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On the convergence result for pseudo-parabolic equations with fractional time derivatives

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Abstract

The main goal of this note is to investigate the convergence of solutions of the pseudo-parabolic equation with the Riemann–Liouville derivative when the order tends to 1^-. This paper is a continuation of the paper [L.D. Long, D. O'Regan, Notes on Convergence Results for Parabolic Equations with Riemann-Liouville Derivatives, Mathematics, 2022] where a special case of the theory below is presented (see Section 1 for a discussion).

Key words: Fractional diffusion equation, Riemman-Liouville, regularity, convergence rate
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1 Introduction

In this note, we are interested in examining the following problem of fractional pseudo-parabolic equations

$$\begin{cases} D_t^\beta y - y_{xx} - kD_t^\beta y_{xx} = 0, \ \ (x,t) \in (0,\pi) \times (0,1), \\ y(0,t) = y(\pi,t) = 0, \ \ (x,t) \in \partial\Omega \times (0,1), \\ t^{1-\beta} y|_{t=0} = \psi(x), \ \ 0 < x < \pi, \end{cases} \tag{1}$$

where $D_t^\beta f$ denotes the Riemann-Liouville fractional derivative of f with the order $\beta \in (0,1)$; note

$$D_t^\beta f(t) = \frac{d}{dt}\Big(\frac{1}{\Gamma(1-\beta)} \int_0^t (t-r)^{-\beta} f(r) dr \Big)$$

and $D_t^\beta f(t) =: \frac{d}{dt} f(t)$ if $\beta = 1$. Pseudo-parabolic equations have been studied extensively and describe a number of important physical properties. They describe physical phenomena such as

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one-way propagation of long waves, nonlinearity, dispersion, and agglomeration of populations, and analyze non-stationary processes in semiconductors. Equations of time fractional reactions occur in describing "memory" in physics, for example, plasma turbulence, fractal geometry [1], and single-molecular protein dynamics [2]. There are a few papers concerning differential equations involving the Riemann-Louville derivative. In [3], the authors studied the following equation with the Riemann–Liouville derivative

$$\begin{cases} D_t^\beta y - y_{xx} - kD_t^\beta y_{xx} = 0, & (x,t) \in (0,\pi) \times (0,1), \\ y(0,t) = y(\pi,t) = 0, & (x,t) \in \partial\Omega \times (0,1), \\ t^{1-\beta} y|_{t=0} = \psi(x), & 0 < x < \pi, \end{cases} \tag{2}$$

where $\lambda \geq 0$ and $T > 0$ is a positive real number. In [4], C. Zhai and R. Jiang studied the non-local problem for systems with a Riemann—Liouville derivative. In [5], the authors studied the existence and continuation of solutions to the general fractional differential equation (FDE) with the Riemann-Liouville derivative, and several global existence results for FDE's are constructed in this paper. In Ngoc-Tuan-Zhou-O'Regan [6] the authors considered the nonlinear diffusion equation with the Riemann–Liouville derivative

$$D_t^\beta u - u_{xx} = F(x,t,u),$$

and they obtained the existence (and regularity) of mild solutions using Banach fixed point theory. There are also many papers on fractional time derivatives, however, we only note the contents on mild solutions as described in [7, 8].

Let us now return to our problem (1). If $k = 0$ then problem (1) was recently investigated in [9] and our paper improves, complements and extends results in [9]. In addition we present the theory in a more logical and readable way and also an error estimate is given for a solution to a fractional pseudo-parabolic equation with a solution to a classical pseudo-parabolic equation. Since the solution of the problem depends on the fractional order β, we denote the solution as y_β. The main goal of this note is to consider the convergence of y_β when $\beta \to 1^-$. This question is inherently difficult and was solved carefully in [9] when $k = 0$. Our paper uses some techniques from some previous papers [8, 10].

Our paper is the first study noting that the solution y_β to our problem converges to the solution of problem (1) with the classical derivative. We refer the reader to Theorem 2.3 where you see the converence rate.

2 Preliminaries

Let us denote the Mittag-Leffler function by

$$E_{\alpha,\beta}(z) = \sum_{n=0}^{\infty} \frac{z^n}{\Gamma(n\alpha + \beta)}$$

($z \in \mathbb{C}$), for $\alpha > 0$ and $\beta \in \mathbb{R}$ (see [6, 11]).

