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Existence of solutions for a class of Kirchhoff-type equations with indefinite potential

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Abstract

This study explores the existence of solutions to the following Kirchhoff-type problem

$$\begin{cases} -\left(a+b\int_{\mathbb{R}^3}|\nabla u|^2dx\right)\Delta u+V(x)u=f(x,u), \text{ in } \mathbb{R}^3, \\ u\in H^1(\mathbb{R}^3), \end{cases}$$

where a and b are postive constants, and the potential $V(x)$ is continuous and indefinite in sign. With suitable assumptions on $V(x)$ and f, we establish the existence of solutions using the Symmetric Mountain Pass Theorem.

Keywords: Kirchhoff-type equations, $(C)_c$-condition, Symmetric Mountain Pass Theorem
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1 Introduction and main result

This study explores the existence of solutions to the following Kirchhoff-type problem

$$\begin{cases} -\left(a+b\int_{\mathbb{R}^3}|\nabla u|^2dx\right)\Delta u+V(x)u=f(x,u), & \text{in } \mathbb{R}^3, \\ u\in H^1(\mathbb{R}^3), \end{cases} \tag{1}$$

where a and b are postive constants, and the potential $V(x)$ is continuous and indefinite in sign. The nonlinear term $\int_{\mathbb{R}^3}|\nabla u|^2dx$ appears in (1), which implies that (1) is not a pointwise identity. This has resulted in certain mathematical challenges, rendering this research especially intriguing. (1) possesses a fascinating physical background. When $V(x)=0$, and a bounded domain $\Omega\subset\mathbb{R}^N$ is substituted $\mathbb{R}^3$, then we obtain the following nonlocal Kirchhoff-type problem

$$\begin{cases} -\left(a+b\int_{\Omega}|\nabla u|^2dx\right)\Delta u=f(x,u), & \text{in } \Omega, \\ u=0, & \text{on } \partial\Omega. \end{cases} \tag{2}$$

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The problem (2) is regard to the stationary analogue of the equation

$$u_{tt} - \left(a + b\int_{\Omega} |\nabla u|^2 dx\right)\Delta u = f(x,u), \tag{3}$$

which was presented by Kirchhoff in [1], and (3) extends the classical D'Alembert wave equation, applied to the free vibration of elastic strings. Since Lions introduced an abstract framework for problem (3) in [2], it has garnered increasing attention. For the background on the physics and mathematics of this problem, see [3,4,5].

In recent years, there has been a great deal of researches on the Schrödinger-Kirchhoff equations, numerous papers have made different assumptions about $V(x)$ and f, refer to [6-23]. In [6-15], the potential $V(x)$ is assumed to be positive definite. In [6], Wu used a Symmetric Montain Theorem obtained nontrivial and high energy solutions for equations similar to (1) in $\mathbb{R}^N$. In [12], by applying Ekeland's variational principle and the Montain Pass Theorem, Cheng obtained a large number of nontrivial solutions for the nonhomogeneous Schrödinger Kirchhoff type problem in $\mathbb{R}^N$. The indefiniteness of the potential $V(x)$ has been discussed in [16-23]. In [18], Chen and Wu got a nontrivial solution and an unbounded sequence of solutions for the problem (1) in $\mathbb{R}^N$ via the Morse Theory and the Fountain Theorem. In [22], using the Local Linking Theorem and Clark's Theorem, Jiang and Liu obtained the existence of multiple solutions for problem (1).

Set $F(x,u) = \int_0^u f(x,s)ds$, $V^{\pm}(x) = \max\{\pm V(x), 0\}$, then $V(x) = V^+(x) - V^-(x)$. Before stating our main result, we make the following assumptions:

(V_1) $V(x) \in C(\mathbb{R}^3, \mathbb{R})$, $V(x)$ is bounded from below, and there is $M > 0$ such that the set $\{x \in \mathbb{R}^3 | V^+(x) < M\}$ is nonempty and has finite measure.

