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Exponential Stability of Viscoelastic Structure with Second Sound

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Abstract

This manuscript deals with a thermo-viscoelastic system describing the vibrations of a flexible structure. We study this structure's stabilization problem when subjected to Kelvin-Voigt damping and the second sound. Cattaneo-Vernotte's law governs the thermal effect, eliminating the physical paradox of equal speed of waves in the classical thermoelastic theory. Semigroup theory proves the solution's existence and uniqueness. Exponential stability is proved by the energy method.

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We consider the longitudinal displacement of a flexible structure with viscoelastic material given by Kelvin-Voigt damping (see [9], pp. 22-26). The problem of vibration control in such structures is suppressing their oscillatory disturbances, which can eventually damage the structures. Due to its wide application in structural engineering and industry, the vibration stability analysis of such problems is a rich subject area from both theoretical and practical perspectives. In recent years, several structural vibration control systems have been developed to improve the performance and protection of structures against vibration events of natural origin or man-made. These control mechanisms are classified as passive, semi-active, active, and hybrid control [18, 27]. Passive control systems can contain springs, masses, and dampers, which generate a controlled force in response to structural motion and are most widely present in civil structures due to their simplicity and do not depend on external forces.

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Passive control is incorporated as damping in a structure to absorb part of the input energy, reduce structural damage and stabilize vibrations (see also Škvorc et al. [24]). Some of these controls are incorporated through a viscoelastic element, a thermal source, or a heat flux. These dampers are essential because they dissipate energy at all levels of distortion and over a wide range of frequencies. In this context, this manuscript deals with non-homogeneous flexible structures in thermo-viscoelasticity with second sound, see [22], given by

$$\rho(x)\varphi_{tt} - (s(x)\varphi_x + 2\delta(x)\varphi_{xt})_x + \alpha\vartheta_x + \beta q_x = 0, \tag{1}$$

$$\vartheta_t + kq_x + k\tau q_{tx} + \alpha\varphi_{xt} = 0, \tag{2}$$

$$\tau q_t + q + k\vartheta_x + \beta\varphi_{xt} = 0, \tag{3}$$

with initial conditions

$$\varphi(x,0) = \varphi_0(x), \;\; \varphi_t(x,0) = \varphi_1(x), \;\; \vartheta(x,0) = \vartheta_0(x), \;\; q(x,0) = q_0(x) \tag{4}$$

and boundary conditions

$$\varphi(0,t) = \varphi(\ell,t) = 0, \;\; \vartheta(0,t) = \vartheta(\ell,t) = q(0,t) = q(\ell,t) = 0. \tag{5}$$

In the sequel, we present a brief mathematical formulation.

0.1 Mathematical formulation

According to Kirchhoff's hypothesis and neglecting the rotatory inertia, as in [16], for longitudinal displacement, φ of a clamped elastic structures of length ℓ made of a viscoelastic material, the gradient of the displacement φ_x represent the strain. The equation of balance of linear momentum is

$$\rho\varphi_{tt} - \sigma_x = 0 \tag{6}$$

where the coefficients ρ and σ, depending on the strain, are the mass per unit length and the stress. The equation

$$\sigma = \sigma(\varphi_x, \varphi_{tx}) = su_x + a\varphi_{xt}. \tag{7}$$

gives the velocity viscoelastic material with Kelvin-Voigt constitutive relation.

From (6) and (7), the longitudinal vibrations of the structure are modeled by the following equations and initial-boundary conditions

$$\rho\varphi_{tt}(x,t) - (s\varphi_x(x,t) + a\varphi_{xt}(x,t))_x = 0, \;\text{in}\; (0,\ell)\times(0,\infty), \tag{8}$$

$$\varphi(x,0) = \varphi_0(x), \;\; \varphi_t(x,0) = \varphi_1(x), \;\; x \in (0,\ell), \tag{9}$$

$$\varphi(0,t) = \varphi(\ell,t) = 0, \;\; t \in (0,\infty), \tag{10}$$

with the coefficients $\rho,\ s,\ a \geq c_0 > 0$.

New advances in material science lead to the reduction of vibrations with patches made of smart material to the structure under passive and active damping. Equations (8) model these phenomena by taking the coefficients with particular forms, see [4]. The material properties, such as the density, Young's modulus, and damping coefficients, are changed due to the patches. Considering a non-uniform body with the mass per unit length not constant, in the case of the stress depending on the position in the body, we have

$$\sigma = \sigma(\varphi_x, \varphi_{tx}) = s(x)\varphi_x + 2\delta(x)\varphi_{xt}.$$

From (8) the longitudinal vibrations is given by

$$\rho(x)\varphi_{tt}(x,t) - (s(x)\varphi_x(x,t) + 2\delta(x)\varphi_{xt}(x,t))_x = 0. \tag{11}$$

