x)\frac{t^{5\beta}}{\beta^5\Gamma(6)}. \qquad (65)
\end{aligned}
$$

Hence, the system described in Eq. (55) has an FPS solution, which is outlined below.

$$
Y(x,t) = \sin(x)\Big(\sum_{\jmath=0}^{\infty}\frac{(-1)^{\jmath}}{\Gamma(2\jmath+1)}\Big(\frac{3t^{\beta}}{2\beta}\Big)^{2\jmath} - \mathfrak{i}\sum_{\jmath=0}^{\infty}\frac{(-1)^{\jmath}}{\Gamma(2\jmath+2)}\Big(\frac{3t^{\beta}}{2\beta}\Big)^{2\jmath+1}\Big). \qquad (66)
$$

The Ex-S when $\beta = 1.0$ is $Y(x,t) = \sin(x)\exp^{-\frac{3\mathfrak{i}t}{2}}$. An identical result was obtained by [37].

5 Graphical and Numerical Outcome

In this section, we analyze the numerical and graphical results of the Ex-Ss and App-Ss for the linear and nonlinear problems presented in the fourth section of this research study. To assess the EADM's accuracy, we use two error functions, the Abs-E and Rel-E functions. Delineating the errors in the App-Ss is essential since EADM provides an approximation expressed in terms of an infinite FPS.

First, we present some useful notation for Ex-S and App-S, along with formulas for error functions, which we utilize in this section to analyze the reliability and correctness of our approach.

$$\mathrm{Y}(\mathrm{x},\mathrm{t}) \approx \mathrm{Y}^{\kappa}(\mathrm{x},\mathrm{t}),\ \kappa = 1,\ 2,\ 3, \cdots,$$

where $\mathrm{Y}(\mathrm{x},\mathrm{t})$ and $\mathrm{Y}^{\kappa}(\mathrm{x},\mathrm{t})$ denote the Ex-S and App-S of the nonlinear problems 4.2 and 4.3 obtained by EADM.

Abs-E plays a crucial role in quantifying the accuracy of these App-Ss and is indispensable for validating and refining numerical techniques. The Abs-E represents the magnitude of the absolute difference between the App-S $(\mathrm{Y}^{\kappa}(\mathrm{x},\mathrm{t}))$ obtained through approximate methods and the Ex-S $(\mathrm{Y}(\mathrm{x},\mathrm{t}))$, if available. Smaller Abs-E values indicate higher accuracy in the approximation. The Abs-E is defined as follows:

$$Abs.E^{\kappa}(\mathrm{x},\mathrm{t}) = |\mathrm{Y}(\mathrm{x},\mathrm{t}) - \mathrm{Y}^{\kappa}(\mathrm{x},\mathrm{t})|,\ \kappa = 1,\ 2,\ 3, \cdots,$$

when κ increases to infinity, it frequently happens that $\mathrm{Abs.E}^{\kappa}(\mathrm{x},\mathrm{t})$ gets decreasing, eventually decreasing almost to zero.

The Rel-E serves as a powerful tool for evaluating the effectiveness of the approach that generates App-Ss. It is calculated as the ratio of the Abs-E to the magnitude of the Ex-S at each point within the solution domain. This provides valuable insights into the degree of alignment between the App-S and the behavior of the Ex-S. A smaller Rel-E indicates higher accuracy in the approximation. Mathematically, it is defined as follows:

$$Rel.E^{\kappa}(\mathrm{x},\mathrm{t}) = \frac{|\mathrm{Y}(\mathrm{x},\mathrm{t}) - \mathrm{Y}^{\kappa}(\mathrm{x},\mathrm{t})|}{|\mathrm{Y}(\mathrm{x},\mathrm{t})|},\ \kappa = 1,\ 2,\ 3, \cdots,$$

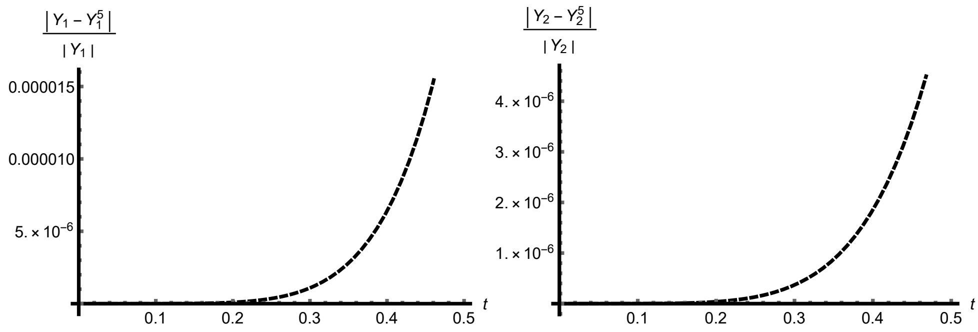
where the Rel-E for the κth-step App-S is represented by $\mathrm{Rel.E}^{\kappa}(\mathrm{x},\mathrm{t})$ for the Ex-S $(\mathrm{Y}(\mathrm{x},\mathrm{t}))$. In fact, it often happens that as κ goes to infinity, $\mathrm{Rel.E}^{\kappa}(\mathrm{x},\mathrm{t})$ gets ever smaller until it almost reaches zero.

