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Multiplicity results for a Kirchhoff type equations with general potential

Linsong Chen[1], **Tianqun Hu**[1] **and Jian Zhou**[1,*]

1. School of Mathematical Sciences, Guizhou Normal University, Guiyang 550025, China

[*]Corresponding author

Abstract

This research we examine a Kirchhoff type equation in $\mathbb{R}^3$ involving a potential that changes sign. By imposing appropriate conditions on V and making spectral assumptions, we successfully establish the existence of multiple solutions for this particular issue using variational methods.

Key words: Palais-Smale condition, Morse index, Kirchhoff type equation, Variational methods
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1 Introduction and main result

This study focuses on the existence of multiple solutions for the following equation of Kirchhoff type.

$$-\left(a+b\int_{\mathbb{R}^3}|\nabla u|^2\,dx\right)\Delta u+V(x)u=Q(x)g(x,u),\quad \text{in } \mathbb{R}^3. \tag{1}$$

where $a>0, b\geq 0$ are constants.

The Kirchhoff equation has a strong physical background and the following equation is obtained when studying changes in elastic chord length

$$\rho\frac{\partial^2 u}{\partial t^2}-\left(\frac{P_0}{h}+\frac{E}{2L}\int_0^L\left|\frac{\partial u}{\partial x}\right|^2 dx\right)\frac{\partial^2 u}{\partial x^2}=0$$

first introduced by Kirchhoff [14]. This model is an extension of the classical d'Alembert wave equation regarding the length variation of strings during vibration, where L represents the chord length, h represents the cross-sectional area, E is the Young's modulus of matter, ρ is density, and P_0 represents the initial tension. In addition, it has extensive applications in

Contact: Linsong Chen 2777456670@qq. com; Tianqun Hu 2433201629@qq. com; Jian Zhou zhoujiandepict@163. com

various fields such as cosmic physics, elasticity theory, and non Newtonian mechanics. For more information on its physical background, we can refer to literature [4, 5, 6, 7]. Problem (1) has attracted significant attention since Lions [15] introduced an abstract framework for it. Moreover, in [13] Xu L, Li F, and Xie Q investigate the existence and multiplicity of normalized solutions for the Kirchhoff equation with unknown parameters.

First, assuming that the functions $Q(x)$ and $V(x)$ meet the requirements:

(Q) $\inf\limits_{x\in\mathbb{R}^3} Q(x) = Q_0 > 0$, where Q_0 is a positive constant, and there is a positive constant $0 < \alpha \leqslant 1$, such that $\sup\limits_{x\in\mathbb{R}^3} Q(x) \leqslant \alpha$.

(V_1) $V(x) \in L^q_{loc}(\mathbb{R}^3)$ is real valued function, and denote that $V^- := -\min\{V, 0\} \in L^\infty(\mathbb{R}^3) \cap L^q(\mathbb{R}^3)$ where $q \geq 2$.

From the condition (V_1), it is known that the Schrödinger operator $T := -a\Delta + V$ is self-adjoint and semi-bounded on $L^2(\mathbb{R}^3)$ (see [1, 12]). Denote the spectrum of T is $\sigma(T)$, the the essential spectrum of T is $\sigma_{ess}(T)$, and the pure point spectrum of T is $\sigma_d(T)$. Moreover, we make the following general spectrum assumption:

(V_2) $\gamma := \inf \sigma(T)$, $\delta := \inf \sigma_{ess}(T)$, $-\infty < \gamma < \delta$, and $\delta > 0$.

(g_1) $g \in C(\mathbb{R}^3 \times \mathbb{R})$, and $\dfrac{g(x,u)}{u}$ is bounded on $\mathbb{R}^3 \times (\mathbb{R} \setminus \{0\})$.

(g_2) $g^* := \limsup\limits_{|x|\to+\infty} \sup\limits_{u\neq 0} \dfrac{g(x,u)}{u} < \delta$.