Lemma 2.1. *Let β be such that $0 < \beta < 1$. Then there exists a constant $\overline{C}_\beta$ such that*

$$0 \leq E_{\beta,\beta}(-z) \leq \frac{\overline{C}_\beta}{1+z}, \quad z > 0.$$

For a positive number $r \geq 0$, we define the Hilbert scale space

$$X^m(0,\pi) = \left\{ \theta \in L^2(0,\pi) : \sum_{n=1}^{\infty} n^{2m} \langle \theta, \sqrt{\frac{2}{\pi}} \sin(nx) \rangle^2 < +\infty \right\},$$

associated with the following norm

$$\|\theta\|_{X^m(0,\pi)} = \left(\sum_{n=1}^{\infty} n^{2m} \langle \theta, \sqrt{\frac{2}{\pi}} \sin(nx) \rangle^2 \right)^{\frac{1}{2}}.$$

Lemma 2.2. *Let $3/4 \le \beta < 1$. Then for any $z < 0$,*

$$\left| E_{\beta,\beta}(z) - e^z \right| \le \frac{C(1-\beta)}{1+|z|},$$

here the constant C is independent of β, z.

The proof can be found in [9, 12].

Theorem 2.3. *Let the input data $\psi \in \mathbb{X}^m(\Omega)$ for any $m \ge 0$. Let y_β $(3/4 \le \beta < 1)$ be the mild solution to Problem (1). Let y^* be the mild solution of the classical pseudo-parabolic equation*

$$\begin{cases} u_t - u_{xx} - k u_{xxt} = 0, & (x,t) \in (0,\pi) \times (0,1), \\ u(0,t) = u(\pi,t) = 0, & 0 < t < 1, \\ u(x,0) = \psi(x), & 0 < x < \pi. \end{cases}$$

Then

$$\begin{aligned} &\left\| y_\beta - y^* \right\|_{L^1(0,1;\mathbb{X}^m(\Omega))} \\ &\le \left[\frac{C\Gamma(\beta)(1+k)^\gamma (1-\beta)}{\beta - \beta\gamma} + C(\theta,\mu,\beta)(1-\beta)^\mu + (1+k)^\vartheta \left| \Gamma(\beta) - 1 \right| \right] \left\| \psi \right\|_{\mathbb{X}^m(\Omega)} \end{aligned} \tag{3}$$

here $0 < \mu < \frac{\beta}{2}$, $0 < \gamma < 1$ and $\vartheta > 0$.

Remark 2.4. We can check immediately that the right hand side of (3) tends to zero when $\beta \to 1^-$.

Proof. Note

$$\int_\Omega y_\beta(x,t) \sqrt{\frac{2}{\pi}} \sin(nx) dx = \Gamma(\beta) t^{\beta-1} E_{\beta,\beta} \left(-\frac{n^2}{1+kn^2} t^\beta \right) \psi_n, \tag{4}$$

here ψ_n is the Fourier coefficient of the function ψ which is given by

$$\psi_n = \int_\Omega \psi(x) \sqrt{\frac{2}{\pi}} \sin(nx) dx.$$

The mild solution to the classical problem is defined by

$$\int_\Omega y^*(x,t) \sqrt{\frac{2}{\pi}} \sin(nx) dx = \exp\left(-\frac{n^2}{1+kn^2} t \right) \psi_n. \tag{5}$$

From (4) and (5), we derive that

$$\begin{aligned} &\int_\Omega \left(y_\beta(x,t) - y^*(x,t) \right) \sqrt{\frac{2}{\pi}} \sin(nx) dx \\ &= \Gamma(\beta) t^{\beta-1} \left(E_{\beta,\beta} \left(-\frac{n^2}{1+kn^2} t^\beta \right) - \exp\left(-\frac{n^2}{1+kn^2} t \right) \right) \psi_n \\ &+ \Gamma(\beta) \left(t^{\beta-1} - 1 \right) \exp\left(-\frac{n^2}{1+kn^2} t \right) \psi_n + \left(\Gamma(\beta) - 1 \right) \exp\left(-\frac{n^2}{1+kn^2} t \right) \psi_n. \end{aligned}$$

Hence, one has the following equality

$$\begin{aligned} y_\beta(x,t) - y^*(x,t) &= \sum_{n=1}^{\infty} \Gamma(\beta) t^{\beta-1} \left(E_{\beta,\beta}\left(-\frac{n^2}{1+kn^2} t^\beta\right) - \exp\left(-\frac{n^2}{1+kn^2} t\right) \right) \psi_n \sqrt{\frac{2}{\pi}} \sin(nx) \\ &+ \sum_{n=1}^{\infty} \Gamma(\beta) \left(t^{\beta-1} - 1\right) \exp\left(-\frac{n^2}{1+kn^2} t\right) \psi_n \sqrt{\frac{2}{\pi}} \sin(nx) \\ &+ \sum_{n=1}^{\infty} \left(\Gamma(\beta) - 1\right) \exp\left(-\frac{n^2}{1+kn^2} t\right) \psi_n \sqrt{\frac{2}{\pi}} \sin(nx) \\ &= \mathscr{J}_1(x,t) + \mathscr{J}_2(x,t) + \mathscr{J}_3(x,t). \end{aligned}$$

