EJAM
www.ejamjournal.com

Electronic Journal of Applied Mathematics

Research article

Mersin University Publishing
DOI: 10.61383/ejam.20242263
2024, Vol. 2, No. 2, 22–31

Reconstruct the unknown source on the right hand side of time fractional diffusion equation with Caputo-Hadamard derivative

Ngo Ngoc Hung[1], **Danh Hua Quoc Nam**[2] and **Le Dinh Long**[3*]

1. Faculty of Fundamental Science, Industrial University of Ho Chi Minh City, Vietnam
2. Thu Dau Mot University, Binh Duong province, Vietnam
3. Falculty of Maths, FPT University HCM Saigon Hi-tech Park, Ho Chi Minh City, Vietnam

[*]Corresponding author

Abstract

The Caputo-Hadamard derivative was used to investigate the problem of functional recovery in this study. This problem is ill-posed, we propose a novel Quasi-reversibility for reconstructing the sought function and show that the regularization solution depends on space. After that, the convergence rates are established under a priori and posterior choice rules of regularization parameters, respectively.

Key words: Inverse source problem, parabolic equation, regularization method, error estimate
2020 Mathematics Subject Classification: 35K99, 47J06, 47H10, 35K05
Article history: Received 04 Dec 2023; Accepted 07 Jun 2024; Online 15 Jun 2024

1 Introduction

In last decades, fractional derivatives have been widely used in many scientific fields, such as biology, physics, and chemistry [1–11]. In 1892, Hadamard proposed another powerful and valuable essential differential [12]. In 2012, Jarad et al. proposed another subsidiary, which was changed by Caputo subordinate and is called Caputo-Hadamard subsidiary [13]. The presence and uniqueness of arrangement of C-H fragmentary differential condition are concentrated in [14]. In this work, we study the following problem with Caputo–Hadamard fractional

Contact: Ngo Ngoc Hung ngongochung@iuh.edu.vn; Danh Hua Quoc Nam namdhq@tdmu.edu.vn ; Le Dinh Long longld13@fe.edu.vn

derivative [1, 2] as follows:

$$\begin{cases} {}_{CH}\mathcal{D}^{\gamma}_{d,t}\theta(x,t) + (-\Delta)\theta(x,t) = \mathcal{S}(x)\mathcal{Q}(t), & x \in \Omega, t \in (d,\mathscr{T}], 0<\gamma<1,\ d>1, \\ \theta(x,d) = \mathcal{G}(x), & x \in \Omega, \\ \theta(x,t) = 0, & x \in \partial\Omega, t \in (d,\mathscr{T}], \\ \theta(x,\mathscr{T}) = \mathcal{H}(x), & x \in \Omega, \end{cases} \tag{1}$$

where $\mathcal{H}$ is the terminal data, where $0 < \mathcal{Q}_1 \le \mathcal{Q}(t) \le \mathcal{Q}_2$, and ${}_{CH}\mathcal{D}^{\gamma}_{d,t}$ is the Caputo-Hadamard fractional derivative of the order γ defined by [15]

$${}_{CH}\mathcal{D}^{\gamma}_{d,t}\theta(x,t) = \frac{1}{\Gamma(1-\gamma)}\int_d^t \big(\log t - \log s\big)^{-\gamma}\delta\theta(x,s)ds, 0<\gamma<1,$$

where $\delta = t\frac{d}{dt}$, $\Gamma(x)$ is a Gamma function. In problem (1), the source term $\mathcal{S}(x)$ is unknown, and the initial value $\mathcal{G}(x)$ is known. The final value $\theta(x,\mathscr{T}) = \mathcal{H}(x)$ is the given data. This paper uses the final value $\mathcal{H}(x)$ to identify the source term $\mathcal{S}(x)$. Assume that the couple exact data $\{\mathcal{G},\mathcal{H}\}$ and the measurement data $\{\mathcal{G}^{\epsilon},\mathcal{H}^{\epsilon}\}$ satisfy

$$\|\mathcal{G}^{\delta} - \mathcal{G}\|_2 + \|\mathcal{H}^{\delta} - \mathcal{H}\|_2 \le \delta. \tag{2}$$

where $\|\cdot\|_2$ is the $L^2(\Omega)$ norm, and $\epsilon > 0$ is the measurement error.

The inverse source problems have concerned many researchers during the past decade years, these problems aim to identify a term in the right-hand side of parabolic equations from some over-specific measurements that are known to be ill-posed, see in [16–18]. Regarding the regularization methods used, we introduce some of the following methods : the logarithmic-type regularization method [19], a quasi-boundary value method [20], the generalize Tikhonov method [21], a fractional Tikhonov method [22], the exponential Tikhonov method [23], a finite difference method [24]. However, the Quasi-reversibility method is the method we use in this study, with the following ideas [25] and their references.

The paper is organized as follows. In Section 2, we introduce paramilitaries. In Section 3, we show the mild solution of the problem (1), the condition of f is given by and Theorem 3.1. Next, in Section 4, based on the quasi-reversibility Method, we show the convergent rate in two cases: A priori parameter choice rule as posteriori parameter choice rule.

2 Preliminaries

We begin this section by introducing some notations and assumptions that are needed for our analysis in the next sections.