Corresponding to (g_2), we assume P is continuous, bounded and real-valued function, define a set as follows:

$$\Gamma := \left\{ P | P^* := \limsup_{|x|\to+\infty} P(x) < \delta \right\}. \tag{2}$$

And suppose there exist $P_1(x), P_2(x) \in \Gamma$ such that

(g_3) $g(x,u) = P_1(x)u + o(|u|)$ with $u \to 0$ uniformly in $x \in \mathbb{R}^3$,

(g_4) $g(x,u) = P_2(x)u + R_u(x,u)$, with $R(x,u) \leq 0$ for all (x,u), where $R(x,u) = \int_0^u R_u(x,t)dt$,

Function $i(P)$ can be defined as the dimension of the negative eigenspace corresponding to $-a\Delta + V - P$. It is also noted that $i(P)$ is nondecreasing with respect to P. The $i(P)$ is called the index of P and the nullity of P is the $\nu(P)$. Denote

$$\nu(P) = \ker(-a\Delta + V - P).$$

We have obtained some characters with respect to P in the next Section. For more details about this, it can refer to [1].

We have the following main results regarding the existence of a solution to this problem (1)

Theorem 1.1. *Suppse that the equation satisfied* (Q), (V_1), (V_2), *and* (g_1) − (g_4), *we have a conclusion that the problem (1) has at least* $i(P_0)$ *pairs of nontrivial solutions.*

In studying periodic solutions of Hamiltonian systems, the index theory was widely applied (see [2, 16, 17]). The theory of classification presented in this article exhibits a strong correlation with the aforementioned indicator theory. Although our methods of constructing the classification differ from those employed in the indicator theory, they are still technically distinct in. [2, 16, 17]. The essential spectrum poses several additional challenging issues (see [1]) that need to be addressed. .

Let:

$$\lambda_n := \inf_{Y_n} \sup_{u \in Y_n \setminus \{0\}} \frac{\int_{\mathbb{R}^3} (a|\nabla u|^2 + V(x)u^2) dx}{\int_{\mathbb{R}^3} u^2 dx},$$

the family of n-dimensional subspace of $C_0^\infty(\mathbb{R}^3)$ denoted by Y_n (see [18]). And

$$\lambda_\infty = \lim_{n\to\infty} \lambda_n, \text{ and } \lambda_\infty = \inf \sigma_{ess}(T),$$

if $\lambda_\infty < +\infty$ and the inequality $\lambda_n < \lambda_\infty$ be satisfied, then $\lambda_n \in \sigma_d(T)$. Thus, if $\lambda_\infty > 0$, obviously, T satisfies condition (V_2).

When $\lambda_k < 0 \leqslant \lambda_{k+1}$ for some k, problem (1) has achieved some results for the existence of solutions. In [9, 10, 11] the authors study the existence of multiple solutions by using variant Clark's Theorem, Three Critical Points Theorem and Clark's Theorem. In this article, to deal with the problem (1), we need to encounter various difficulties. On the one hand, due to the indefinite of $V(x)$ in sign, the functional I does not have mountain path geometry, then the Mountain Pass Theorem can not be applied. On the other hand, under our assumptions, the Sobolev embedding $H^1(\mathbb{R}^3) \hookrightarrow L^2(\mathbb{R}^3)$ is not compact, which is crucial in verifying the Palais–Smale condition.

2 Preliminaries

Let $L^p(\mathbb{R}^3)$ be the standard L^p space for $1 \le p < \infty$ associated with the norm

$$|u|_p = \left(\int_{\mathbb{R}^3} |u|^p dx \right)^{1/p}, \quad u \in L^p(\mathbb{R}^3),$$

and let $H^1(\mathbb{R}^3)$, $D^{1,2}(\mathbb{R}^3)$ be the usual Sobolev space with a norm

$$||u||_{H^1(\mathbb{R}^3)} = \left(\int_{\mathbb{R}^3} \left[|\nabla u|^2 + u^2 \right] dx \right)^{1/2}, \ ||u||_{D^{1,2}} = \left(\int_{\mathbb{R}^3} |\nabla u|^2 dx \right)^{1/2}.$$

Due to 0 is, at most, an eigenvalue with finite multiplicity, then it assume that $0 \notin \sigma(T)$. Furthermore, based on the spectral property (V_2), we can induce an orthogonal decomposition as follows

$$L^2 = L^- \oplus L^+, \quad u = u^- + u^+,$$

such that T is negative definite on L^- and positive definite on L^+. The notation $\mathcal{N} = D(|T|^{1/2})$ introduces a Hilbert space equipped with an inner product as follows, where $|T|$ is the absolute value of T.

$$(u, w) = (|T|^{1/2} u, |T|^{1/2} w)_2,$$

and norm

$$||u||^2 = (u, u).$$

Meanwhile, we can obtain the following orthogonal decomposition

$$\mathcal{N} := \mathcal{N}^- \oplus \mathcal{N}^+,$$

where

$$\mathcal{N}^\pm = \mathcal{N} \cap L^\pm.$$

The space $\mathcal{N} \hookrightarrow H^1(\mathbb{R}^3)$ is continuous embedding, and then, $\mathcal{N} \hookrightarrow L^p(\mathbb{R}^3)$ is continuous embedding for $p \in [2, 6]$.