The mass per unit length $\rho(x)$, the stiffness related to the velocity of longitudinal vibrations $s(x)$, and the coefficient of internal damping $\delta(x)$ are positive functions responsible for the non-homogeneous properties of the structure. From now on, we assume that

$$\begin{gathered}\rho(x),\ s(x),\ \delta(x) \in W^{1,\infty}(0,\ell), \\ \operatorname{ess\,inf}(\rho),\ \operatorname{ess\,inf}(s),\ \ \delta_0 := \operatorname{ess\,inf}(\delta) > 0. \end{gathered} \tag{12}$$

0.2 Heat conduction

The viscosity reflects several damping situations intrinsic to the constitutive relationships of the material. As we consider the temperature difference between the material and the medium $\vartheta(x,t) := T(x,t) - T_0$, given by the Fourier law [7, 25]

$$\vartheta_t(x,t) + kq_x(x,t) = 0 \quad \text{and} \quad q(x,t) + k\vartheta_x(x,t) = 0,\ k > 0 \text{ a constant}, \tag{13}$$

where $q = q(x,t)$ is the heat flux, in equation (13), the temperature has an infinite speed.

Modeling the thermo-diffusion phenomenon with (13) differs from reality. To avoid this inconsistency, we take the modified coupling coefficient via the Cattaneo-Vernotte law, introducing the parameter $\tau > 0$ describing the time lag in the response for the temperature, which leads to an equation

$$\vartheta_t(x,t) + kq_x(x,t) = 0 \quad \text{and} \quad q(x,t+\tau) + k\vartheta_x(x,t) = 0. \tag{14}$$

After retaining the first order term of τ in the Taylor series of $q(x,t+\tau)$ about t, we get approximately

$$\vartheta_t(x,t) + kq_x(x,t) = 0 \quad \text{and} \quad q(x,t) + \tau q_t(x,t) + k\vartheta_x(x,t) = 0. \tag{15}$$

Consider another modification: $q(x,t+\tau) = q(x,t) + \tau q_t(x,t)$ introduced in both equations of (13), so we have

$$\vartheta_t(x,t) + kq_x(x,t) + k\tau q_{tx}(x,t) = 0, \tag{16}$$
$$q(x,t) + \tau q_t(x,t) + k\vartheta_x(x,t) = 0. \tag{17}$$

Equations (16) and (17) coupled with (11) is a hyperbolic system that has a finite speed of propagation, and then the paradox of equal speed of waves is eliminated.

0.3 Related models

Fourier's law cannot be applied to specific scientific areas such as nano-scale heat transfer analysis like small periodic heat flow, laser heating, etc. [3, 6, 21, 26]. Maxwell first introduced the most popular modification to avoid this paradox for non-Fourier heat conduction. Cattaneo and Vernotte [5, 7] generalized by introducing the second sound phenomenon. The above mathematical concepts lead to a vibration model of a flexible viscoelastic structure (1)-(3).

The uniform exponential decay of the initial value and boundary problem associated with equation

$$m(x)u_{tt} - (p(x)u_x + 2\delta(x)u_{xt})_x = f.$$

describing the longitudinal vibrations of an non-homogeneous structure with Kelvin-Voigt damping, was proved in [10].

In [11] was proved exponential stability for the non-homogeneous viscoelastic system with Cattaneo-Vernotte's law

$$\begin{aligned} m(x)u_{tt} - (p(x)u_x + 2\delta(x)u_{xt})_x + \theta_{xt} &= F(x,t), \\ \tau\theta_{tt} + \theta_t - \beta\theta_{xx} + \eta u_{xt} &= 0, \end{aligned}$$

where $F : (0, L) \times (0, \infty) \to \mathbb{R}$ is appeared as the distributed input disturbing force in the model.

In [20], was proved exponential stability for the non-homogeneous viscoelastic system with Fourier's law given by

$$\begin{aligned} m(x)u_{tt} - (p(x)u_x + 2\delta(x)u_{xt})_x + k\theta_x &= f, \\ \theta_t - \theta_{xx} + ku_{xt} &= 0. \end{aligned}$$

Memory damping combined with heat conduction given by the Cattaneo-Vernotte law was considered in [12]

$$\begin{aligned} m(x)u_{tt} - (p(x)u_x + 2\delta(x)u_{xt})_x + \eta\theta_x + \int_0^\infty g(s)u_{xx}(x,t-s)ds &= 0, \\ \theta_t + kq_x + \eta u_{xt} &= 0, \\ \tau q_t + \beta q + k\theta_x &= 0, \end{aligned}$$

and under suitable conditions on the relaxation kernel $g(s)$ was obtained the exponential decay.

In [13] was considered micro-temperature effect

$$\begin{aligned} m(x)u_{tt} - (p(x)u_x + 2\delta(x)u_{xt})_x + \eta\theta_x + dw_x &= 0, \\ c\theta_t - k\theta_{xx} + \eta u_{xt} + k_1 w_x &= 0, \\ \tau w_t - k_3 w_{xx} + k_2 w + k_1\theta_x + du_{tx} &= 0, \end{aligned}$$

and proved the exponential stability.