In Figures 1-4 for Problems 4.2 and 4.3, 2D curves are employed to compare the App-Ss and Ex-Ss in terms of Rel-E and Abs-E. The comparative analysis reveals a high degree of similarity between the fifth-step App-Ss and the Ex-Ss. The Abs-E is presented on the graphs to demonstrate the excellent precision of EADM.

The 2D graphs of the App-Ss obtained from five iterations and the Ex-Ss derived by EADM for $\beta = 0.6, 0.7, 0.8, 0.9$, and 1.0 are depicted in Figures 5 and 6 for Problems 4.2 and 4.3. These graphs illustrate how, as $\beta \to 1.0$, the App-Ss converge to the Ex-Ss. The interaction between the App-Ss and Ex-Ss when $\beta = 1.0$ demonstrates the accuracy of the proposed approach.

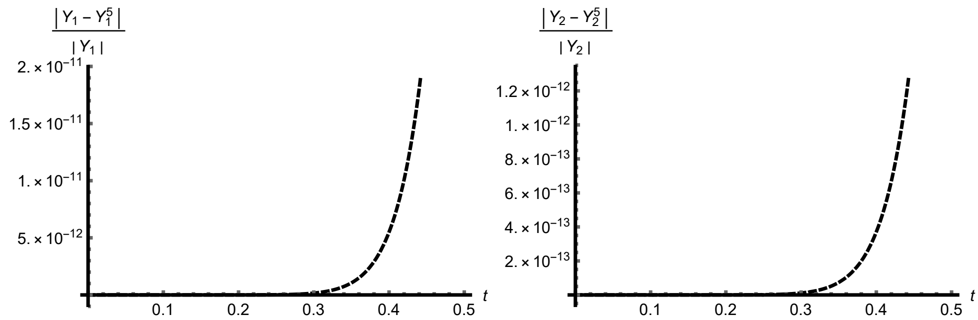
Figures 7-14 for Problems 4.2 and 4.3 display the 3D graphs of the App-Ss obtained from five iterations and the Ex-Ss determined by EADM for $\beta = 0.7, 0.8, 0.9, 1.0$ and Ex-Ss. These graphs demonstrate how the App-Ss converge to the Ex-Ss as β approaches 1.0. The interaction between the App-Ss and Ex-Ss when $\beta = 1.0$ illustrates the accuracy of the proposed method.

The Abs-E and Rel-E for specified locations between the Ex-Ss and fifth-order App-Ss derived by EADM in Problems 4.2 and 4.3 at $\beta = 1.0$ are presented in Tables 1 and 2. These tables demonstrate that the App-Ss and Ex-Ss are nearly in agreement, confirming the accuracy of EADM. From these tables, it is observed that the Abs-E and Rel-E for all problems in the fifth-step App-Ss is very small. The findings presented in this section, depicted in both graphs and tables, demonstrate that EADM is a useful and effective technique for solving FSDEs, requiring fewer calculations and iterations.



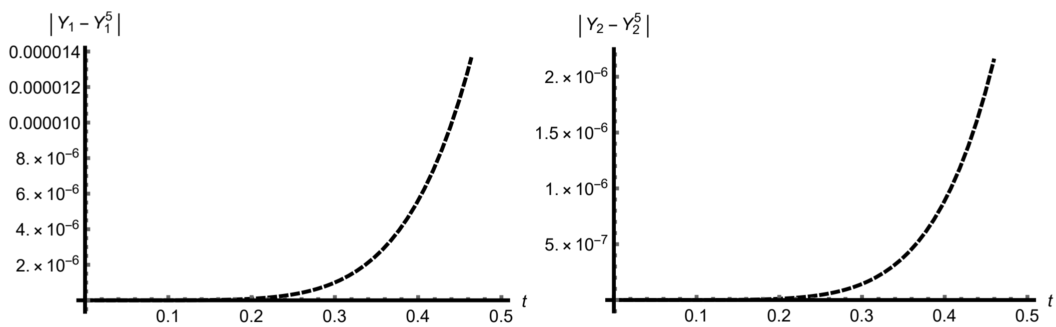
(a) (b)

Figure 1: 2D graphs of Rel-E in the range $t \in [0, 0.5]$ comparing the fifth-step App-S and Ex-S for $x = 0.1$ with $\beta = 1.0$ in problem 4.2: (a) $Y_1(x, t)$; (b) $Y_2(x, t)$.