It is easy to know that problem (1) has the energy functional

$$I(u) = \frac{1}{2}\left|\left|u^{+}\right|\right|^{2} - \frac{1}{2}\left|\left|u^{-}\right|\right|^{2} + \frac{b}{4}\left(\int_{\mathbb{R}^3} |\nabla u|^2 \, dx\right)^2 - \int_{\mathbb{R}^3} Q(x)G(x,u)dx, \ u \in \mathcal{N}.$$

where

$$G(x,u) = \int_0^u g(x,s)ds.$$

I is of class C^1 on $\mathcal{N}$ and the derivative function of I has the following form

$$\left(I^{'}(u), v\right) = \left(u^{+}, w^{+}\right) - \left(u^{-}, w^{-}\right) + b\int_{\mathbb{R}^3} |\nabla u|^2 \, dx \int_{\mathbb{R}^3} \nabla u \cdot \nabla v dx - \int_{\mathbb{R}^3} Q(x)g(x,u)v dx. \quad (3)$$

Recall Γ defined in (2), we can define the following form of quadratic

$$q_P(u,w) = \frac{(u^{+}, w^{+}) - (u^{-}, w^{-}) - (Pu, w)_2}{2}, \quad \forall u, w \in \mathcal{N}.$$

Remark 2.1. In the reference [8], introducing an important concept–the PS-condition. If condition $I(u_n) \to c$ and $I^{'}(u_n) \to 0$, as $n \to \infty$ satisfied, we say that this sequence is a $(PS)_c$ sequence. Moreover, if each such sequence contains a convergent subsequence, then we say that the function I satisfies $(PS)_c$ condition. As an important theoretical support, this is crucial for us to study the existence of solutions to the Kirchhoff equation. Meanwhile, (PS) conditions play an important role in nonlinear analysis theory and are the core of the entire theoretical analysis framework.

Theorem 2.2 ([3]). *Let $I \in C^1(\mathcal{N}, \mathbb{R}^3)$ be an even functional on a Banach space $\mathcal{N}$. Assume that $I(0) = 0$ and I satisfies the (PS) condition. If*

(I_1) *there exists a set $\mathcal{N}_1 \subset \mathcal{N}$, $\dim \mathcal{N}_1 = k_1$, and $\rho > 0$ such that*

$$\sup_{u \in \mathcal{N}_1 \bigcap S_\rho} I(u) < 0;$$

(I_2) *there exists a set $\mathcal{N}_2 \subset \mathcal{N}$, codim$\mathcal{N}_2 = k_2 < k_1$ such that*

$$\inf_{u \in \mathcal{N}_2} I(u) > -\infty,$$

then I has at least $k_1 - k_2$ pairs of critical points with negative critical values.

3 Proof of main result

Lemma 3.1. *Assume that (V_1), (V_2), (Q) and (g_4) are satisfied. Then the functional I is coercive.*

Proof. We adopt the method of proof by contradiction, we can choose $\{u_n\} \subset \mathcal{N}$, and $M > 0$ such that $I(u_n) \leq M$ as $||u_n|| \to +\infty$. Let $v_n = \frac{u_n}{||u_n||}$, then $||v_n|| = 1$. From condition (Q) and