Definition 2.1. [26] The Mittag-Leffler function is

$$E_{\gamma,\beta}(z) = \sum_{k=0}^{\infty}\frac{z^k}{\Gamma(\gamma k+\beta)}, \quad z \in \mathbb{C},$$

where $\gamma > 0$ and $\beta \in R$ are arbitrary constants.

Definition 2.2. The Hilbert scale space $\mathcal{H}^s$, $(s>0)$ defined by

$$\mathcal{H}^s(\Omega) := \Big\{f \in L^2(\Omega) : \sum_{n=1}^{\infty}\lambda_n^{2s}\langle f,\phi_n\rangle_2^2 < \infty\Big\}.$$

Lemma 2.3. *If $\lambda > 0$, then the following equation holds*

$$\int_d^{\infty} e^{-s\log\frac{t}{d}}\Big(\log(t/d)\Big)^{\beta k+\gamma-1}E^{(k)}_{\beta,\gamma}\big(\pm\lambda\big(\log(t/d)\big)^{\beta}\big)\frac{dt}{t} = \frac{k!s^{\beta-\gamma}}{\big(s^{\beta}\mp\lambda\big)^{k+1}}, \mathrm{Re}(s) > |\lambda|^{\frac{1}{\beta}},$$

where $E^{(m)}_{\beta,\gamma}(y) := \frac{d^m}{dy^m}E_{\beta,\gamma}(y)$. *The Lemma 2.3 means that the Laplace transformation of*

$$\big(\log\frac{t}{c}\big)^{\beta k+\gamma-1}E^{(k)}_{\beta,\gamma}\Big(\pm\lambda\big(\log\frac{t}{c}\big)^{\beta}\Big) \text{ is } \frac{k!s^{\beta-\gamma}}{(s^{\beta}\mp\lambda)^{k+1}}.$$

Lemma 2.4. *For $0 < \beta < 1, z > 0$, we have $0 \le E_{\beta,1}(-z) < 1$. Moreover, $E_{\beta,1}(-z)$ is completely monotonic, that is,*

$$(-1)^n\frac{d^n}{dz^n}E_{\beta,1}(-z) \ge 0, z \ge 0.$$

Lemma 2.5. *[15] For any λ_n that satisfies $0 < \lambda_1 \le \lambda_n$, there is a positive number $\mathcal{C}$ that depends on $\gamma, d, \mathscr{T}$, and λ_1 such that*

$$\frac{\mathcal{Q}_1\mathcal{C}}{\lambda_n} \le \int_d^{\mathscr{T}}\big(\log\mathscr{T}-\log y\big)^{\gamma-1}E_{\gamma,\gamma}\big(-\lambda_n\big(\log\mathscr{T}-\log y\big)^{\gamma}\big)\mathcal{Q}(y)\frac{dy}{y} \le \frac{\mathcal{Q}_2}{\lambda_n},$$

where $\mathcal{C} = 1 - E_{\gamma,1}\big(-\lambda_1\big(\log\mathscr{T}-\log d\big)^{\gamma}\big)$.

3 The mild solution of problem

The solution of problem (1) is obtained by using the separated variable method as follows:

$$\begin{aligned}\theta(x,t) = \sum_{n=1}^{\infty}\Bigg(&E_{\gamma,1}\big(-\lambda_n\big(\log t-\log d\big)^{\gamma}\big)\langle\mathcal{G},\phi_n\rangle\\ &+\int_d^t\big(\log t-\log y\big)^{\gamma-1}E_{\gamma,\gamma}\Big(-\lambda_n\big(\log t-\log y\big)^{\gamma}\Big)\mathcal{Q}(y)\frac{dy}{y}\Bigg)\langle\mathcal{S},\phi_n\rangle\phi_n(x).\end{aligned}\tag{3}$$

From (3), we know that

$$\begin{aligned}\big\langle\theta(.,t),\phi_n\big\rangle = \Bigg(&E_{\gamma,1}\big(-\lambda_n\big(\log t-\log d\big)^{\gamma}\big)\langle\mathcal{G},\phi_n\rangle\\ &+\int_d^t\big(\log t-\log y\big)^{\gamma-1}E_{\gamma,\gamma}\Big(-\lambda_n\big(\log t-\log y\big)^{\gamma}\Big)\mathcal{Q}(y)\frac{dy}{y}\Bigg)\langle\mathcal{S},\phi_n\rangle.\end{aligned}$$

Using the condition $\big\langle\theta(\cdot,\mathscr{T}),\phi_n\big\rangle = \mathcal{H}(x)$, we receive

$$\begin{aligned}\big\langle\mathcal{H},\phi_n\big\rangle = \big\langle\theta(\cdot,\mathscr{T}),\phi_n\big\rangle &= \big\langle\mathcal{G},\phi_n\big\rangle\Big(E_{\gamma,1}\big(-\lambda_n\big(\log\mathscr{T}-\log d\big)^{\gamma}\big)\Big)\\ &+\big\langle\mathcal{S},\phi_n\big\rangle\int_d^{\mathscr{T}}\big(\log\mathscr{T}-\log y\big)^{\gamma-1}E_{\gamma,\gamma}\Big(-\lambda_n\big(\log\mathscr{T}-\log y\big)^{\gamma}\Big)\mathcal{Q}(y)\frac{dy}{y}.\end{aligned}\tag{4}$$