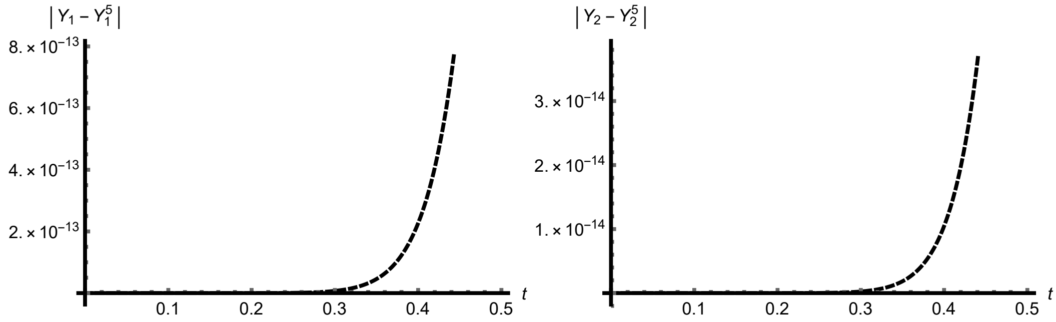
(a) (b)

Figure 2: 2D graphs of Rel-E in the range $t \in [0, 0.5]$ comparing the fifth-step App-S and Ex-S for $x = 0.1$ with $\beta = 1.0$ in problem 4.3: (a) $Y_1(x, t)$; (b) $Y_2(x, t)$.



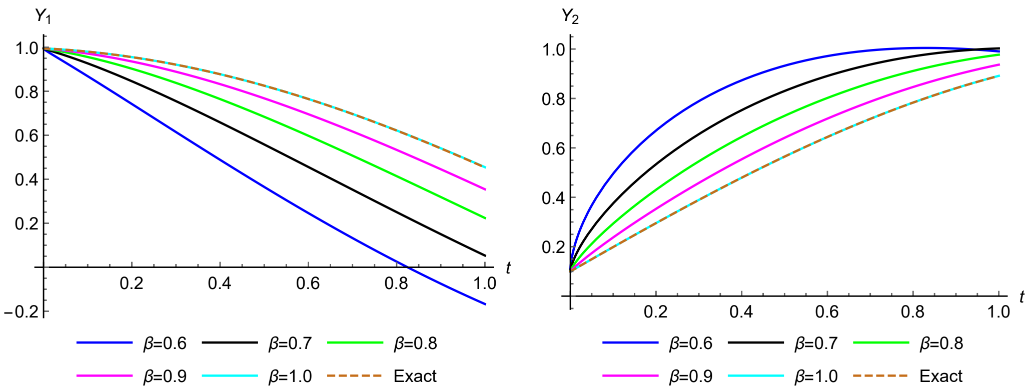
(a) (b)

Figure 3: 2D graphs of Abs-E in the range $t \in [0, 0.5]$ comparing the fifth-step App-S and Ex-S for $x = 0.1$ with $\beta = 1.0$ in problem 4.2: (a) $Y_1(x, t)$; (b) $Y_2(x, t)$.



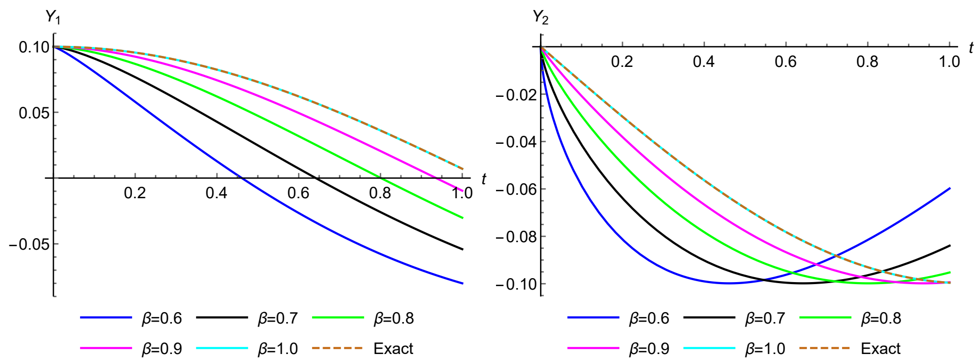
(a) (b)

Figure 4: 2D graphs of Abs-E in the range $t \in [0, 0.5]$ comparing the fifth-step App-S and Ex-S for $x = 0.1$ with $\beta = 1.0$ in problem 4.3: (a) $Y_1(x,t)$; (b) $Y_2(x,t)$.