Let us focus first on the quantity $\mathscr{J}_1$. Using Lemma 2.2, one has

$$\left| E_{\beta,\beta}\left(-\frac{n^2}{1+kn^2} t^\beta\right) - \exp\left(-\frac{n^2}{1+kn^2} t\right) \right| \le \frac{C(1-\beta)}{1 + \frac{n^2}{1+kn^2} t^\beta}.$$

For any $0 < \gamma < 1$, we get the following bound

$$\frac{C}{1 + \frac{n^2}{1+kn^2} t^\beta} \le \frac{C}{(1 + \frac{n^2}{1+kn^2} t^\beta)^\gamma} \le C \left(\frac{1+kn^2}{n^2} \right)^\gamma t^{-\beta\gamma} \le C(1+k)^\gamma t^{-\beta\gamma}.$$

This implies that

$$\begin{aligned} \left\| \mathscr{J}_1(.,t) \right\|^2_{\mathbb{X}^m(\Omega)} &= \sum_{n=1}^{\infty} n^{2m} |\Gamma(\beta)|^2 t^{2\beta-2} \left| E_{\beta,\beta}\left(-\frac{n^2}{1+kn^2} t^\beta\right) - \exp\left(-\frac{n^2}{1+kn^2} t\right) \right|^2 (\psi_n)^2 \\ &\le C^2 |\Gamma(\beta)|^2 (1+k)^{2\gamma} t^{2\beta-2-2\beta\gamma} (1-\beta)^2 \sum_{n=1}^{\infty} n^{2m} |\psi_n|^2. \end{aligned}$$

Thus

$$\left\| \mathscr{J}_1(.,t) \right\|_{\mathbb{X}^m(\Omega)} \le C\Gamma(\beta)(1+k)^\gamma t^{\beta-1-\beta\gamma} (1-\beta) \left\| \psi \right\|_{\mathbb{X}^m(\Omega)},$$

and so

$$\begin{aligned} \left\| \mathscr{J}_1 \right\|_{L^1(0,1;\mathbb{X}^m(\Omega))} = \int_0^1 \left\| \mathscr{J}_1(.,t) \right\|_{\mathbb{X}^m(\Omega)} dt &\le C\Gamma(\beta)(1+k)^\gamma (1-\beta) \left\| \psi \right\|_{\mathbb{X}^m(\Omega)} \left(\int_0^1 t^{\beta-1-\beta\gamma} dt \right) \\ &= \frac{C\Gamma(\beta)(1+k)^\gamma (1-\beta) \left\| \psi \right\|_{\mathbb{X}^m(\Omega)}}{\beta - \beta\gamma}. \end{aligned}$$

Next note

$$\left\| \mathscr{J}_2(.,t) \right\|^2_{\mathbb{H}^s(\Omega)} = \sum_{n=1}^{\infty} |\Gamma(\beta)|^2 \left(t^{\beta-1} - 1\right)^2 \exp\left(-2\frac{n^2}{1+kn^2} t\right) |\psi_n|^2. \tag{6}$$

Let us now consider the term $\left|t^{\beta-1} - 1\right|$. Let $\mu > 0$. From [9], we have

$$\left| t^{\beta-1} - 1 \right| \le C_\mu (1-\beta)^\mu t^{\beta-1-\mu}, \tag{7}$$

for any $\mu > 0$. In view of the inequality $e^{-z} \le C_\theta z^{-\theta}$, one has

$$\exp\left(-2\frac{n^2}{1+kn^2} t\right) \le C_\theta t^{-2\theta} \left(\frac{1+kn^2}{n^2} \right)^{2\theta} \le C_\theta t^{-2\theta} (1+k)^{2\theta}. \tag{8}$$