(V_2) There is a constant $\tau_0 > 1$ such that

$$\tau_1 := \inf_{u\in H^1(\mathbb{R}^3)\setminus\{0\}} \frac{\int_{\mathbb{R}^3}\left(a|\nabla u|^2 + V^+u^2\right)dx}{\int_{\mathbb{R}^3} V^- u^2 dx} \geq \tau_0.$$

(f_1) $f \in C^1(\mathbb{R}^3, \mathbb{R})$, and there are constants $q \in (2,6)$ and $c > 0$ such that

$$|f(x,u)| \leq c(1 + |u|^{q-1}), \quad \forall (x,u) \in \mathbb{R}^3 \times \mathbb{R}.$$

(f_2) $f(x,u) = o(u)$ as $u \to 0$ uniformly in $x \in \mathbb{R}^3$, and is 4-superlinear at infinity,

$$\lim_{|u|\to\infty} \frac{F(x,u)}{u^4} = +\infty.$$

(f_3) There exist $a_0, b_0 > 0$ and $\theta \in (0, \theta_*)$ such that

$$0 < (4 + \frac{1}{a_0|u|^{\theta} + b_0})F(x,u) \leq uf(x,u), \quad \text{for } x \in \mathbb{R}^3 \text{ and } u \neq 0,$$

where $\theta_* := \min\{q', 5q' - 6\}$, $\frac{1}{q} + \frac{1}{q'} = 1$.

(f_4) $\lim_{|x|\to\infty} \sup_{|u|\leq l} \frac{|f(x,u)|}{|u|} = 0$ for every $l > 0$.

We are now prepared to present the main result of this paper:

Theorem 1.1. *Under assumptions (V_1), (V_2) and (f_1)-(f_4), if $f(x,u)$ is odd in u, then problem (1) has infinitely many solutions.*

2 Preliminaries

We work in the Hilbert space

$$E := \left\{ u \in H^1(\mathbb{R}^3) : \int_{\mathbb{R}^3} V^+(x) |u|^2 \, dx < +\infty \right\},$$

and define the inner product

$$\langle u, w \rangle = \int_{\mathbb{R}^3} (a \nabla u \nabla w + V^+(x) u w) \, dx, \ \ \forall u, w \in E,$$

with the norm

$$\|u\| = \langle u, u \rangle^{1/2}, \ \ \forall u \in E.$$

The problem (1) possesses a variational structure, thus a weak solution to problem (1) is a critical point of the following functional $\Psi : E \to \mathbb{R}$

$$\Psi(u) = \frac{1}{2} \int_{\mathbb{R}^3} (a|\nabla u|^2 + V(x)u^2) \, dx + \frac{b}{4} \left(\int_{\mathbb{R}^3} |\nabla u|^2 dx \right)^2 - \int_{\mathbb{R}^3} F(x,u) dx. \tag{4}$$

Then under assumptions (V_1), (f_1) and (f_2), the functional $\Psi \in C^1(E, \mathbb{R})$ and for any $u, w \in E$,

$$\langle \Psi'(u), w \rangle = \int_{\mathbb{R}^3} (a \nabla u \nabla w + V(x) u w) \, dx + b \int_{\mathbb{R}^3} |\nabla u|^2 dx \int_{\mathbb{R}^3} \nabla u \nabla w dx - \int_{\mathbb{R}^3} f(x,u) w dx. \tag{5}$$

For any $t \in [2, 6]$, since the continuous embedding $E \hookrightarrow L^t(\mathbb{R}^3)$, there exists a constant $\zeta_t > 0$ such that

$$|u|_t \le \zeta_t \|u\|, \ \ \forall u \in E. \tag{6}$$

Forthermore, it follows from (V_2) that

$$\|u\| \ge \int_{\mathbb{R}^3} \left(a |\nabla u|^2 + V |u|^2 \right) dx \ge \frac{\tau_0 - 1}{\tau_0} \|u\|. \tag{7}$$

We use the following Symmetric Mountain Pass Theorem to prove theorem 1.1.

Theorem 2.1. *([24]) Let X be an infinite dimensional Banach space, $X = Y \oplus Z$, where Y is finite dimensional. If $I \in C^1(X, \mathbb{R})$ satisfies $(C)_c$-condition for all $c > 0$, and*

(i_1) *$I(0) = 0, I(-u) = I(u), \forall u \in X$;*
(i_2) *there exist constants $\alpha, \rho > 0$, such that $I|_{\partial B_\rho \cap Z} \ge \alpha$;*
(i_3) *for any finite dimensional subspace $\tilde{X} \subset X$, there is $R = R(\tilde{X}) > 0$, such that $I(u) \le 0$ on $\tilde{X} \setminus B_R$;*
then I possesses an unbounded sequence of critical values.