Through simple transformations for (4), we get

$$\begin{aligned}\big\langle\mathcal{S},\phi_n\big\rangle &= \big\langle\mathcal{H},\phi_n\big\rangle - \big\langle\mathcal{G},\phi_n\big\rangle E_{\gamma,1}\big(-\lambda_n(\log\mathscr{T}-\log d)^{\gamma}\big)\\ &\times\Bigg(\int_d^{\mathscr{T}}\big(\log\mathscr{T}-\log y\big)^{\gamma-1}E_{\gamma,\gamma}\big(-\lambda_n\big(\log(T/y)\big)^{\gamma}\big)\mathcal{Q}(y)\frac{dy}{y}\Bigg)^{-1}.\end{aligned}\tag{5}$$

To simplify the formula (5), we denote it

$$\mathcal{B}_n^{\gamma}\big(\log t - \log y\big) = \frac{1}{y}\big(\log t - \log y\big)^{\gamma-1} E_{\gamma,\gamma}\Big(-\lambda_n\Big(\log t - \log y\Big)^{\gamma}\Big).$$

This leads to

$$\langle \mathcal{S}, \phi_n \rangle = \frac{\big(\langle \mathcal{H}, \phi_n \rangle - \langle \mathcal{G}, \phi_n \rangle E_{\gamma,1}\big(-\lambda_n(\log \mathscr{T} - \log d)^{\gamma}\big)\big)}{\left(\int_d^{\mathscr{T}} \mathcal{B}_n^{\gamma}\big(\log \mathscr{T} - \log y\big)\mathcal{Q}(y)dy\right)}. \tag{6}$$

From (6), it yields

$$\mathcal{S}(x) = \sum_{n=1}^{\infty} \frac{\langle \mathcal{H}, \phi_n \rangle - \langle \mathcal{G}, \phi_n \rangle E_{\gamma,1}\big(-\lambda_n(\log \mathscr{T} - \log d)^{\gamma}\big)}{\left(\int_d^{\mathscr{T}} \mathcal{B}_n^{\gamma}\big(\log \mathscr{T} - \log y\big)\mathcal{Q}(y)dy\right)} \phi_n(x)\cdot$$

Denote

$$\langle \mathcal{H}, \phi_n \rangle - \langle \mathcal{G}, \phi_n \rangle E_{\gamma,1}\big(-\lambda_n\big(\log \mathscr{T} - \log d\big)^{\gamma}\big) = \langle \ell, \phi_n \rangle.$$

This implies that

$$\mathcal{S}(x) = \sum_{n=1}^{\infty} \langle \ell, \phi_n \rangle \phi_n(x) \left(\int_d^{\mathscr{T}} \mathcal{B}_n\big(\log \mathscr{T} - \log y\big)\mathcal{Q}(y)dy\right)^{-1}. \tag{7}$$

Defining operator $\mathcal{K} : \mathcal{S} \to \ell$, then problem (1) can be write to the following operator equation: $\mathcal{K}\mathcal{S}(x) = \xi(x)$. Its singular value is

$$\{\sigma_n\}_{n=1}^{\infty}, \sigma_n = \int_d^{\mathscr{T}} \mathcal{B}_n^{\gamma}\big(\log \mathscr{T} - \log y\big)\mathcal{Q}(y)dy.$$

When $n \to \infty, \lambda_n \to \infty, \sigma_n^{-1} \to \infty$, so from formula (7), the small perturbation of $\ell(x)$ will cause a great change in the source term $\mathcal{S}(x)$. Hence, the inverse problem is ill-posed, see [27]. Next, we give the conditional stability.

Theorem 3.1. *[15] Let $\mathcal{S}$ satisfies $\|\mathcal{S}\|_{\mathcal{H}^s} \le \mathcal{M}$, $\mathcal{Q}(t) \in L^{\infty}(d, \mathscr{T})$, for any $t \in [d, \mathscr{T}]$, we have*

$$\|\mathcal{S}\|_2 \le \left(\frac{1}{\mathcal{Q}_1 \mathcal{C}}\right)^{\frac{s}{s+1}} \mathcal{M}^{\frac{1}{s+1}} \|\ell\|^{\frac{s}{s+1}}, \tau > 0.$$

4 A Quasi-reversibility method and convergence rates

This section provides two convergence estimates under an a priori regularization parameter choice rule and a posteriori regularization parameter choice rule, respectively, and suggests a quasi-reversibility approach to solve issue (1). Let $\theta_\delta(x,t)$ be the solution of the following regularized problem

$$\begin{cases} {}_{CH}\mathcal{D}_{d,t}^{\gamma}\theta_\delta(x,t) + (-\Delta)\theta_\delta(x,t) = \mathcal{S}_\delta(x)\mathcal{Q}(t) - \beta(\Delta\mathcal{S}_\delta)(x)\mathcal{Q}(t), & x \in \Omega, t \in (d, \mathscr{T}],\ d > 1, \\ \theta_\delta(x,d) = \mathcal{G}_\delta(x), & x \in \Omega, \\ \theta_\delta(x,t) = 0, & x \in \partial\Omega, t \in (d, \mathscr{T}], \\ \theta_\delta(x,\mathscr{T}) = \mathcal{H}_\delta(x), & x \in \Omega, \end{cases} \tag{8}$$

where $0 < \gamma < 1$, and $\beta > 0$ is a regularization parameter. By the separation of variables, thanks to the Laplace transform and inverse Laplace transform, showing the $\theta_\delta(x,t)$ as follows

$$\langle \theta_\delta(\cdot,t), \phi_n \rangle = \sum_{n=1}^{\infty} \Big(E_{\gamma,1}\big(-\lambda_n \big(\log t - \log d\big)^\gamma\big) \langle \mathcal{G}_\delta, \phi_n \rangle \\ + (1+\beta\lambda_n) \int_d^t \big(\log t - \log y\big)^{\gamma-1} E_{\gamma,\gamma}\Big(-\lambda_n \big(\log t - \log y\big)^\gamma\Big) \mathcal{Q}(y) \frac{dy}{y} \Big) \langle \mathcal{S}_\delta, \phi_n \rangle.$$