(a) (b)

Figure 5: For problem 4.2, the 2D diagrams of App-S for various levels of β and Ex-S in the range $t \in [0, 1.0]$ at $x = 0.1$ are shown for: (a) $Y_1(x,t)$; (b) $Y_2(x,t)$.



(a) (b)

Figure 6: For problem 4.3, the 2D diagrams of App-Ss for various levels of β and Ex-Ss in the range $t \in [0, 1.0]$ at $x = 0.1$ are shown for: (a) $Y_1(x,t)$; (b) $Y_2(x,t)$.

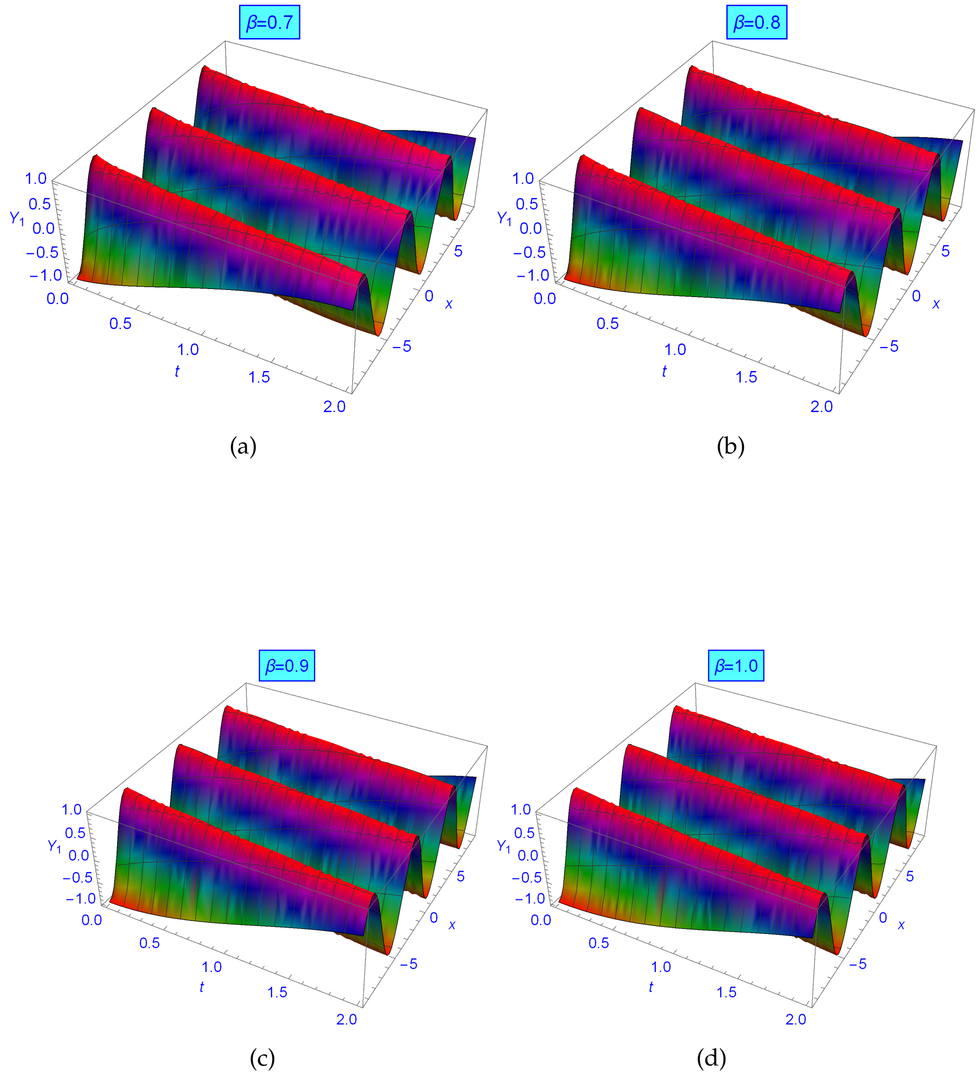


Figure 7: The 3D diagrams of App-S for various levels of β in the range $t \in [0, 2.0]$ and $-3\pi \leq x \leq 3\pi$ for Problem 4.2 of $Y_1(x, t)$ are as follows: (a) $\beta = 0.7$; (b) $\beta = 0.8$; (c) $\beta = 0.9$; and (d) $\beta = 1.0$.