Definition 2.2. ([25]) Let E be a Banach space, and $\Phi \in C^1(E, \mathbb{R})$. For given $c \in \mathbb{R}$, a sequence $\{u_n\} \subset E$ is called a Cerami sequence of Φ at a level c (shortly, $(C)_c$ sequence) if

$$\Phi(u_n) \to c, \quad (1 + \|u_n\|) \left\| \Phi'(u_n) \right\| \to 0. \tag{8}$$

We say that Φ satisfies the Cerami condition at level c (shortly, $(C)_c$-condition) if every $(C)_c$ sequence of Φ contains a convergent subsequence. If Φ satisfies $(C)_c$-condition for every $c \in \mathbb{R}$, then we say that Φ satisfies the Cerami condition..

3 Proof of main results

In this section, we will first prove that the functional Ψ satisfies $(C)_c$-condition, subsequently show the Ψ satisfies (i_2) and (i_3) of Theorem 2.1, and finally, demonstrate problem (1) has infinitely many solutions.

Lemma 3.1. *Suppose that (V_1) and (f_1)-(f_3) are satisfied and $c \in \mathbb{R}$. Then any $(C)_c$ sequence of Ψ is bounded.*

Proof. From (f_3), we know that

$$uf(x,u) - 4F(x,u) \geq \frac{1}{4a_0 |u|^\theta + 4b_0 + 1} uf(x,u) > 0, \quad \forall u \neq 0, x \in \mathbb{R}^3.$$

Let $\{u_n\}$ be a $(C)_c$ sequence of Ψ. Set $\Omega_n := \{x \in \mathbb{R}^3 : |u_n(x)| < 1\}$ and $\Omega_n^c := \mathbb{R}^3 \setminus \Omega_n$. Then there exist two constants $c_0, c_1 > 0$ such that

$$4a_0 |u_n|^\theta + 4b_0 + 1 \leq 1/c_0, \quad \forall x \in \Omega_n,$$

and

$$4a_0 |u_n|^\theta + 4b_0 + 1 \leq |u_n|^\theta / c_1, \quad \forall x \in \Omega_n^c.$$

For large n, there exists $M_1 > 0$, such that

$$\begin{aligned} M_1 &\geq 4\Psi(u_n) - \langle \Psi'(u_n), u_n \rangle \\ &\geq \frac{\tau_0 - 1}{\tau_0} \|u_n\|^2 + \int_{\mathbb{R}^3} (u_n f(x,u_n) - 4F(x,u_n))\, dx \\ &\geq \int_{\mathbb{R}^3} (u_n f(x,u_n) - 4F(x,u_n))\, dx \\ &\geq \int_{\mathbb{R}^3} \frac{u_n f(x,u_n)}{4a_0 |u_n|^\theta + 4b_0 + 1} dx \\ &\geq c_0 \int_{\Omega_n} u_n f(x,u_n) dx + c_1 \int_{\Omega_n^c} |u_n|^{-\theta} u_n f(x,u_n) dx. \end{aligned} \tag{9}$$

Note that $\theta < 5q' - 6$ by (f_3). We have

$$\frac{1}{q'} < \frac{6}{5q'} < \frac{6}{6+\theta} \text{ and } \frac{2}{2+\theta} < \frac{6}{6+\theta}.$$

Then we can chose a constant $s \in (0,1)$ such that

$$\max\left\{\frac{6}{5q'}, \frac{2}{2+\theta}\right\} \leq s \leq \frac{6}{6+\theta}. \tag{10}$$

Let $r := s/(1-s) > 0$, then $\frac{1}{s} + \frac{1}{-r} = 1$. By (9) and the inverse Hölder inequality we have

$$\begin{aligned} M_1 &\geq c_0 \int_{\Omega_n} u_n f(x,u_n) dx + c_1 \int_{\Omega_n^c} |u_n|^{-\theta} u_n f(x,u_n) dx \\ &\geq c_0 \int_{\Omega_n} u_n f(x,u_n) \\ &\quad + c_1 \left(\int_{\Omega_n^c} (u_n f(x,u_n))^s\, dx \right)^{1/s} \left(\int_{\Omega_n^c} |u_n|^{\theta r}\, dx \right)^{1/(-r)} \\ &\geq c_0 \int_{\Omega_n} u_n f(x,u_n) dx + c_1 \frac{\left(\int_{\Omega_n^c} (u_n f(x,u_n))^s\, dx \right)^{1/s}}{|u_n|_{\theta r}^\theta}. \end{aligned} \tag{11}$$