From the terminal data in (8), we get

$$\langle \mathcal{H}_\delta, \phi_n \rangle = E_{\gamma,1}\big(-\lambda_n \big(\log \mathscr{T} - \log d\big)^\gamma\big) \langle \mathcal{G}_\delta, \phi_n \rangle \\ + \Big((1+\beta\lambda_n) \int_d^{\mathscr{T}} \big(\log \mathscr{T} - \log y\big)^{\gamma-1} E_{\gamma,\gamma}\big(-\lambda_n \big(\log \mathscr{T} - \log y\big)^\gamma\big) \mathcal{Q}(y) \frac{dy}{y} \Big) \langle \mathcal{S}_\delta, \phi_n \rangle.$$

This implies that

$$\langle \mathcal{S}_\delta, \phi_n \rangle = \frac{\langle \mathcal{H}_\delta, \phi_n \rangle - E_{\gamma,1}\big(-\lambda_n (\log \mathscr{T} - \log d)^\gamma\big) \langle \mathcal{G}_\delta, \phi_n \rangle}{(1+\beta\lambda_n) \int_d^{\mathscr{T}} \big(\log \mathscr{T} - \log y\big)^{\gamma-1} E_{\gamma,\gamma}\big(-\lambda_n \big(\log \mathscr{T} - \log y\big)^\gamma\big) \mathcal{Q}(y) \frac{dy}{y}}, \tag{9}$$

which leads to

$$\mathcal{S}_{\delta,\beta}(x) = \sum_{n=1}^{\infty} \frac{\langle \mathcal{H}_\delta, \phi_n \rangle - E_{\gamma,1}\big(-\lambda_n (\log \mathscr{T} - \log d)^\gamma\big) \langle \mathcal{G}_\delta, \phi_n \rangle}{(1+\beta\lambda_n) \int_d^{\mathscr{T}} \big(\log \mathscr{T} - \log y\big)^{\gamma-1} E_{\gamma,\gamma}\big(-\lambda_n \big(\log \mathscr{T} - \log y\big)^\gamma\big) \mathcal{Q}(y) \frac{dy}{y}} \phi_n(x), \tag{10}$$

and

$$\mathcal{S}_\beta(x) = \sum_{n=1}^{\infty} \frac{\langle \mathcal{H}, \phi_n \rangle - E_{\gamma,1}\big(-\lambda_n (\log \mathscr{T} - \log d)^\gamma\big) \langle \mathcal{G}, \phi_n \rangle}{(1+\beta\lambda_n) \int_d^{\mathscr{T}} \big(\log \mathscr{T} - \log y\big)^{\gamma-1} E_{\gamma,\gamma}\big(-\lambda_n \big(\log \mathscr{T} - \log y\big)^\gamma\big) \mathcal{Q}(y) \frac{dy}{y}} \phi_n(x). \tag{11}$$

After that, two convergence estimates for $\|\mathcal{S}_{\delta,\beta} - \mathcal{S}\|_2$ by using an a priori and an a posteriori choice rule.

5 Convergence estimate

5.1 An a priori regularization parameter choice rule

Theorem 5.1. *Suppose the a priori condition in Theorem 3.1 and the noise assumption (2) hold, then*

$$\|\mathcal{S}_{\delta,\beta} - \mathcal{S}\|_2 \le \begin{cases} \Big(\mathcal{P}(\mathcal{D}, \mathcal{Q}_1, \mathcal{C}) + \mathcal{C}_1\Big) \mathcal{M}^{\frac{1}{s+1}} \delta^{\frac{s}{s+1}}, & 0 < s < 1, \\ \Big(\mathcal{P}(\mathcal{D}, \mathcal{Q}_1, \mathcal{C}) + \mathcal{C}_2\Big) E^{\frac{1}{2}} \delta^{\frac{1}{2}}, & s \ge 1. \end{cases}$$

Proof. By the triangle inequality, we know

$$\|\mathcal{S}_{\delta,\beta} - \mathcal{S}\|_2 \le \|\mathcal{S}_{\delta,\beta} - \mathcal{S}_\beta\|_2 + \|\mathcal{S}_\beta - \mathcal{S}\|_2. \tag{12}$$