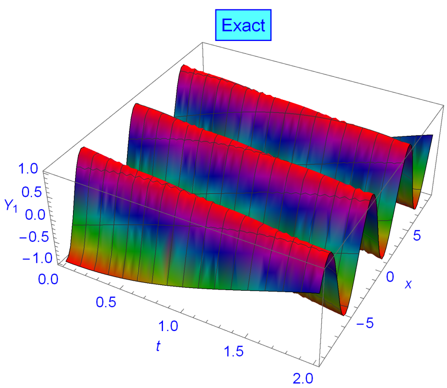
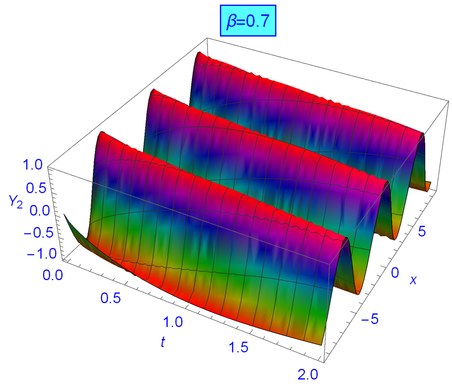
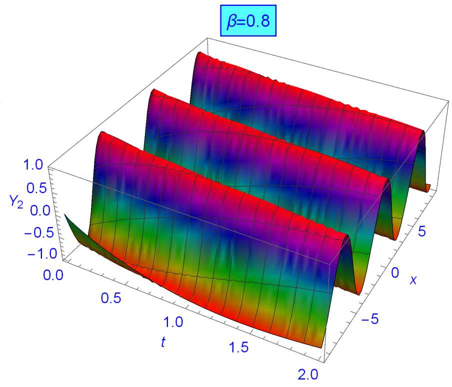


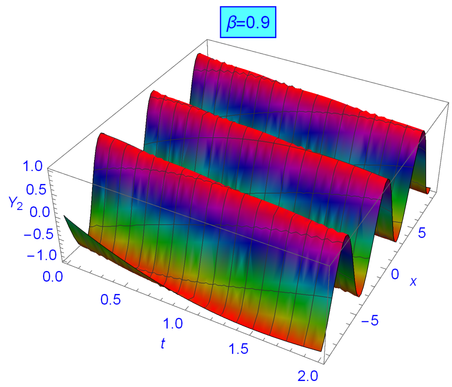
Figure 8: For problem 4.2, the 3D diagrams of Ex-S in the range $t \in [0, 2.0]$ and $-3\pi \leq x \leq 3\pi$ for $Y_1(x, t)$ are shown.



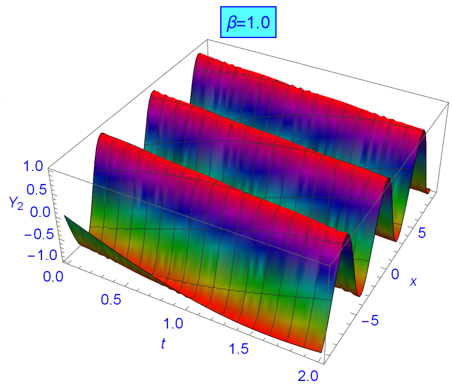
(a)



(b)



(c)



(d)

Figure 9: The 3D diagrams of App-S for various levels of β in the range $t \in [0, 2.0]$ and $-3\pi \leq x \leq 3\pi$ for Problem 4.2 of $Y_2(x,t)$ are as follows: (a) $\beta = 0.7$; (b) $\beta = 0.8$; (c) $\beta = 0.9$; and (d) $\beta = 1.0$.