(g_4), there exists $P_\infty \in \Gamma$ such that

$$
\begin{aligned}
\frac{M}{||u_n||^2} &\geq \frac{I(u_n)}{||u_n||^2} \\
&= \frac{\frac{1}{2}||u_n^+||^2 - \frac{1}{2}||u_n^-||^2 + \frac{b}{4}\left(\int_{\mathbb{R}^3} |\nabla u_n|^2 dx\right)^2 - \int_{\mathbb{R}^3} Q(x)G(x,u_n)dx}{||u_n||^2} \\
&= \frac{1}{2}||v_n^+||^2 - \frac{1}{2}||v_n^-||^2 - \frac{1}{2}(P_\infty v_n, v_n)_2 - \frac{\int_{\mathbb{R}^3} Q(x)R(x,u_n)dx}{||u_n||^2} \\
&\quad + \frac{b\left(\int_{\mathbb{R}^3} |\nabla u_n|^2 dx\right)^2}{4||u_n||^2} \\
&= q_{P_\infty}(v_n, v_n) - \frac{\int_{\mathbb{R}^3} Q(x)R(x,u_n)dx}{||u_n||^2} + \frac{b\left(\int_{\mathbb{R}^3} |\nabla u_n|^2 dx\right)^2}{4||u_n||^2}.
\end{aligned}
$$

It follows from $R(x, u_n) \leq 0$, thus

$$o(1) \geq q_{P_\infty}(v_n, v_n).$$

Based on the definition of the set Γ and by using [3] Proposition 1 (iv), we can choose $\varepsilon > 0$ small enough such that $P_\infty + \varepsilon \in \Gamma$ and $\nu(P_\infty + \varepsilon) = 0$. Moreover, by calculating we have

$$q_{P_\infty+\varepsilon}(v_n, v_n) = q_{P_\infty}(v_n, v_n) - \frac{\varepsilon}{2}|v_n|_2^2 \leq o(1). \tag{4}$$

By [3] Proposition 1 and $\nu(P_\infty + \varepsilon) = 0$, we have the following $q_{P_\infty+\varepsilon}$-orthogonal decomposition of v_n:

$$v_n = v_{n,1} + v_{n,2} \in \mathcal{N}^-(P_\infty + \varepsilon) \oplus \mathcal{N}^+(P_\infty + \varepsilon),$$

and

$$q_{P_\infty+\varepsilon}(v_n, v_n) = q_{P_\infty+\varepsilon}(v_{n,1}, v_{n,1}) + q_{B_\infty+\varepsilon}(v_{n,2}, v_{n,2}).$$

Then, by (4) we have

$$-q_{P_\infty+\varepsilon}(v_{n,1}, v_{n,1}) + o(1) \geq q_{P_\infty+\varepsilon}(v_{n,2}, v_{n,2}).$$

Recalling $||v_n|| = 1$, we may assume $v_n \rightharpoonup v$ in $\mathcal{N}$, since $\dim \mathcal{N}^-(P_\infty + \varepsilon) < +\infty$, we have $v_{n,1} \to v_1$ and $v_{n,2} \rightharpoonup v_2$.

We claim that $v_1 \neq 0$. If $v_1 = 0$, then $v_{n,1} \to 0$ in $\mathcal{N}$ and $q_{P_\infty+\varepsilon}(v_{n,1}, v_{n,1}) \to 0$. By [3] Proposition 1 (v), there exists constant c such that

$$q_{P_\infty}(v_{n,2}, v_{n,2}) \geq c||v_{n,2}||^2,$$

which implies that $v_{n,2} \to v_2 = 0$ and $v_n \to 0$. Contradiction with $||v_n|| = 1$. Thus, $v_1 \neq 0$ and $v \neq 0$. By the Fatou's lemma, let $n \to \infty$

$$
\begin{aligned}
o(1) &= \frac{I(u_n)}{||u_n||^4} \\
&= \frac{q_{P_\infty}(u_n, u_n) - \int_{\mathbb{R}^3} Q(x)R(x,u_n)dx + \frac{b}{4}(\int_{\mathbb{R}^3} |\nabla u_n|^2 dx)^2}{||u_n||^4} \\
&\geq o(1) + \frac{b}{4}(\int_{\mathbb{R}^3} |\nabla v|^2 dx)^2 \\
&> 0.
\end{aligned}
$$

it's a contradiction.