In [8] memory and a thermal effect given by the Green-Naghdi type III was considered and the exponential stability was proved. A particular case of (1)-(3) with $\beta = 0$ was considered in [2].

Others comprehensive study can be found in [14, 15] including the time delay term. Recently, a numerical simulation using the finite difference method with an stability results for coupled wave equations and with the thermal diffusion equation are available in [19, 23].

Now, we briefly describe how this paper is organized: in Section 1, we prove the well-posedness using the semigroup theory. Our main results are demonstrated in Section 2. We prove exponential stability using the energy method and suitable multipliers to construct a Lyapunov functional.

1 Well-Posedness

We use the Sobolev spaces, see [1]. $L^2(0,\ell)$ denotes the usual Lebesgue space with the inner product and induced norm defined by

$$\langle \phi, \psi \rangle_{L^2(0,\ell)} = \int_0^\ell \phi\overline{\psi}\, dx, \quad \text{and} \quad \|\phi\|^2_{L^2(0,\ell)} = \int_0^\ell |\phi|^2\, dx.$$

Denoting $v = \varphi_t$ and $U = (\varphi, v, \vartheta, q)^T$ the system (1)–(4) is write as an abstract Cauchy problem

$$\frac{\mathrm{d}\,U}{\mathrm{d}t} = \mathcal{A}U(t), \text{ with } U(0) = U_0 = (\varphi_0, v_0, \vartheta_0, q_0)^T, \tag{18}$$

being $\mathcal{A} : D(\mathcal{A}) \subset H \to H$ the differential operator defined by

$$\mathcal{A}U = \begin{pmatrix} v \\ \dfrac{1}{\rho(x)}\left[(s(x)\varphi_x + 2\delta(x)v_x)_x - \alpha\vartheta_x - \beta q_x\right] \\ -\left[kq_x + \alpha v_x + k\tau q_{tx}\right] \\ -\dfrac{1}{\tau}\left[q + k\vartheta_x + \beta v_x\right] \end{pmatrix},$$

where $\mathcal{H}$ is the phase space defined by

$$\mathcal{H} = H_0^1(0,\ell) \times L^2(0,\ell) \times L^2(0,\ell) \times L^2(0,\ell),$$

which we endow with the inner product

$$\langle U, U^* \rangle_{\mathcal{H}} = \int_0^\ell s(x)\varphi_x \overline{\varphi_x^*}\,\mathrm{d}x + \int_0^\ell \rho(x) v\overline{v^*}\,\mathrm{d}x + \int_0^\ell \vartheta\overline{\vartheta^*}\,\mathrm{d}x + \tau\int_0^\ell q\overline{q^*}\,\mathrm{d}x, \tag{19}$$

where $U = (\varphi, v, \vartheta, q)^T$, $U^* = (\varphi^*, v^*, \vartheta^*, q^*)^T \in \mathcal{H}$.

Hypothesis (12) on the functions ρ, s and δ allow us to prove that (19) defines an inner product on $\mathcal{H}$ and makes it a Hilbert space relative to the induced norm

$$\|\varphi\|_{\mathcal{H}}^2 = \|\sqrt{s(x)}\varphi_x\|_{L^2(0,\ell)}^2 + \|\sqrt{\rho(x)}v\|_{L^2(0,\ell)}^2 + \|\vartheta\|_{L^2(0,\ell)}^2 + \|\sqrt{\tau}q\|_{L^2(0,\ell)}^2.$$

The domain $D(\mathcal{A})$ of the operator $\mathcal{A}$ is defined by

$$D(\mathcal{A}) = \left\{(\varphi, v, \vartheta, q) \in \mathcal{H} \;\middle|\; s(x)\varphi_x + 2\delta(x)v_x \in H^1(0,\ell),\, v, \vartheta, q \in H_0^1(0,\ell)\right\}.$$

For the existence and uniqueness of the solution of the evolution problem (18), we apply the Lumer-Phillips theorem. First, we show that the operator $\mathcal{A}$ is dissipative.

Lemma 1.1. *For $U \in D(\mathcal{A})$ and $\beta \leq \sqrt{2\delta_0}$, assuming the hypotheses (12), the operator $\mathcal{A}$ is dissipative. More precisely, we have*

$$\mathrm{Re}\langle \mathcal{A}U, U\rangle \leq -(2\delta_0 - \beta^2)\int_0^\ell |v_x|^2\,\mathrm{d}x \leq 0.$$

Proof.