We first give an estimate for the first term. From (9), (10) and (11), we have

$$\begin{aligned}
&\left\|\mathcal{S}_{\delta,\beta}-\mathcal{S}_{\delta}\right\|_2\\
&=\left\|\sum_{n=1}^{\infty}\frac{\langle\mathcal{H}_\delta-\mathcal{H},\phi_n\rangle+E_{\gamma,1}\big(-\lambda_n(\log\mathscr{T}-\log d)^\gamma\big)\langle\mathcal{G}-\mathcal{G}_\delta,\phi_n\rangle}{\big(1+\beta\lambda_n\big)\int\limits_d^{\mathscr{T}}\big(\log\mathscr{T}-\log y\big)^{\gamma-1}E_{\gamma,\gamma}\big(-\lambda_n\big(\log\mathscr{T}-\log y\big)^\gamma\big)\mathcal{Q}(y)\frac{dy}{y}}\phi_n(x)\right\|_2\\
&\le\delta\frac{(1+\mathcal{D})}{\mathcal{Q}_1\mathcal{C}}\sup_{n\in\mathbb{N}}\frac{\lambda_n}{(1+\beta\lambda_n)}\le\frac{\delta}{\beta}\frac{(1+\mathcal{D})}{\mathcal{Q}_1\mathcal{C}}\sup_{n}\frac{\lambda_n}{(1+\beta\lambda_n)}\le\frac{\delta}{\beta}\mathcal{P}(\mathcal{D},\mathcal{Q}_1,\mathcal{C}).
\end{aligned}\tag{13}$$

Now we estimate the second term in (12). By (10) and (11), we can deduce that

$$\begin{aligned}
&\left\|\mathcal{S}_{\beta}-\mathcal{S}\right\|_2\\
&=\left\|\sum_{n=1}^{\infty}\frac{\beta\lambda_n}{1+\beta\lambda_n}\frac{\langle\mathcal{H}_\delta-\mathcal{H},\phi_n\rangle+E_{\gamma,1}\big(-\lambda_n(\log\mathscr{T}-\log d)^\gamma\big)\langle\mathcal{G}-\mathcal{G}_\delta,\phi_n\rangle}{\int\limits_d^{\mathscr{T}}\big(\log\mathscr{T}-\log y\big)^{\gamma-1}E_{\gamma,\gamma}\big(-\lambda_n\big(\log\mathscr{T}-\log y\big)^\gamma\big)\mathcal{Q}(y)\frac{dy}{y}}\phi_n(x)\right\|_2\\
&\le\mathcal{M}\sup_{n\in\mathbb{N}}\frac{\beta\lambda_n^{1-s}}{1+\beta\lambda_n}\le\begin{cases}\mathcal{C}_1\mathcal{M}\beta^s, & 0<s<1,\\ \mathcal{C}_2\mathcal{M}\beta, & s\ge 1.\end{cases}
\end{aligned}$$

Combining the above two inequalities, we obtain

$$\left\|\mathcal{S}_{\delta,\beta}-\mathcal{S}\right\|_2\le\frac{\delta}{\beta}\mathcal{P}(\mathcal{D},\mathcal{Q}_1,\mathcal{C})+\begin{cases}\mathcal{C}_1\mathcal{M}\beta^s, & 0<s<1,\\ \mathcal{C}_2\mathcal{M}\beta, & s\ge 1.\end{cases}$$

We choice β by

$$\beta=\begin{cases}\left(\frac{\delta}{\mathcal{M}}\right)^{\frac{1}{s+1}}, & 0<s<1,\\ \left(\frac{\delta}{\mathcal{M}}\right)^{\frac{1}{2}}, & s\ge 1,\end{cases}$$

then we have

$$\left\|\mathcal{S}_{\delta,\beta}-\mathcal{S}\right\|_2\le\begin{cases}\Big(\mathcal{P}(\mathcal{D},\mathcal{Q}_1,\mathcal{C})+\mathcal{C}_1\Big)\mathcal{M}^{\frac{1}{s+1}}\delta^{\frac{s}{s+1}}, & 0<s<1,\\ \Big(\mathcal{P}(\mathcal{D},\mathcal{Q}_1,\mathcal{C})+\mathcal{C}_2\Big)E^{\frac{1}{2}}\delta^{\frac{1}{2}}, & s\ge 1.\end{cases}$$

The proof is completed. □

5.2 Convergence estimate under an a posteriori regularization parameter choice rule

We use the discrepancy principle in the following form:

$$\left\|\beta(-\Delta)(1-\beta\Delta)^{-1}\big(\mathcal{K}\mathcal{S}_{\delta,\beta}-\ell_\delta\big)\right\|_2=\tau\delta,\tag{14}$$

where $\tau>1$ is a constant. According to the following lemma, we know there exists a unique solution for (14) if $\|\ell_\delta\|_2>\tau\delta>0$.

Lemma 5.2. *Set $\rho(\beta)=\|\beta(-\Delta)(1-\beta\Delta)^{-1}\left(\mathcal{K}\mathcal{S}_{\delta,\beta}-\ell\right)\|_2$. If $\left\|\ell_\delta\right\|_2>\delta>0$, then*

- *$\rho(\beta)$ is a continuous function;*
- *$\lim_{\beta\to 0}\rho(\beta)=0$;*
- *$\lim_{\beta\to+\infty}\rho(\beta)=\left\|\ell_\delta\right\|_2$;*

- *$\rho(\beta)$ is a strictly increasing finction over $(0,\infty)$.*

The proof is straightforward results by virtue of

$$\rho(\beta) = \left(\sum_{n=1}^{\infty}\left(\frac{\beta\lambda_n}{1+\beta\lambda_n}\right)^4 |\langle \ell_\delta, \phi_n\rangle|^2\right)^{\frac{1}{2}}.$$

Theorem 5.3. *Suppose the a priori condition (3.1) and the noise assumption (2) hold, and there exists $\tau > 1+\mathcal{D}$ such that $\|\ell_\delta\|_2 > \tau\delta > 0$. The regularization parameter $\beta > 0$ is chosen by discrepancy principle (14). Then*

$$\|\mathcal{S}_{\delta,\beta} - \mathcal{S}\|_2 \ \text{is of order}\ \ \delta^{\frac{s}{s+1}}.$$

Proof.