By (f_1) and (f_2) we have

$$|f(x,u)|^{q's} \leq \left(c_2 |u|^{(q-1)(q'-1)} |f(x,u)|\right)^s = c_3 (uf(x,u))^s, \ \forall \ |u| \geq 1,$$

$$|f(x,u)|^2 \leq c_4 |u| |f(x,u)| = c_4 uf(x,u), \ \forall \ |u| < 1.$$

Then by (11), we have

$$\left(\int_{\Omega_n^c} |f(x,u_n)|^{q's} dx\right)^{1/q's} \leq c_5 |u_n|_{\theta r}^{\theta/q'}, \tag{12}$$

$$\left(\int_{\Omega_n} |f(x,u_n)|^2 dx\right)^{1/2} \leq c_6. \tag{13}$$

Taking into account (10), we can easily detect that $q's > 1$, $\theta r \in [2,6]$ and $(q's)' \in (2,6]$, where $(q's)' = q's/(q's-1)$. Consequently, by (12) and (13), for n large enough,

$$\begin{aligned}
\int_{\mathbb{R}^3} \left(a|\nabla u_n|^2 + V(x)u_n^2\right) dx &= \langle \Psi'(u_n), u_n\rangle - b\left(\int_{\mathbb{R}^3} |\nabla u_n|^2 dx\right)^2 + \int_{\mathbb{R}^3} f(x,u_n)u_n dx \\
&\leq \|u_n\| + \int_{\mathbb{R}^3} f(x,u_n)u_n dx \\
&\leq \|u_n\| + \left(\int_{\Omega_n} |f(x,u_n)|^2 dx\right)^{1/2} |u_n|_2 \\
&\quad + \left(\int_{\Omega_n^c} |f(x,u_n)|^{q's} dx\right)^{1/q's} |u_n|_{(q's)'} \\
&\leq \|u_n\| + c_6 |u_n|_2 + c_5 |u_n|_{\theta r}^{\theta/q'} |u_n|_{(q's)'} \\
&\leq c_7 \|u_n\| + c_8 \|u_n\| \|u_n\|^{\theta/q'},
\end{aligned}$$

where $c_7, c_8 > 0$ are some constants.
Therefore by (7) and constants $c_9, c_{10} > 0$, we have

$$\|u_n\| \leq c_9 + c_{10} \|u_n\|^{\theta/q'},$$

where $\theta < q'$. Then it is easy to verify that $\{u_n\}$ is bounded. □

Lemma 3.2. *Suppose that (V_1) and (f_1)-(f_4) are satisfied and $c \in \mathbb{R}$. Then Ψ satisfies $(C)_c$-condition.*

Proof. Let $\{u_n\} \subset E$ be any bounded $(C)_c$ sequence of Ψ. Then, passing to a subsequence, we may assume that $u_n \rightharpoonup u$ in E.
Note that, by (5)

$$\begin{aligned}
\langle \Psi'(u_n), u_n - u\rangle &= \int_{\mathbb{R}^3} \left(a|\nabla u_n|^2 + V^+ u_n^2\right) dx - \int_{\mathbb{R}^3} \left(a\nabla u_n \nabla u + V^+ u_n u\right) dx \\
&\quad - \int_{\mathbb{R}^3} V^- u_n^2 dx + \int_{\mathbb{R}^3} V^- u_n u dx - \int_{\mathbb{R}^3} f(x,u_n)(u_n - u) dx. \\
&\quad + b\int_{\mathbb{R}^3} |\nabla u_n|^2 dx \int_{\mathbb{R}^3} \nabla u_n \nabla(u_n - u) dx \\
&= \langle u_n, u_n - u\rangle - b\int_{\mathbb{R}^3} |\nabla u_n|^2 dx \int_{\mathbb{R}^3} \nabla u_n \nabla(u - u_n) dx \\
&\quad - \int_{\mathbb{R}^3} V^- u_n (u_n - u) dx - \int_{\mathbb{R}^3} f(x,u_n)(u_n - u) dx,
\end{aligned}$$

we have

$$0 \le \limsup_{n\to\infty} \left(\|u_n\|^2 - \|u\|^2\right) = \limsup_{n\to\infty} \langle u_n, u_n - u\rangle$$
$$= \limsup_{n\to\infty} [\langle \Psi'(u_n), u_n - u\rangle + \int_{\mathbb{R}^3} V^- u_n (u_n - u)\, dx \tag{14}$$
$$+ b\int_{\mathbb{R}^3} |\nabla u_n|^2 dx \int_{\mathbb{R}^3} \nabla u_n \nabla (u - u_n) dx + \int_{\mathbb{R}^3} f(x, u_n)(u_n - u) dx].$$