$$\begin{aligned}
\langle \mathcal{A}U, U\rangle_H &= \int_0^\ell s(x)v_x\overline{\varphi_x}\,\mathrm{d}x + \int_0^\ell \rho(x)v_t\overline{v}\,\mathrm{d}x + \int_0^\ell \vartheta_t\overline{\vartheta}\,\mathrm{d}x + \tau\int_0^\ell q_t\overline{q}\,\mathrm{d}x \\
&= \int_0^\ell s(x)v_x\overline{\varphi_x}\,\mathrm{d}x + \int_0^\ell \left[(s(x)\varphi_x + 2\delta(x)v_x)_x - \alpha\vartheta_x - \beta q_x\right]\overline{v}\,\mathrm{d}x \\
&\quad - \int_0^\ell (kq_x + k\tau q_{tx} + \alpha v_x)\overline{\vartheta}\,\mathrm{d}x - \int_0^\ell (q + k\vartheta_x + \beta v_x)\overline{q}\,\mathrm{d}x \\
&= 2i\,\mathrm{Im}\int_0^\ell \left[s(x)v_x\overline{\varphi_x} - kq_x\overline{\vartheta} - \alpha\vartheta_x\overline{v} - \beta q_x\overline{v}\right]\mathrm{d}x \\
&\quad + k\int_0^\ell (q + k\vartheta_x + \beta v_x)_x\overline{\vartheta}\,\mathrm{d}x - 2\int_0^\ell \delta(x)|v_x|^2\,\mathrm{d}x - \int_0^\ell |q|^2\,\mathrm{d}x.
\end{aligned}$$

Taking the real part,

$$\begin{aligned}\operatorname{Re}\langle \mathcal{A}U, U\rangle &= \operatorname{Re}\left(k\int_0^\ell q\overline{\vartheta_x}\,\mathrm{d}x + k\beta\int_0^\ell v_x\overline{\vartheta_x}\,\mathrm{d}x\right) - k^2\int_0^\ell \vartheta_x^2\,\mathrm{d}x\\
&\quad -2\int_0^\ell \delta(x)|v_x|^2\,\mathrm{d}x - \int_0^\ell |q|^2\,\mathrm{d}x\\
&= -\int_0^\ell \operatorname{Re}\left[(k\overline{\vartheta_x})\cdot(q+\beta v_x)\right]\mathrm{d}x - k^2\int_0^\ell |\vartheta_x|^2\,\mathrm{d}x\\
&\quad -2\int_0^\ell \delta(x)|v_x|^2\,\mathrm{d}x - \int_0^\ell |q|^2\,\mathrm{d}x\\
&\le \frac{k^2}{2}\int_0^\ell |\vartheta_x|^2\,\mathrm{d}x + \int_0^\ell (q^2+\beta^2 v_x^2)\,\mathrm{d}x - k^2\int_0^\ell |\vartheta_x|^2\,\mathrm{d}x\\
&\quad -2\int_0^\ell \delta(x)|v_x|^2\,\mathrm{d}x - \int_0^\ell |q|^2\,\mathrm{d}x.\end{aligned}$$

Then

$$\operatorname{Re}\langle \mathcal{A}U, U\rangle \le -(2\delta_0-\beta^2)\int_0^\ell |v_x|^2\,\mathrm{d}x \le 0,$$

for $\beta \le \sqrt{2\delta_0}$, and we conclude that the operator $\mathcal{A}$ is dissipative. □

Lemma 1.2. *If $\varrho(\mathcal{A})$ is the resolvent set of the operator $\mathcal{A}$, then $0 \in \varrho(\mathcal{A})$.*

Proof. By definition, $D(\mathcal{A})$ is dense in $\mathcal{H}$. For a given $\mathbf{F} = (f_1, f_2, f_3, f_4)^T \in \mathcal{H}$, we must show that there is a unique $\mathbf{U} = (\varphi, v, \vartheta, q)^T \in \mathcal{D}(\mathcal{A})$ such that $\mathcal{A}\mathbf{U} = \mathbf{F}$, that means,

$$v = f_1 \quad \in \quad H_0^1(0,\ell), \tag{20}$$

$$(s(x)\varphi_x + 2\delta(x)v_x)_x - \alpha\vartheta_x - \beta q_x = \rho(x) f_2 \quad \in \quad L^2(0,\ell), \tag{21}$$

$$-\left[kq_x + \alpha v_x + k\tau q_{tx}\right] = f_3 \quad \in \quad L^2(0,\ell), \tag{22}$$

$$-\frac{1}{\tau}\left[q + k\vartheta_x + \beta v_x\right] = \tau f_4 \quad \in \quad L^2(0,\ell). \tag{23}$$

From (20) and (22) we deduce that there is a unique solution q of (22) such that $(q+\tau q_t)_x \in L^2(0,\ell)$ and then

$$q \in H^1(0,\ell). \tag{24}$$

By using similar procedure, from (20) and (24) we get $\vartheta_x \in L^2(0,\ell)$, and then, due to boundary conditions (5) we deduce that ϑ is a unique solution of (23) in $H_0^1(0,\ell)$.