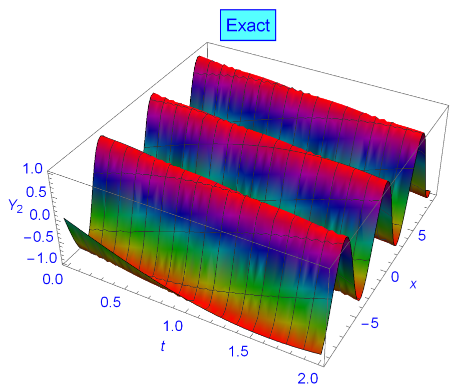
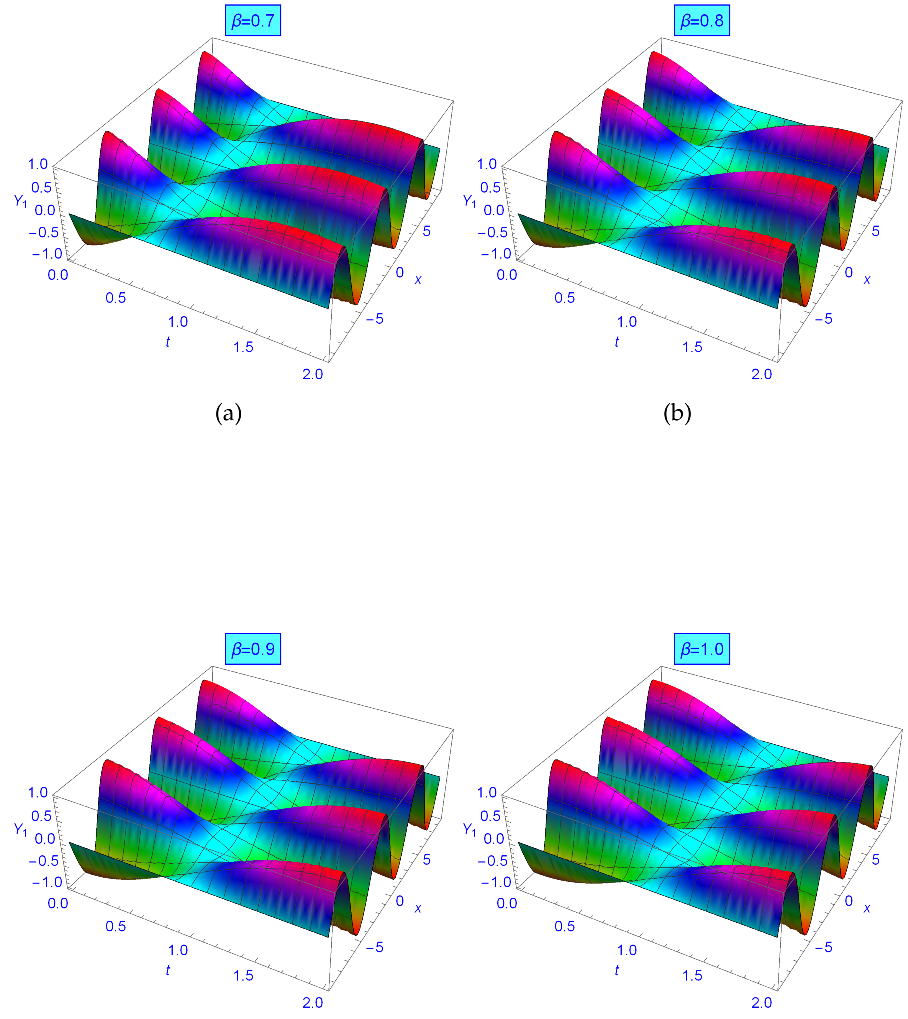


Figure 10: For problem 4.2, the 3D diagrams of Ex-S in the range $t \in [0, 2.0]$ and $-3\pi \leq x \leq 3\pi$ for $Y_2(x,t)$ are shown.



(a) (b)

(c) (d)

Figure 11: The 3D diagrams of App-S for various levels of β in the range $\mathrm{t} \in [0, 2.0]$ and $-3\pi \leq \mathrm{x} \leq 3\pi$ for Problem 4.3 of $\mathrm{Y}_1(\mathrm{x}, \mathrm{t})$ are as follows: (a) $\beta = 0.7$; (b) $\beta = 0.8$; (c) $\beta = 0.9$; and (d) $\beta = 1.0$.

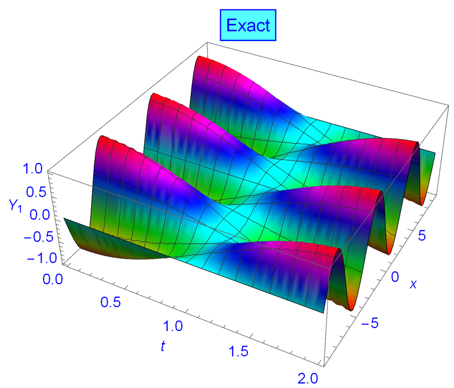


Figure 12: For problem 4.3, the 3D diagrams of Ex-S in the range $\mathrm{t} \in [0, 2.0]$ and $-3\pi \leq \mathrm{x} \leq 3\pi$ for $\mathrm{Y}_1(\mathrm{x}, \mathrm{t})$ are shown.