□

Lemma 3.2. *Assume that* (V_1), (V_2), (Q), *and* (g_1), (g_3) *can be held. Then there exist* $\varepsilon, \rho > 0$ *and* $P_0(x) \in \Gamma$, *such that*

$$\sup_{\mathcal{N}^-(P_0-\varepsilon)\cap S_\rho} I(u) < 0.$$

Proof. By condition (g_3), we have

$$g(x,u) = P_1(x)u + g_1(x,u),$$

and

$$g_1(x,u) = o(|u|).$$

as $|u| \to 0$ uniformly in $x \in \mathbb{R}^3$. Set

$$G_1(x,u) = \int_0^u g_1(x,t)dt.$$

Fix any $s \in (2,6)$. For any $\varepsilon > 0$, there is a $C_\varepsilon > 0$ such that

$$g_1(x,u) \geqslant -\varepsilon u - C_\varepsilon |u|^{s-1},$$

which implies that

$$G_1(x,u) \geqslant -\frac{\varepsilon}{2}|u|^2 - \frac{C_\varepsilon}{s}|u|^s.$$

Since the embedding $\mathcal{N} \hookrightarrow L^s(\mathbb{R}^3)$ is continuous, there exists a positive constant C_ε^* which depends on s, such that

$$\int_{\mathbb{R}^3} G_1(x,u)dx \geqslant -\frac{\varepsilon}{2}|u|_2^2 - C_\varepsilon^* ||u||^s.$$

Let $P_0(x) = Q(x)P_1(x)$, then $P_0(x) \in \Gamma$. Noting that $\mathcal{N}^-(P_0 - \varepsilon)$ is finite dimensional, by the Proposition 2. 2. (v), there exist constants c_1, c_2 such that

$$c_1 ||u||^2 \leqslant -q_{P_0-\varepsilon}(u,u) \leqslant c_2 ||u||^2.$$

Due to the equivalence of norms in finite dimensional spaces, there exists $C > 0$ such that for all $u \in \mathcal{N}^-(P_0 - \varepsilon)$, we have

$$\begin{aligned} I(u) &= \frac{1}{2}||u^+||^2 - \frac{1}{2}||u^-||^2 + \frac{b}{4}\left(\int_{\mathbb{R}^3}|\nabla u|^2 dx\right)^2 - \int_{\mathbb{R}^3} Q(x)G(x,u)dx \\ &\leqslant q_{P_0-\varepsilon}(u,u) + \frac{b}{4}|\nabla u|_2^4 + C_\varepsilon^* ||u||^s \\ &\leqslant -c_1||u||^2 + C||u||^4 + C_\varepsilon^*||u||^s. \end{aligned}$$

Thus, this lemma follows by choosing ρ small enough.

□

Lemma 3.3. *Under our assumptions, the functional I fulfills the Palais-Smale condition.*

Proof. We can choose a sequence $\{u_n\}$ which is a $(PS)_c$ sequence, i. e. $I(u_n) \to c$ and $I'(u_n) \to 0$. By Lemma 3. 1, we know that I satisfies coercive condition, then $I(u_n) \to c$ implies that $\{u_n\}$ is bounded in $\mathcal{N}$. Suppose $u_n \rightharpoonup u$ in $\mathcal{N}$. Set $\omega_n = u_n - u$. Then

$$\omega_n \rightharpoonup 0, \quad \omega_n^+ \rightharpoonup 0, \quad \omega_n^- \rightharpoonup 0, \quad \text{in } \mathcal{N},$$

by (3), we have

$$\begin{aligned} o(1) &= \left(I'(u_n), \omega_n^+ - \omega_n^-\right) \\ &= o(1) + ||\omega_n||^2 - \int_{\mathbb{R}^3} Q(x) f(x, u_n)(\omega_n^+ - \omega_n^-) dx \\ &\quad + b \int_{\mathbb{R}^3} |\nabla u_n|^2 \, dx \int_{\mathbb{R}^3} \nabla u_n \nabla (\omega_n^+ - \omega_n^-) dx. \end{aligned} \tag{5}$$

By the Hölder inequality, let $n \to \infty$ and $R \to \infty$, then we have

$$\begin{aligned} &\left| b \int_{\mathbb{R}^3} |\nabla u_n|^2 \, dx \int_{\mathbb{R}^3} \nabla u_n \nabla (\omega_n^+ - \omega_n^-) dx \right| \\ &\leqslant \left| b \int_{|x| \leqslant R} |\nabla u_n|^2 \, dx \int_{|x| \leqslant R} \nabla u_n \nabla (\omega_n^+ - \omega_n^-) dx \right| \\ &\quad + \left| b \int_{|x| \geqslant R} |\nabla u_n|^2 \, dx \int_{|x| \geqslant R} \nabla u_n \nabla (\omega_n^+ - \omega_n^-) dx \right| \\ &\leqslant \left| b \int_{|x| \leqslant R} |\nabla u_n|^2 \, dx \right| |\nabla u_n|_{L^2(P_R(o))} \left|\nabla (\omega_n^+ - \omega_n^-)\right|_{L^2(B_R(o))} \\ &\quad + \left| b \int_{|x| \geqslant R} |\nabla u_n|^2 \, dx \right| |\nabla u_n dx|_{L^2(P_R^c(o))} \left|\nabla (\omega_n^+ - \omega_n^-)\right|_{L^2(P_R^c(o))} \\ &\to 0 \end{aligned} \tag{6}$$