Now, by using (20) and (21) we have a unique φ solution of (21) such that

$$(s(x)\varphi_x + 2\delta(x)v_x)_x = \rho(x)f_2 - \alpha\vartheta_x - \beta q_x \in L^2(0,\ell),$$

from where follows

$$s(x)\varphi_x + 2\delta(x)v_x \in H^1(0,\ell).$$

Thus, there exists a unique $\mathbf{U} = (\varphi, v, \vartheta, \phi)^T \in \mathcal{D}(\mathcal{A})$ such that $\mathcal{A}\mathbf{U} = F$. Moreover, It is easily established using the Poincaré inequality that there exists a positive constant c satisfying $\|\mathbf{U}\|_{\mathcal{H}} \le c\|F\|_{\mathcal{H}}$. This implies that $0 \in \varrho(\mathcal{A})$. □

The well-posedness is ensured by the following theorem.

Theorem 1.3. *For $U_0 \in \mathcal{H}$, there exists a unique weak solution $\mathbf{U} = (\varphi, v, \vartheta, \phi)^T$ of (18) satisfying*

$$\mathbf{U} \in C([0,\infty);\mathcal{H}). \tag{25}$$

Moreover, if $U_0 \in D(\mathcal{A})$, then

$$\mathbf{U} \in C([0,\infty);D(\mathcal{A})) \cap C^1([0,\infty);\mathcal{H}). \tag{26}$$

Proof. $D(\mathcal{A})$ dense in $\mathcal{H}$. By lemma 1.1, $\mathcal{A}$ is dissipative. By lemma 1.2 $0 \in \varrho(\mathcal{A})$. Using Lumer-Phillips theorem (see Theorem 1.2.2, page 3, in [17]) it follows that $\mathcal{A}$ generates a C_0-semigroup of contractions $S(t) = e^{t\mathcal{A}}$ on $\mathcal{H}$. From semigroup theory, $\mathbf{U}(t) = e^{t\mathcal{A}}U_0$ is the unique solution of (18) in the conditions (25) and (26). The proof is complete. □

2 Exponential stability

In this section, under the hypothesis $\beta \le \sqrt{2\delta_0}$, we will prove our main result.

2.1 The energy of the system

Given an initial data $U_0 \in D(\mathcal{A})$, with the solution $\mathbf{U} = (\varphi, v, \vartheta, \phi)^T$ that exists thanks to Theorem 1.3, we define

$$E(t) := \frac{1}{2}\left[\int_0^\ell s(x)|\varphi_x|^2\,\mathrm{d}x + \int_0^\ell \rho(x)|\varphi_t|^2\,\mathrm{d}x + \int_0^\ell |\vartheta|^2\,\mathrm{d}x + \tau\int_0^\ell |q|^2\,\mathrm{d}x\right].$$

as the energy functional $E : \mathbb{R}^+ \to \mathbb{R}^+$.

Differentiating with respect to t, applying the integration by parts formula to the first integral and grouping with the second integral, we get

$$\frac{d}{dt}E(t) = \int_0^\ell \left[\rho(x)\varphi_{tt} - \Big(s(x)\varphi_x\Big)_x\right]\varphi_t\,\mathrm{d}x + \int_0^\ell \vartheta\vartheta_t\,\mathrm{d}x + \tau\int_0^\ell qq_t\,\mathrm{d}x$$

Replacing equations (1)-(3), we obtain

$$\begin{aligned}\frac{d}{dt}E(t) = &\int_0^\ell 2\Big(\delta(x)\varphi_{xt}\Big)_x\varphi_t\,\mathrm{d}x - \alpha\int_0^\ell \vartheta_x\varphi_t\,\mathrm{d}x - \beta\int_0^\ell q_x\varphi_t\,\mathrm{d}x - k\int_0^\ell \vartheta q_x\,\mathrm{d}x\\ &- k\tau\int_0^\ell \vartheta q_{tx}\,\mathrm{d}x - \alpha\int_0^\ell \vartheta\varphi_{xt}\,\mathrm{d}x - \int_0^\ell |q|^2\,\mathrm{d}x - k\int_0^\ell q\vartheta_x\,\mathrm{d}x - \beta\int_0^\ell qu_{xt}\,\mathrm{d}x\end{aligned}$$

If we group the second with the sixth integral, the third with the ninth, and the fourth with the eighth, we see that after integrating by parts, the above formula reduces to

$$\frac{d}{dt}E(t) = -\int_0^\ell 2\delta(x)|\varphi_{xt}|^2\,\mathrm{d}x + k\tau\int_0^\ell \vartheta_x q_t\,\mathrm{d}x - \int_0^\ell |q|^2\,\mathrm{d}x.$$

Replacing (3), we get

$$\begin{aligned}\frac{d}{dt}E(t) = &-\int_0^\ell 2\delta(x)|\varphi_{xt}|^2\,\mathrm{d}x - \int_0^\ell |q|^2\,\mathrm{d}x - k^2\int_0^\ell |\vartheta_x|^2\,\mathrm{d}x\\ &- k\int_0^\ell \vartheta_x q\,\mathrm{d}x - k\beta\int_0^\ell \vartheta_x\varphi_{xt}\,\mathrm{d}x\end{aligned}$$