From (8)

$$\langle \Psi'(u_n), u_n - u\rangle \to 0, \ \text{ as } \ n \to \infty. \tag{15}$$

Moreover, through the boundedness of $\{u_n\}$ and $u_n \rightharpoonup u$ in E, we have

$$b\int_{\mathbb{R}^3} |\nabla u_n|^2 dx \int_{\mathbb{R}^3} \nabla u_n \nabla (u - u_n) dx \to 0, \ \text{ as } \ n \to \infty. \tag{16}$$

According to the definition of $V^-(x)$ and (V_1), there is $V^- \in L^\infty(\mathbb{R}^3)$. Furthermore, by (V_1) that $\{V^+ = 0\}$ has finite measure, that is to say meas $\{V^-(x) > 0\} < +\infty$. Since $u_n \rightharpoonup u$ in E, then $u_n \to u$ in $L^t_{loc}(\mathbb{R}^3)$, $t \in [2, 6)$, we have

$$\int_{\mathbb{R}^3} V^- u_n (u_n - u) dx \le \|V^-\|_\infty \int_W |u_n|\, |u_n - u|\, dx$$
$$\le \|V^-\|_\infty \left(\int_W |u_n|^2 dx\right)^{1/2} \left(\int_W |u_n - u|^2 dx\right)^{1/2} \to 0, \ \text{ as } \ n \to \infty. \tag{17}$$

where $W := suppV^-$.
Next, let $\varepsilon > 0$, for $l \ge 1$, by (f_1) and Hölder inequality that

$$\int_{|u_n|\ge l} f(x, u_n)(u_n - u) dx \le 2cl^{q-6} \int_{|u_n|\ge l} |u_n|^5 |u_n - u|\, dx$$
$$\le 2cl^{q-6} |u_n|_6^5 |u_n - u|_6,$$

since $q < 6$, we may fix l as large enough, then

$$\int_{|u_n|\ge l} f(x, u_n)(u_n - u) dx \le \frac{\varepsilon}{3} \tag{18}$$

for all n. And, from (f_4) there is $L > 0$ such that

$$\int_{|u_n|\le l, |x|\ge L} f(x, u_n)(u_n - u) dx \le |u_n|_2 |u_n - u|_2 \sup_{|u_n|\le l, |x|\ge L} \frac{|f(x, u_n)|}{|u_n|} \le \frac{\varepsilon}{3} \tag{19}$$

for all n. For $\varepsilon > 0$, according to (f_1) and (f_2), there is $C_\varepsilon > 0$ such that

$$|f(x, u)| \le \varepsilon |u| + C_\varepsilon |u|^{q-1}, \tag{20}$$

and

$$|F(x, u)| \le \frac{\varepsilon}{2} |u|^2 + \frac{C_\varepsilon}{q} |u|^q, \tag{21}$$

where $2 < q < 6$. Since $u_n \to u$ in $L^t(B_L(0))$ for $t \in [2, 6)$, using (20) we have

$$\int_{|u_n|\le l, |x|\le L} f(x, u_n)(u_n - u) dx$$
$$\le (\varepsilon + C_\varepsilon) \int_{|u_n|\le l, |x|\le L} \left(|u_n| + |u_n|^{q-1}\right) |u_n - u|\, dx$$
$$\le (\varepsilon + C_\varepsilon) |u_n|_2 |u_n - u|_{L^2(B_L(0))} + (\varepsilon + C_\varepsilon) |u_n|_q^{q-1} |u_n - u|_{L^q(B_L(0))} \tag{22}$$
$$\le \frac{\varepsilon}{3}$$

for n large enough. Combining (18), (19), (22), since ε is arbitrary, we conclude that

$$\int_{\mathbb{R}^3} f(x,u_n)(u_n-u)dx \to 0, \text{ as } n\to\infty. \tag{23}$$

Therefore, by (14)-(17) and (23), we obtain $\|u_n\| \to \|u\|$. Thus, $u_n \to u$ in E. □

Proof of Theorem 1.1

Let $\{e_i\}$ is a total orthonormal basis of E and define $X_i = \mathbb{R}e_i$,

$$Y_m = \bigoplus_{i=1}^{m} X_i, \quad Z_m = \bigoplus_{i=m+1}^{\infty} X_i, \ m\in\mathbb{Z}.$$

Proof. Obviously, $\Psi(0)=0$ and Ψ is even due to f is odd, we will verify that Ψ satisfies the remain conditions of Theorem 2.1.