Here θ is chosen later such that $\theta < \beta$. By collecting some previous results (6), (7) and (8), we deduce that

$$\begin{aligned}\Big\|\mathscr{I}_2(.,t)\Big\|^2_{\mathbb{X}^m(\Omega)} &\le C(\theta,\mu)(1+k)^{\theta}(1-\beta)^{2\mu}t^{2\beta-2-2\mu-2\theta}\sum_{n=1}^{\infty} n^{2m}|\psi_n|^2 \\ &= C(\theta,\mu)(1+k)^{2\theta}(1-\beta)^{2\mu}t^{2\beta-2-2\mu-2\theta}\Big\|\psi\Big\|^2_{\mathbb{X}^m(\Omega)},\end{aligned}$$

and so

$$\Big\|\mathscr{I}_2(.,t)\Big\|_{\mathbb{X}^m(\Omega)} \le C(\theta,\mu)(1-\beta)^{\mu}t^{\beta-1-\mu-\theta}\Big\|\psi\Big\|_{\mathbb{X}^m(\Omega)},$$

where $C(\theta,\mu)$ represents a constant that depends on μ,θ. Therefore, we have the following estimate

$$\Big\|\mathscr{I}_2\Big\|_{L^1(0,T;\mathbb{X}^m(\Omega))} = \int_0^1 \Big\|\mathscr{I}_2(.,t)\Big\|_{\mathbb{X}^m(\Omega))}dt \le C(\theta,\mu)(1-\beta)^{\mu}\Big\|\psi\Big\|_{\mathbb{X}^m(\Omega)}\Big(\int_0^1 t^{\beta-1-\mu-\theta}dt\Big).$$

Let us choose $\theta = \frac{\beta}{2}$ and since $\mu < \beta-\theta = \frac{\beta}{2}$, we have

$$\int_0^1 t^{\beta-1-\mu-\theta}dt = \frac{1}{\beta-\mu-\theta} = \frac{1}{\frac{\beta}{2}-\mu}.$$

From the above we have

$$\Big\|\mathscr{I}_2\Big\|_{L^1(0,T;\mathbb{H}^s(\Omega))} \le C(\mu,\beta)(1-\beta)^{\mu}\Big\|\psi\Big\|_{\mathbb{X}^m(\Omega)}.$$

Next using Parseval's equality, we derive that

$$\Big\|\mathscr{I}_3(.,t)\Big\|^2_{\mathbb{X}^m(\Omega)} = \sum_{n=1}^{\infty} n^{2m}\Big(\Gamma(\beta)-1\Big)^2 \exp\left(-2\frac{n^2}{1+kn^2}t\right)|\psi_n|^2. \quad (9)$$

Using the inequality $e^{-y} \le C_\vartheta y^{-\vartheta}$ for any $\vartheta > 0$, we get that

$$\exp\left(-2\frac{n^2}{1+kn^2}t\right) \le C_\vartheta\left(\frac{1+kn^2}{n^2}\right)^{2\vartheta} t^{-2\vartheta} \le C_\vartheta\,(1+k)^{2\vartheta}\,t^{-2\vartheta}.$$

This inequality together with (9) gives

$$\Big\|\mathscr{I}_3(.,t)\Big\|_{\mathbb{X}^m(\Omega)} \le C_\vartheta\,(1+k)^{\vartheta}\,\big|\Gamma(\beta)-1\big|t^{-\vartheta}\Big\|\psi\Big\|_{\mathbb{X}^m(\Omega)}.$$

This bound implies that

$$\begin{aligned}\Big\|\mathscr{I}_3\Big\|_{L^1(0,T;\mathbb{X}^m(\Omega))} &= \int_0^1 \Big\|\mathscr{I}_3(.,t)\Big\|_{\mathbb{X}^m(\Omega)}dt \le \Big|\Gamma(\beta)-1\Big|\Big\|\psi\Big\|_{\mathbb{X}^m(\Omega)}.\Big(\int_0^1 t^{-\vartheta}dt\Big) \\ &= C_\vartheta\,(1+k)^{\vartheta}\,\frac{\Big|\Gamma(\beta)-1\Big|}{1-\vartheta}\Big\|\psi\Big\|_{\mathbb{X}^m(\Omega)} \lesssim (1+k)^{\vartheta}\,\Big|\Gamma(\beta)-1\Big|\|\psi\|_{\mathbb{X}^m(\Omega)}.\end{aligned}$$

The proof of our theorem is complete.

□

3 Conclusion

In this paper, we investigate the continuity with respect to the parameter of the derivative in pseudo-parabolic equations with fractional in the time derivative under the initial value condition $\psi \in \mathbb{X}^m(\Omega)$. Also an error estimate is given for a solution to a fractional pseudo-parabolic equation with a solution to a classical pseudo-parabolic equation.

4 Declarations

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Competing Interests

I declare that there are no competing interests.

Ethical Approval

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