Applying the Cauchy-Schwarz inequality to the last two integrals, we get

$$\begin{aligned}\frac{d}{dt}E(t) \le &-\int_0^\ell 2\delta(x)|\varphi_{xt}|^2\,\mathrm{d}x - \int_0^\ell |q|^2\,\mathrm{d}x - k^2\int_0^\ell |\vartheta_x|^2\,\mathrm{d}x + \frac{k^2}{2}\int_0^\ell |\vartheta_x|^2\,\mathrm{d}x\\ &+ \frac{1}{2}\int_0^\ell |q|^2\,\mathrm{d}x + \frac{k^2}{4}\int_0^\ell |\vartheta_x|^2\,\mathrm{d}x + \beta^2\int_0^\ell |\varphi_{xt}|^2\,\mathrm{d}x,\end{aligned}$$

that is

$$\frac{d}{dt}E(t) \le -(2\delta_0 - \beta^2) \int_0^\ell |\varphi_{xt}|^2 \, \mathrm{d}x$$

Then, condition $\beta^2 \le 2\delta_0$, guarantees that

$$\frac{d}{dt}E(t) \le 0, \quad \text{for all } t > 0.$$

Since ρ, s, $\delta \in W^{1,\infty}(0,\ell)$, the Sobolev embedding $W^{1,\infty}(0,\ell) \subset C(0,\ell)$ implies that these functions are continuous on $[0,\ell]$. Moreover, φ, φ_t, φ_x, $\varphi_{xt} \in L^2(0,\ell)$, so their squares are integrable on $[0,\ell]$. Then, thanks to the Mean Value Theorem for integrals, there exist $\xi_i \in (0,\ell)$, $i = 1, \ldots, 5$, such that

$$\int_0^\ell \rho(x)|\varphi|^2 \, \mathrm{d}x = \rho(\xi_1) \int_0^\ell |\varphi|^2 \, \mathrm{d}x, \tag{27}$$

$$\int_0^\ell \rho(x)|\varphi_t|^2 \, \mathrm{d}x = \rho(\xi_2) \int_0^\ell |\varphi_t|^2 \, \mathrm{d}x, \tag{28}$$

$$\int_0^\ell s(x)|\varphi_x|^2 \, \mathrm{d}x = s(\xi_3) \int_0^\ell |\varphi_x|^2 \, \mathrm{d}x, \tag{29}$$

$$\int_0^\ell \delta(x)|\varphi_x|^2 \, \mathrm{d}x = \delta(\xi_4) \int_0^\ell |\varphi_x|^2 \, \mathrm{d}x, \tag{30}$$

$$\int_0^\ell \delta(x)|\varphi_{xt}|^2 \, \mathrm{d}x = \delta(\xi_5) \int_0^\ell |\varphi_{xt}|^2 \, \mathrm{d}x. \tag{31}$$

Here, we will use Cauchy-Schwarz inequality

$$\int_0^\ell fg \, \mathrm{d}x \le \frac{1}{2}\left(a\int_0^\ell |f|^2 \, \mathrm{d}x + \frac{1}{a}\int_0^l |g|^2 \, \mathrm{d}x\right), \quad \text{for all } f,g \in L^2(0,\ell),\ a > 0,$$

and Poincaré inequality

$$\int_0^\ell |f|^2 \, \mathrm{d}x \le \frac{L^2}{\pi^2}\int_0^\ell |f_x|^2 \, \mathrm{d}x, \quad \text{for all } f \in H_0^1(0,\ell).$$

Then, applying these inequalities and the identity (27), we obtain

$$\begin{aligned}\left|\int_0^\ell \rho(x)\varphi\varphi_t \, \mathrm{d}x\right| &\le \frac{1}{2}\left(a\int_0^\ell \rho(x)|\varphi|^2 \, \mathrm{d}x + \frac{1}{a}\int_0^\ell \rho(x)|\varphi_t|^2 \, \mathrm{d}x\right) \\ &= \frac{1}{2}\left(a\rho(\xi_1)\int_0^\ell |\varphi|^2 \, \mathrm{d}x + \frac{1}{a}\rho(\xi_2)\int_0^\ell |\varphi_t|^2 \, \mathrm{d}x\right)\end{aligned}$$

from where follows

$$\left|\int_0^\ell \rho(x)\varphi\varphi_t \, \mathrm{d}x\right| \le \frac{1}{2}\left(a\rho(\xi_1)\frac{L^2}{\pi^2}\int_0^\ell |\varphi_x|^2 \, \mathrm{d}x + \frac{1}{a}\rho(\xi_2)\frac{L^2}{\pi^2}\int_0^\ell |\varphi_{tx}|^2 \, \mathrm{d}x\right) \tag{32}$$

and

$$\left|\int_0^\ell \rho(x)\varphi\varphi_t \, \mathrm{d}x\right| \le \frac{1}{2}\left(a\rho(\xi_1)\frac{L^2}{\pi^2}\int_0^\ell |\varphi_x|^2 \, \mathrm{d}x + \frac{1}{a}\rho(\xi_2)\int_0^\ell |\varphi_t|^2 \, \mathrm{d}x\right) \le \lambda_0 E(t) \tag{33}$$

for some λ_0.