Here, we have used

$$|\nabla u_n|_{L^2(P_R^c(0))} \to 0, \quad \text{as } R \to \infty.$$

Since $\{u_n\}$ is bounded, and $\mathcal{N}$ locally compact embedding to $L^2(B_R(0))$, thus

$$\left|\nabla (\omega_n^+ - \omega_n^-)\right|_{L^2(B_R(0))} \to 0, \quad \text{as } n \to \infty.$$

Combining (5) and (6) , we know that

$$o(1) = ||\omega_n||^2 - \int_{\mathbb{R}^3} Q(x) f(x, u_n)(\omega_n^+ - \omega_n^-) dx.$$

Let P_1 be the spectral projection of T associated with $(\delta - \varepsilon, +\infty]$, and P_2 be the spectral projection of T associated with $(-\infty, \delta - \varepsilon)$. Then P_2 is finite dimensional with $P_2 \omega_n \to 0$ and

$$||\omega_n||^2 = ||P_1 \omega_n||^2 + o(1), \quad ||P_1 \omega_n||^2 \geqslant (\delta - \varepsilon) \, |P_1 \omega_n|_2^2.$$

Consequently,

$$\begin{aligned} o(1) &= ||P_1 \omega_n||^2 - \int_{\mathbb{R}^3} Q(x) \frac{f(x, u_n)}{u_n} (\omega_n^+ - \omega_n^-) u_n dx \\ &= o(1) + ||P_1 \omega_n||^2 - \int_{\mathbb{R}^3} Q(x) \frac{f(x, u_n)}{u_n} (P_1 \omega_n)^2 dx \\ &\geqslant o(1) + ||P_1 \omega_n||^2 - \alpha f^* \int_{\mathbb{R}^3} (P_1 \omega_n)^2 dx \\ &\geqslant o(1) + (1 - \frac{\alpha f^*}{\delta - \varepsilon}) \, ||P_1 \omega_n||^2 . \end{aligned} \tag{7}$$

That implies $||P_1\omega_n|| \to 0$. Hence, $u_n \to u$ in $\mathcal{N}$.

□

Proof of Theorem 1.1. Let g is even funciton, then I is even. According to Lemma 3.2. we know that I satisfies condition (I_1) of Theorem 2.2. with $\mathcal{N}_1 = \mathcal{N}^-(P_0 - \varepsilon)$. By (ii) and (iv) of Proposition 2. 2. we have

$$\dim \mathcal{N}_1 = i(P_0 - \varepsilon) = i(P_0),$$

by choosing ε small enough. Moreover, from Lemma 3.3. we know that I satisfies the Palais-Smale condition. By Lemma 3.1, condition (I_2) of Theorem 2.2 holds by $\mathcal{N}_2 = \mathcal{N}$ with $codim\mathcal{N}_2 = 0$. Thus, according to the Theorem 2.2 we can conclude that I has at least $i(P_0)$ pairs of nontrivial critical points.

□

4 Conclusions

In this article, we obtained some results on the existence of solutions to the Kirchhoff equation. Due to the appearance of nonlinear terms $\int_{\mathbb{R}^3} |\Delta u|^2 \, dx$ in this paper, which mean that (1) is not a pointwise identity. This has brought some mathematical difficulties to our research. This leads to some classical theories no longer being applicable. In order to overcome these difficulties, we utilized the methods mentioned in the reference [3] which studies a class of Schrödinger-Poisson equations (This is different from our research), important conclusion on the existence of solutions to the Kirchhoff equation be obtained.

5 Declarations

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Competing Interests

The authors declare that they have no competing interests.

Ethical Approval

Not applicable.

Author's Contributions

Linsong Chen writing-original draft preparation, Tianqun Hu, Jian Zhou writing-review and modify

Availability Data and Materials

Not applicable.

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