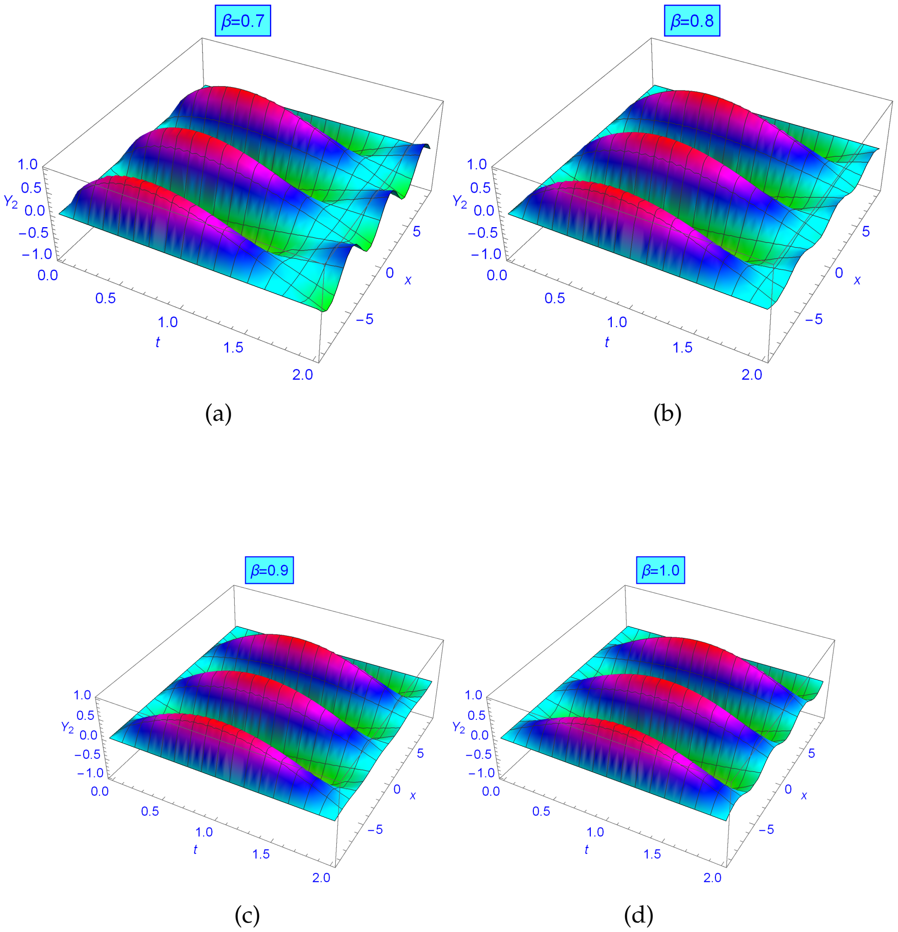


Figure 13: The 3D diagrams of App-S for various levels of β in the range $t \in [0, 2.0]$ and $-3\pi \leq x \leq 3\pi$ for Problem 4.3 of $Y_2(x,t)$ are as follows: (a) $\beta = 0.7$; (b) $\beta = 0.8$; (c) $\beta = 0.9$; and (d) $\beta = 1.0$.

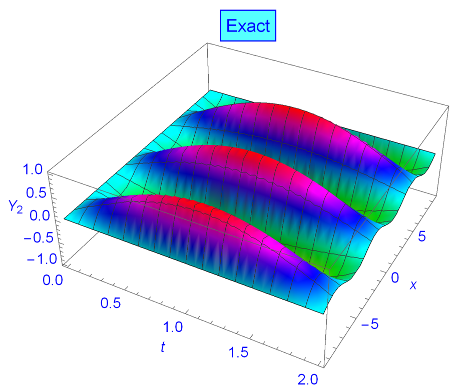


Figure 14: For problem 4.3, the 3D diagrams of Ex-S in the range $t \in [0, 2.0]$ and $-3\pi \leq x \leq 3\pi$ for $Y_2(x,t)$ are shown.

The numerical values of Abs-E and Rel-E are shown in Tables 1 and 2 for the 5th-step App-S and Ex-S of the real $(Y_1(x,t))$ and imaginary $(Y_2(x,t))$ parts of $Y(x,t)$ for problems 4.2 and 4.3, respectively, with $\beta = 1$ and $x = 0.2$. These tables demonstrate that the App-Ss and Ex-Ss are nearly in agreement, confirming the accuracy of EADM.

Table 1: The Abs-E and Rel-E for $Y_1(x,t)$ and $Y_2(x,t)$ at $\beta = 1$ for problem 4.2.

t	$\lvert Y_1 - Y_1^5\rvert$	$\frac{\lvert Y_1 - Y_1^5\rvert}{\lvert Y_1\rvert}$	$\lvert Y_2 - Y_2^5\rvert$	$\frac{\lvert Y_2 - Y_2^5\rvert}{\lvert Y_2\rvert}$
0.1	1.35702×10^{-9}	1.42046×10^{-9}	2.95323×10^{-10}	9.99334×10^{-10}
0.2	8.65506×10^{-8}	9.39683×10^{-8}	2.01346×10^{-8}	5.17042×10^{-8}
0.3	9.82114×10^{-7}	1.11911×10^{-6}	2.43305×10^{-7}	5.07492×10^{-7}
0.4	5.49515×10^{-6}	6.65808×10^{-6}	1.44488×10^{-6}	2.55892×10^{-6}
0.5	2.08672×10^{-5}	2.72831×10^{-5}	5.80614×10^{-6}	9.01270×10^{-6}
0.6	6.20037×10^{-5}	8.89954×10^{-5}	1.82078×10^{-5}	2.53818×10^{-5}
0.7	1.55526×10^{-4}	2.50199×10^{-4}	4.80863×10^{-5}	6.13872×10^{-6}
0.8	3.44589×10^{-4}	6.37771×10^{-4}	1.11933×10^{-4}	1.33020×10^{-4}
0.9	6.94386×10^{-4}	1.53085×10^{-3}	2.36508×10^{-4}	2.65379×10^{-4}
1.0	1.29829×10^{-3}	3.58289×10^{-3}	4.62838×10^{-6}	4.96586×10^{-4}