- For $0 < s \le 1$, we get

$$\|\mathcal{S}_{\delta,\beta} - \mathcal{S}\|_2 \le \|\mathcal{S}_{\delta,\beta} - \mathcal{S}_\beta\|_2 + \|\mathcal{S}_\beta - \mathcal{S}\|_2.$$

From (14), there holds

$$\begin{aligned}
\tau\delta &= \left\|\sum_{n=1}^{\infty}\left(\frac{\beta\lambda_n}{1+\beta\lambda_n}\right)^2\langle \ell_\delta,\phi_n\rangle\phi_n(x)\right\|_2 \\
&\le \left\|\sum_{n=1}^{\infty}\left(\frac{\beta\lambda_n}{1+\beta\lambda_n}\right)^2\langle \ell_\delta-\ell,\phi_n\rangle\phi_n(x)\right\|_2 + \left\|\sum_{n=1}^{\infty}\left(\frac{\beta\lambda_n}{1+\beta\lambda_n}\right)^2\langle \ell,\phi_n\rangle\phi_n(x)\right\|_2 \\
&\le \delta(1+\mathcal{D}) + \left\|\sum_{n=1}^{\infty}\left(\frac{\beta\lambda_n}{1+\beta\lambda_n}\right)^2\right. \\
&\quad \times \left.\left(\int_d^{\mathscr{T}}(\log\mathscr{T}-\log y)^{\gamma-1}E_{\gamma,\gamma}\big(-\lambda_n(\log\mathscr{T}-\log y)^{\gamma}\big)\mathcal{Q}(y)\frac{dy}{y}\right)\lambda_n^{-s}\lambda_n^{s}\langle\mathcal{S},\phi_n\rangle\phi_n(x)\right\|_2 \\
&\le \delta(1+\mathcal{D}) + \mathcal{M}\sup_{n\in\mathbb{N}}\left(\frac{\beta\lambda_n}{1+\beta\lambda_n}\right)^2 \\
&\qquad \times\left(\int_d^{\mathscr{T}}(\log\mathscr{T}-\log y)^{\gamma-1}E_{\gamma,\gamma}\big(-\lambda_n(\log\mathscr{T}-\log y)^{\gamma}\big)\mathcal{Q}(y)\frac{dy}{y}\right)\lambda_n^{-s} \\
&\le \delta(1+\mathcal{D}) + \mathcal{M}\mathcal{Q}_2\sup_{n\in\mathbb{N}}\left(\frac{\beta\lambda_n^{1-\frac{s+1}{2}}}{1+\beta\lambda_n}\right)^2 \\
&\le \delta(1+\mathcal{D}) + \mathcal{M}\mathcal{Q}_2\beta^{s+1}.
\end{aligned}$$

This yields

$$\frac{1}{\beta} \le \left(\frac{1}{\tau-1-\mathcal{D}}\right)^{\frac{1}{s+1}}\left(\frac{\mathcal{M}\mathcal{Q}_2}{\delta}\right)^{\frac{1}{s+1}}. \tag{15}$$

Substituting (15) into (18), we get

$$\|\mathcal{S}_{\delta,\beta} - \mathcal{S}_\beta\|_2 \le \delta^{\frac{s}{s+1}}\left(\frac{\mathcal{M}\mathcal{Q}_2}{\tau-1-\mathcal{D}}\right)^{\frac{1}{s+1}}\mathcal{P}(\mathcal{D},\mathcal{Q}_1,\mathcal{C}). \tag{16}$$