By similar way for some $a_2,\ a_3 > 0$, we get

$$\left| \int_0^\ell \vartheta_x (q + \beta\varphi_{xt})\, \mathrm{d}x \right| \le \frac{1}{2} \left(a_2 \int_0^\ell |\vartheta_x|^2\, \mathrm{d}x + \frac{2}{a_2} \int_0^\ell \left(|q|^2 + \beta^2 |\varphi_{xt}|^2 \right) \mathrm{d}x \right) \tag{34}$$

and

$$\left| \int_0^\ell \varphi_x (\alpha\vartheta + \beta q)\, \mathrm{d}x \right| \le \frac{1}{2} \left(a_3 \int_0^\ell |\vartheta_x|^2\, \mathrm{d}x + \frac{2}{a_3} \int_0^\ell \left(\alpha^2 |\vartheta|^2 + \beta^2 |q|^2 \right) \mathrm{d}x \right). \tag{35}$$

In view of (30), we deduce that

$$\int_0^\ell \delta(x) |\varphi_x|^2\, \mathrm{d}x \le \lambda_1 E(t), \text{ for some } \lambda_1. \tag{36}$$

Introducing the function

$$G(t) = \int_0^\ell \rho(x) \varphi\varphi_t\, \mathrm{d}x + \int_0^\ell \delta(x) |\varphi_x|^2\, \mathrm{d}x \tag{37}$$

and by applying the estimates (33) and (36) into (37), we obtain

$$-\lambda_0 E(t) \le G(t) \le (\lambda_0 + \lambda_1) E(t),\ \ t \ge 0. \tag{38}$$

Differentiating $G(t)$, using equation (1) and replacing the boundary conditions, we get

$$\begin{aligned}
\frac{d}{dt} G(t) &= \int_0^\ell \rho(x) |\varphi_t|^2\, \mathrm{d}x + \int_0^\ell \varphi \left[\left(s(x)\varphi_x + 2\delta(x)\varphi_{xt} \right)_x - \alpha\vartheta_x - \beta q_x \right] \mathrm{d}x \\
&\quad + \int_0^\ell 2\delta(x) \varphi_{xt}\varphi_x\, \mathrm{d}x \\
&= \int_0^\ell \rho(x) |\varphi_t|^2\, \mathrm{d}x - \int_0^\ell s(x) |\varphi_x|^2\, \mathrm{d}x - \int_0^\ell 2\delta(x)\varphi_{xt}\varphi_x\, \mathrm{d}x \\
&\quad + \int_0^\ell \varphi_x (\alpha\vartheta_x + \beta q_x)\, \mathrm{d}x + \int_0^\ell 2\delta(x)\varphi_{xt}\varphi_x\, \mathrm{d}x
\end{aligned}$$

and then,

$$\frac{d}{dt} G(t) = \int_0^\ell \rho(x) |\varphi_t|^2\, \mathrm{d}x - \int_0^\ell s(x) |\varphi_x|^2\, \mathrm{d}x + \int_0^\ell \varphi_x (\alpha\vartheta + \beta q)\, \mathrm{d}x.$$

2.2 A direct approach to exponential stability

In this section, we establish the main result.

Theorem 2.1. *Under the assumption $\beta \le \sqrt{2\delta_0}$, the system (1)-(3) is exponentially stable. That means, there exist some positive constants M and w such that*

$$E(t) \le M E(0) e^{-wt}, \ \ \textit{for all}\ \ t > 0.$$

Proof. We introduce a Lyapunov functional V

$$V(t) = E(t) + \epsilon G(t),\ \ t > 0, \tag{39}$$

$\epsilon > 0$ being a small constant to be fixed later.
Combining (38) and (39), we get

$$(1 - \lambda_0 \epsilon) E(t) \le V(t) \le (1 + (\lambda_0 + \lambda_1)\epsilon) E(t).$$