Table 2: The Abs-E and Rel-E for $Y_1(x,t)$ and $Y_2(x,t)$ at $\beta = 1$ for problem 4.3.

t	$\lvert Y_1 - Y_1^5\rvert$	$\frac{\lvert Y_1 - Y_1^5\rvert}{\lvert Y_1\rvert}$	$\lvert Y_2 - Y_2^5\rvert$	$\frac{\lvert Y_2 - Y_2^5\rvert}{\lvert Y_2\rvert}$
0.1	2.77556×10^{-17}	1.41294×10^{-16}	3.46945×10^{-18}	1.16861×10^{-16}
0.2	2.22045×10^{-16}	1.16991×10^{-15}	0.00000	0.00000
0.3	2.85605×10^{-14}	1.59653×10^{-13}	9.99201×10^{-16}	1.15629×10^{-14}
0.4	9.01057×10^{-13}	5.49529×10^{-12}	4.16195×10^{-14}	3.71016×10^{-13}
0.5	1.30975×10^{-11}	7.98778×10^{-11}	7.55923×10^{-13}	5.58203×10^{-12}
0.6	1.16620×10^{-10}	9.44332×10^{-10}	8.07848×10^{-12}	5.19106×10^{-11}
0.7	7.40353×10^{-10}	7.48950×10^{-9}	5.98459×10^{-11}	3.47274×10^{-10}
0.8	3.66893×10^{-9}	5.09649×10^{-8}	3.39027×10^{-10}	1.83092×10^{-9}
0.9	1.50474×10^{-8}	3.45838×10^{-7}	1.56469×10^{-9}	8.07183×10^{-9}
1.0	5.31541×10^{-8}	3.78232×10^{-6}	6.14324×10^{-9}	3.09996×10^{-8}

6 Conclusions

In this paper, we have developed App-Ss and Ex-Ss for both linear and nonlinear FSDEs involving time-fractional conformable derivatives. The effectiveness of the EADM has been demonstrated through graphical and numerical results, showing that the App-Ss obtained via the EADM have been in complete agreement with the Ex-Ss. Tables 1 and 2 have provided numerical evidence of the correctness of our approach by comparing the App-Ss and Ex-Ss through Abs-E and Rel-E. Additionally, the results from EADM have been compared with those from other techniques, such as HAM and RPSM. This comparison has shown a strong level of agreement, indicating that EADM has been an effective alternative to Cap-FD-based methods for solving FSDEs.

The key features of the EADM distinguish it from other methods that provide approximate solutions. This method eliminates the need for assumptions about physical parameters, making it applicable to both weakly and strongly nonlinear problems and addressing some limitations of perturbation techniques. The EADM allows for the derivation of FPS solutions for FODEs without requiring perturbation, linearization, or discretization, unlike other approximate solution methods. The efficiency of the EADM underscores its computational strength, making it a valuable alternative to Cap-FD-based methods for solving FSDEs. We also found that the Con-FD effectively replaces the Cap-FD for modeling time-FSDEs. Overall, the EADM proves to be user-friendly, accurate, and efficient. Moreover, this method is versatile and can be applied to a range of ordinary and partial FODEs.

We aim to extend the EADM to address FODEs in other scientific and engineering domains in future work.

7 Declarations

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Competing Interests

I declare that there are no competing interests.

Ethical Approval

Not applicable.

Authors's Contributions

Not applicable.

Availability Data and Materials

Not applicable.

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Abbreviation

The following abbreviations are used in this manuscript:

HAM	homotopy analysis method
RPSM	residual power series method
FC	Fractional calculus
FODEs	fractional order differential equations
DEs	differential equations
Cap-FD	Caputo fractional derivatives
Con-FrD	Conformable fractional derivatives
FSDEs	fractional Schrödinger differential equation
SDE	Schrödinger differential equation
ADM	Adomian decomposition method
ET	Elzaki transform
EADM	Elzaki Adomian decomposition method
App-Ss	approximate solutions
Ex-Ss	exact solutions
ABS-E	absolute errors
Rel-E	relative errors

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