From (13), and the a prion bound condition in 3.1, we know

$\|\mathcal{S}_\beta - \mathcal{S}\|_2$

$$
\begin{aligned}
&= \Big\| \sum_{n=1}^{\infty} \frac{-\beta\lambda_n}{1+\beta\lambda_n} \langle \mathcal{S}, \phi_n \rangle \phi_n(x) \Big\|_2 \\
&= \Big\| \sum_{n=1}^{\infty} \Big(\frac{\beta\lambda_n \int_d^{\mathscr{T}} (\log \mathscr{T} - \log y)^{\gamma-1} E_{\gamma,\gamma}\big(-\lambda_n (\log \mathscr{T} - \log y)^{\gamma}\big) \mathcal{Q}(y) \frac{dy}{y}}{1+\mu\lambda_n} \Big)^{s} \Big(\frac{\beta\lambda_n}{1+\beta\lambda_n} \Big)^{1-s} \\
&\quad \times \frac{\langle \mathcal{S}, \phi_n \rangle \phi_n(x)}{\Big(\int_d^{\mathscr{T}} (\log \mathscr{T} - \log y)^{\gamma-1} E_{\gamma,\gamma}\big(-\lambda_n (\log \mathscr{T} - \log y)^{\gamma}\big) \mathcal{Q}(y) \frac{dy}{y} \Big)^{s}} \Big\|_2 \\
&\le \Big\| \sum_{n=1}^{\infty} \Big(\frac{\beta\lambda_n \int_d^{\mathscr{T}} (\log \mathscr{T} - \log y)^{\gamma-1} E_{\gamma,\gamma}\big(-\lambda_n (\log \mathscr{T} - \log y)^{\gamma}\big) \mathcal{Q}(y) \frac{dy}{y}\big)}{1+\beta\lambda_n} \Big)^{s} \\
&\quad \times \Big(\frac{\beta\lambda_n}{1+\beta\lambda_n} \Big)^{1-s} \frac{\langle \mathcal{S}, \phi_n \rangle \phi_n(x)}{\Big(\int_d^{\mathscr{T}} (\log \mathscr{T} - \log y)^{\gamma-1} E_{\gamma,\gamma}\big(-\lambda_n (\log \mathscr{T} - \log y)^{\gamma}\big) \mathcal{Q}(y) \frac{dy}{y} \Big)^{s}} \Big\|_2^{\frac{s}{s+1}} \\
&\quad \times \Big\| \sum_{n=1}^{\infty} \Big(\frac{\beta\lambda_n}{1+\beta\lambda_n} \Big)^{1-s} \frac{\langle \mathcal{S}, \phi_n \rangle \phi_n(x)}{\Big(\int_d^{\mathscr{T}} (\log \mathscr{T} - \log y)^{\gamma-1} E_{\gamma,\gamma}\big(-\lambda_n (\log \mathscr{T} - \log y)^{\gamma}\big) \mathcal{Q}(y) \frac{dy}{y} \Big)^{s}} \Big\|_2^{\frac{1}{s+1}} \\
&\le \Big\| \sum_{n=1}^{\infty} \Big(\frac{\beta\lambda_n}{1+\beta\lambda_n} \Big)^{2} \\
&\quad \times \Big(\int_d^{\mathscr{T}} (\log \mathscr{T} - \log y)^{\gamma-1} E_{\gamma,\gamma}\big(-\lambda_n (\log \mathscr{T} - \log y)^{\gamma}\big) \mathcal{Q}(y) \frac{dy}{y} \Big) \langle \mathcal{S}, \phi_n \rangle \phi_n(x) \Big\|^{\frac{s}{s+1}} \\
&\quad \times \Big\| \sum_{n=1}^{\infty} \frac{\langle \mathcal{S}, \phi_n \rangle}{\Big(\int_d^{\mathscr{T}} (\log \mathscr{T} - \log y)^{\gamma-1} E_{\gamma,\gamma}\big(-\lambda_n (\log \mathscr{T} - \log y)^{\gamma}\big) \mathcal{Q}(y) \frac{dy}{y} \Big)^{s}} \phi_n(x) \Big\|_2^{\frac{1}{s+1}} \\
&\le \Big(\Big\| \sum_{n=1}^{\infty} \Big(\frac{\beta\lambda_n}{1+\beta\lambda_n} \Big)^{2} \langle \ell - \ell_\delta, \phi_n \rangle \phi_n(x) \Big\|_2 + \Big\| \Big(\frac{\beta\lambda_n}{1+\beta\lambda_n} \Big)^{2} \langle \ell_\delta, \phi_n \rangle \phi_n(x) \Big\|_2 \Big)^{\frac{s}{s+1}} \\
&\quad \times \Big\| \sum_{n=1}^{\infty} \lambda_n^{s} \langle \mathcal{S}, \phi_n \rangle \phi_n(x) \\
&\quad \times \Big(\int_d^{\mathscr{T}} (\log \mathscr{T} - \log y)^{\gamma-1} E_{\gamma,\gamma}\big(-\lambda_n (\log \mathscr{T} - \log y)^{\gamma}\big) \mathcal{Q}(y) \frac{dy}{y} \Big)^{-s} \Big\|_2^{\frac{1}{s+1}} \\
&\le \delta^{\frac{s}{s+1}} (1+\mathcal{D}+\tau)^{\frac{s}{s+1}} \mathcal{M}^{\frac{1}{s+1}} (\mathcal{Q}_1 \mathcal{C})^{\frac{-s}{s+1}}.
\end{aligned} \tag{17}
$$

From (13), we have

$$
\|\mathcal{S}_{\delta,\beta} - \mathcal{S}_\beta\|_2 \le \frac{\delta}{\beta} \mathcal{P}(\mathcal{D}, \mathcal{Q}_1, \mathcal{C}). \tag{18}
$$

Combining (17) and (16), we obtain

$$
\|\mathcal{S}_{\delta,\beta} - \mathcal{S}\|_2 \le \delta^{\frac{s}{s+1}} \Big(\frac{\mathcal{M}\mathcal{Q}_2}{\tau - 1 - \mathcal{D}} \Big)^{\frac{1}{s+1}} \mathcal{P}(\mathcal{D}, \mathcal{Q}_1, \mathcal{C}) + \delta^{\frac{s}{s+1}} (1+\mathcal{D}+\tau)^{\frac{s}{s+1}} \mathcal{M}^{\frac{1}{s+1}} (\mathcal{Q}_1 \mathcal{C})^{\frac{-s}{s+1}}.
$$

- For $s > 1$, because $\mathcal{H}^s$ is imbedded into $L^2(\Omega)$, we can get the conclusion and this completes the proof.

□

6 Conclusion

In this work, by using the Quasi Reversibility method, we evaluate the error between the exact solution and the regularized solution and present the errors with the choice of the a priori and a posteriori regularization parameters. In the future, we will conduct a survey of the source function recovery problem and the initial value problem in the same model.