Choosing $0 < \epsilon < \dfrac{1}{\lambda_0}$ so that $V(t) \geq 0$ for all $t > 0$, we have that there exist positive constants C_1, C_2 such that

$$C_1 E(t) \leq V(t) \leq C_2 E(t). \tag{40}$$

Differentiating (39), we obtain

$$\begin{aligned}
\frac{d}{dt}V(t)+2\mu\epsilon V(t) &= \frac{d}{dt}E(t) + \epsilon\frac{d}{dt}G(t) + 2\mu\epsilon[E(t) + \epsilon G(t)] \\
&= -2\int_0^\ell \delta(x)|\varphi_{xt}|^2\,\mathrm{d}x - \int_0^\ell |q|^2\,\mathrm{d}x - k\int_0^\ell |\vartheta_x|^2\,\mathrm{d}x - k\int_0^\ell \vartheta_x(q+\beta\varphi_{xt})\,\mathrm{d}x \\
&\quad + \epsilon\left[\int_0^\ell \rho(x)|\varphi_t|^2\,\mathrm{d}x - \int_0^\ell s(x)|\varphi_x|^2\,\mathrm{d}x + \int_0^\ell \varphi_x(\alpha\vartheta+\beta q)\,\mathrm{d}x\right] \\
&\quad + \mu\epsilon\left[\int_0^\ell s(x)|\varphi_x|^2\,\mathrm{d}x + \int_0^\ell \rho(x)|\varphi_t|^2\,\mathrm{d}x + \int_0^\ell |\vartheta|^2\,\mathrm{d}x + \tau\int_0^\ell |q|^2\,\mathrm{d}x\right] \\
&\quad + 2\mu\epsilon^2\left[\int_0^\ell \rho(x)\varphi\varphi_t\,\mathrm{d}x + \int_0^\ell \delta(x)|\varphi_x|^2\,\mathrm{d}x\right].
\end{aligned}$$

By using (28), (29), (30), (32), (34), (35), (37), we obtain

$$\begin{aligned}
\frac{d}{dt}V(t)+2\mu\epsilon V(t) \leq\ & A\int_0^\ell |\vartheta_x|^2\,\mathrm{d}x + B\int_0^\ell |q|^2\,\mathrm{d}x \\
& + C\int_0^\ell \delta(x)|\varphi_{xt}|^2\,\mathrm{d}x + D\int_0^\ell s(x)|\varphi_x|^2\,\mathrm{d}x
\end{aligned}$$

where

$$\begin{aligned}
A &= \left[-k + \frac{ka_2}{2} + \epsilon\frac{\alpha^2 L^2}{\pi^2} + \epsilon\frac{\mu L^2}{\pi^2}\right], \\
B &= \left[-1 + \frac{k}{a_2} + \epsilon\frac{\beta^2}{a_3} + \epsilon\mu\tau\right], \\
C &= \left[-2 + \frac{k\beta^2}{\alpha_2\delta(\xi_5)} + \epsilon\frac{\rho(\xi_2)}{\delta(\xi_5)} + \epsilon\frac{\mu\rho(\xi_2)}{\delta(\xi_5)} + \epsilon^2\frac{\mu L^2}{\pi^2}\frac{\rho(\xi_2)}{a_1\delta(\xi_5)}\right]
\end{aligned}$$

and

$$D = \epsilon\left[-(1-\mu) + \epsilon\frac{a_3}{2\sqrt{s(\xi_3)}} + \epsilon\frac{\mu a_1 L^2}{\pi_2}\frac{\rho(\xi_1)}{s(\xi_3)} + \epsilon\frac{\mu\delta(\xi_4)}{s(\xi_3)}\right].$$

Now, choosing $k < a_2 < 2$, $\dfrac{k\beta^2}{2\delta(\xi_1)} < \alpha_2$ and taking $\epsilon < \dfrac{1}{\lambda_0}$, $\mu < 1$, $\beta < \sqrt{2\delta_0}$, then there exists some $\epsilon_0 > 0$ for which $A \leq 0$, $B \leq 0$, $C \leq 0$, $D \leq 0$. For a fixed $epsilon < \epsilon_0$, we get

$$\frac{d}{dt}V(t) + 2\mu\epsilon V(t) \leq 0, \quad \text{for all } \ t > 0.$$

Integration over $[0, t]$ leads

$$e^{2\mu\epsilon}V(t) \leq V(0),$$

from where follows, by the equivalence (40), that

$$E(t) \leq ME(0)e^{-wt}, \quad \text{for all } \ t > 0$$

with

$$M = \frac{1 + (\lambda_0 + \lambda_1)\epsilon}{1 - \lambda_0\epsilon}, \quad \text{and } \ w := 2\mu\epsilon.$$

This completes the proof.

□

3 Conclusions

This research is mainly concerned with the mathematical stability for the vibrations of a flexible structure with thermo-diffusion effect after a slight modification of Cattaneo-Vernotte's law. By showing the well-posedness of the problem with semigroup theory, we finally achieve the uniform stability of the system with a specific form of the exponential energy decay estimate $w(= 2\mu\epsilon)$. Theorem 2.1 shows that the solution of the system decays exponentially. The maximum acceptable value ϵ will certainly give the highest energy decay rate w leading to rapid stabilization of the system. Hence, the thermo-viscoelastic system considered here is in particular controllable to rest from an arbitrary initial state.

4 Declarations

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The author(s) declare no competing interests.

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Not applicable.

Authors' Contributions

All authors contributed equally. All the authors read and approved the final manuscript.

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