Firstly, we can check that Ψ satisfies (i_2). By (7) and (21) with $0<\varepsilon\le \frac{\tau_0-1}{2\tau_0\zeta_2^2}$, we have

$$\begin{aligned}
\Psi(u) &\ge \frac{\tau_0-1}{2\tau_0}\|u\|^2 - \int_{\mathbb{R}^3} F(x,u)dx \\
&\ge \frac{\tau_0-1}{2\tau_0}\|u\|^2 - \frac{\varepsilon}{2}|u|_2^2 - \frac{C_\varepsilon}{q}|u|_q^q \\
&\ge \frac{1}{2}\left(\frac{\tau_0-1}{\tau_0} - \varepsilon\zeta_2^2\right)\|u\|^2 - \frac{C_\varepsilon}{q}\zeta_q^q\|u\|^q \\
&\ge \frac{1}{4}\frac{\tau_0-1}{\tau_0}\|u\|^2 - \frac{C_\varepsilon}{q}\zeta_q^q\|u\|^q, \ \forall u\in\partial B_\rho,
\end{aligned}$$

where $B_\rho = \{u\in E : \|u\|<\rho\}$. Hence,

$$\Psi|_{\partial B_\rho\cap Z_k} \ge \frac{1}{4}\frac{\tau_0-1}{\tau_0}\rho^2 - \frac{C_\varepsilon}{q}\zeta_q^q\rho^q := \alpha > 0$$

for ρ small enough.

Secondly, we verify that Ψ satisfies (i_3), for any finite dimensional subspace $\hat{E}\subset E$, there exists a positive integral number k such that $\hat{E}\subset E_k$. Because all norms in finite-dimensional spaces are equivalent, there exists a constant $b_1>0$ such that

$$|u|_4 \ge b_1\|u\|, \ \forall u\in E_k.$$

Based on (f_1) and (f_2), we deduce that for any $M_2 > \frac{b}{4b_1^4}$, there exists a constant $C(M_2)>0$ such that

$$F(x,u) \ge M_2|u|^4 - C(M_2)|u|^2, \ \forall (x,u)\in\mathbb{R}^3\times\mathbb{R}.$$

Therefore

$$\begin{aligned}
\Psi(u) &\le \frac{1}{2}\|u\|^2 + \frac{b}{4}\|u\|^4 - M_2|u|_4^4 + C(M_2)|u|_2^2 \\
&\le \frac{1}{2}\|u\|^2 + \frac{b}{4}\|u\|^4 - M_2 b_1^4\|u\|^4 + C(M_2)\zeta_2^2\|u\|^2 \\
&= (\frac{1}{2} + C(M_2)\zeta_2^2)\|u\|^2 - (M_2b_1^4 - \frac{b}{4})\|u\|^4, \ \forall u\in E_k.
\end{aligned}$$

Hence, there exists a sufficiently large $R=R(\hat{E})>0$ such that $\Psi(u)\le 0$ holds on $\hat{E}\setminus B_R$.

Based on Lemmas 3.1 and 3.2, it follows that Ψ satisfies $(C)_c$-condition, according to Theorem 2.1, problem (1) has infinitely many solutions. □

4 Conclusions

In this paper, we obtain the existence of solutions to a class of Kirchhoff-type equations with indefinite potential. Due to the potential $V(x)$ is continuous and indefinite in sign and the appearance of nonlinear term $\int_{\mathbb{R}^3} |\nabla u|^2 dx$, which lead to two difficulties, namely, verifying the linking constructure and the boundedness of Cerami sequence of Ψ. In order to overcome these difficulties, we utilize the methods of Chen [25] and Sun [26](They are different from our research) to obtain the existance of infinitely many solutions to a class of Kirchhoff-type equations.

5 Declarations

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Competing Interests

The authors state that they have no conflicts of interest.

Ethical Approval

Not applicable.

Authors's Contributions

Linlian Xiao writes and revises this paper, Jiaqian Yuan checks this paper, Jian Zhou and Yunshun Wu provide suggestions for revisions on this paper.

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