7 Declarations

Funding

This research received no external funding.

Competing Interests

The authors declare that they have no competing interests.

Ethical Approval

Not applicable.

Authors's Contributions

All authors contributed equally. All the authors read and approved the final manuscript.

Availability Data and Materials

Not applicable.

Acknowledgements

The authors would like to thank the editors and reviewers for their time, effort, and valuable comments that led to possible improvements to the current version of our manuscript.

References

[1] C Li, Z Li, and Z Wang, *Mathematical analysis and the local discontinuous Galerkin method for Caputo–Hadamard fractional partial differential equation*, J. Scientif. Comput. **85** (2020), no. 2, 1–27.

[2] M Gohar, C Li, and Z Li, *Finite difference methods for Caputo–Hadamard fractional differential equations*, Mediterr. J. Math. **17** (2020), no. 6, 1–26.

[3] A A Kilbas, H M Srivastava, and J J Trujillo, *Theory and applications of fractional differential equations*, Elsevier, 2006.

[4] R C Koeller, *Applications of fractional calculus to the theory of viscoelasticity*, Trans. Asme J. Appl. Mec. **51** (1984), no. 2, 299–307.

[5] C Li and F Zeng, *Numerical methods for fractional calculus*, Vol. 24, CRC Press, 2015.

[6] K S Miller and B Ross, *An introduction to the fractional calculus and fractional differential equations*, Wiley, 1993.

[7] L Oparnica, *Generalized fractional calculus with applications in mechanics*, Matematicki vesnik **54** (2002), no. 3-4, 151–158.

[8] Truong T N, *Classification of blow-up and global existence of solutions to a system of Petrovsky equations*, Electronic Journal of Applied Mathematics, **1**(2), 29-59.

[9] Thi T X D and Thi T H V, *Recovering solution of the Reverse nonlinear time Fractional diffusion equations with fluctuations data*, Electronic Journal of Applied Mathematics, **1**(2), (2023) 60-70.

[10] Pandir Y and Yasmin, *H Optical soliton solutions of the generalized sine-Gordon equation*, Electronic Journal of Applied Mathematics, **1**(2), 71-86, (2023).

[11] Pandir Y and Ekin A, *New solitary wave solutions of the Korteweg-de Vries (KdV) equation by new version of the trial equation method*, Electron. J. Appl. Math, **1**, 101-113, (2023).

[12] J Hadamard, *Essai sur l'étude des fonctions, données par leur développement de Taylor*, Gauthier-Villars, 1892.

[13] F Jarad, T Abdeljawad, and D Baleanu, *Caputo-type modification of the Hadamard fractional derivatives*, Adv. Differ. Equat. (2012), no. **1**, 1–8.

[14] M Gohar, C Li, and C Yin, *On Caputo–Hadamard fractional differential equations*, Int. J. Comput. Math. **97** (2020), no. 7, 1459–1483.

[15] Yang F, Cao Y, and Li X X, *Two regularization methods for identifying the source term of Caputo–Hadamard time-fractional diffusion equation*, Mathematical Methods in the Applied Sciences, **46**(15), 16170-16202, (2023).

[16] Prilepko A I and Tkachenko D S, *Inverse problem for a parabolic equation with integral overdetermination*, J InverseIll-Posed Probl. 2003;**11**:191–218.

[17] Choulli M, Yamamoto M, *Conditional stability in determining a heat source*, J Inverse Ill-Posed Probl.2004;**12**:233–243.

[18] Johansson T, Lesnic D, *Determination of a spacewise dependent heat source*. J Comput Appl Math.2007; **209**(1) :66–80.

[19] Hu B, Xie S, and Wang Z, *Determination of a spacewise-dependent heat source by a logarithmic-type regularization method*, Applicable Analysis, **102**(14), (2023) 3986-4003.

[20] Wei T and Wang J, *A modified quasi-boundary value method for an inverse source problem of the time-fractional diffusion equation* Applied Numerical Mathematics, **78**, 95-111, (2014).

[21] Ma YK, Prakash P, Deiveegan A, *Generalized Tikhonov methods for an inverse source problem of the timefractional diffusion equation*, Chaos Solitons Fractals. 2018; **108** :39–48.

[22] Xiong X, Xue X, *A fractional Tikhonov regularization method for identifying a space-dependent source in thetime-fractional diffusion equation*, Appl Math Comput. 2019; **349** :292–303.

[23] Wang Z, Qiu S and Yu S, *Exponential Tikhonov regularization method for solving an inverse source problem oftime fractional diffusion equations*. J Comput Math. doi:10.4208/jcm.2107-m2020-0133.

[24] Wang Z, Ruan Z, Huang H, *Determination of an unknown time-dependent heat source from A nonlocalmeasurement by finite difference method*. Acta Math Appl Sin, English Ser. 2020; **36**(1) :151–165.

[25] Wang J G and Wei T, *Quasi-reversibility method to identify a space-dependent source for the time-fractional diffusion equation*, Applied Mathematical Modelling, **39**(20), (2015), 6139-6149.

[26] I Podlubny, *Fractional differential equations*, Elsevier, 1999.

[27] A Kirsch, *An Introduction to the Mathematical Theory of Inverse Problems*, Applied Mathematical Sciences (AMS, volume 120).