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Weak Mean Random Attractor of Reversible Selkov Lattice Systems Driven by Locally Lipschitz Lévy Noises

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Abstract

This work, operated across set $\mathbb{Z}$ and influenced by locally Lipschitz Lévy noises, addresses weak pullback mean random attractor of reversible Selkov lattice systems. Firstly, we express the stochastic lattice equations as an abstract system defined in the space $\ell^2 \times \ell^2$ of square-summable sequences. Secondly, we demonstrate the global well-posedness of the systems with locally Lipschitz diffusion terms. Under certain conditions, we demonstrate that the long-term behavior can be described by a mean random attractor, which is weakly compact and weakly attractive in the Bochner space $L^2(\Omega, \ell^2 \times \ell^2)$. To surmount the difficulty caused by nonlinear terms, we utilize a stopping time approach to establish the convergence of solutions in terms of probability. The mean random dynamical systems theory proposed by Wang (DOI: 10.1007/s10884-018-9696-5) is used to overcome the difficulty caused by the nonlinear noise.

Keywords: Selkov systems, Weak pullback mean random attractor, Lévy noises
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1 Introduction

1.1 Statement of problems

It is well known that random processes are primarily propelled by Brownian motion that is continuous. But numerous systems could experience unexpected alteration, which cause a discontinuity in their sample trajectories. To account for such phenomena, Lévy noise is incorporated into models to describe random phenomena that include jumps. For more information about Levy processes, see [1, 2]. Stochastic systems with Lévy noises are discussed in [3, 4], and the α-stable Lévy noise is presented in [5]. Inspired by the above articles, we

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investigate the weak pullback mean random attractor of reversible Selkov lattice system driven by locally Lipschitz Lévy noises defined on the integer set $\mathbb{Z}$ for $t > \lambda$:

$$\begin{cases} d\xi_i(t) = & (d_1(\xi_{i+1}(t) - 2\xi_i + \xi_{i-1}(t)) - a_1\xi_i(t) + b_1\xi_i^{2p}(t)\psi_i(t) \\ & - b_2\xi_i^{2p+1}(t) + f_{1i}(t))dt \\ & + \epsilon_1 \sum_{k=1}^{\infty}(h_{k,i}(t) + \delta_{k,i}\tilde{\sigma}(\xi_i(t)))dW_k(t) \\ & + \epsilon_2 \sum_{k=1}^{\infty} \int_{|z_k|<1} (\kappa_{k,i}(t) + q_{k,i}(\xi_i(t-), z_k))\tilde{L}_k(dt, dz_k) \\ d\psi_i(t) = & (d_2(\psi_{i+1}(t) - 2\psi_i(t) + \psi_{i-1}(t)) - a_2\psi_i(t) - b_1\xi_i^{2p}(t)\psi_i(t) \\ & + b_2\xi_i^{2p+1}(t) + f_{2i})dt \\ & + \gamma_1 \sum_{k=1}^{\infty}(h_{k,i}(t) + \delta_{k,i}\tilde{\sigma}(\psi_i(t)))dW_k(t) \\ & + \gamma_2 \sum_{k=1}^{\infty} \int_{|z_k|<1} (\kappa_{k,i}(t) + q_{k,i}(\psi_i(t-), z_k))\tilde{L}_k(dt, dz_k) \end{cases} \quad (1)$$

with initial conditions

$$\xi_i(\lambda) = \xi_{\lambda,i}, \quad \psi_i(\lambda) = \psi_{\lambda,i}, \quad (2)$$

where $\lambda \in \mathbb{R}$, $i \in \mathbb{Z}$, $\xi = (\xi_i)_{i\in\mathbb{Z}}$, $\psi = (\psi_i)_{i\in\mathbb{Z}} \in \ell^2$, ϵ_1, ϵ_2, γ_1, $\gamma_2 > 0$, $p \geq 1$. d_1, d_2, a_1, a_2, b_1, b_2 are positive constants, $f_1(t) = (f_{1i}(t))_{i\in\mathbb{Z}}$ and $f_2(t) = (f_{2i}(t))_{i\in\mathbb{Z}} \in \ell^2$ are time independent random sequences. $\delta = (\delta_{k,i}(t))_{k\in\mathbb{N},i\in\mathbb{Z}}$ is a sequence of real numbers satisfying $\|\delta\|^2 := \sum_{k\in\mathbb{N},i\in\mathbb{Z}} |\delta_{k,i}|^2 < \infty$, $h(t) = (h_{k,i}(t))_{k\in\mathbb{N},i\in\mathbb{Z}}$, $\kappa(t) = (\kappa_{k,i}(t))_{k\in\mathbb{N},i\in\mathbb{Z}} \in \ell^2$-valued progressively measurable processes. The sequence $\tilde{\sigma}_k$ satisfies $|\tilde{\sigma}_k(s)| \leq \beta(1 + |s|)$, where $\beta > 0$. Let $(W_k)_{k\in\mathbb{N}}$ and $(Z_k)_{k\in\mathbb{N}}$ be two sequences of *mutually* independent *two-sided* real-valued Wiener and Lévy processes defined on a complete filtered probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_\lambda\}_{\lambda\in\mathbb{R}}, \mathbb{P})$, respectively. For every $t \in \mathbb{R}$ and $\mathfrak{A} \in \mathcal{B}(\mathbb{R} - \{0\})$, we define a *two-sided* Poisson counting random measure $L_k(\cdot, \cdot)$ associated with Z_k by

$$L_k(t, \mathfrak{A}) := \begin{cases} \pounds\{0 \leq s \leq t : \Delta Z_k(s) \in \mathfrak{A}\} & \text{if } t \geq 0, \\ \pounds\{t \leq s < 0 : \Delta Z_k(s) \in \mathfrak{A}\} & \text{if } t < 0, \end{cases}$$

where $\pounds$ denotes the cardinality of a set, $\Delta Z_k(s) := Z_k(s) - Z_k(s-)$ is a jump process, and $Z_k(s-)$ is the left-limit of $Z_k(s)$ at s. Define a Borel measure ν_k on $\mathcal{B}(\mathbb{R} - \{0\})$ by $\nu_k(\cdot) := \mathbb{E}[L_k(1, \cdot)]$. Then ν_k is a Lévy measure with $\int_{\mathbb{R}-\{0\}}(|z|^2 \wedge 1)\nu_k(dz) < \infty$, and is also called as an intensity measure associated with Z_k. Define $\tilde{L}_k(dt, dz) := L_k(dt, dz) - \nu_k(dz)dt$ for $t \in \mathbb{R}$. Then $\tilde{L}_k(dt, dz)$ is called a *two-sided* compensated Possion random measure with the compensator $\nu_k(dz)dt$. The pair $((W_k)_{k\in\mathbb{N}}, (L_k)_{k\in\mathbb{N}})$ is called a family of *two-sided Lévy* noises.

1.2 Literature survey.

This equation originates from biology and can be divided into two situations: reversible and irreversible. Selkov was the first to propose the irreversible scenario and simulated two irreversible reaction processes, see [6]. Conversely, the latter pattern is characterized by cubic self catalysis, continuous material supply, and isothermal conditions, see [7]. The reversible system is also called the dual-element Gray-Scott formulas (see [8, 9, 10]) which represents the Brussels Schools led by distinguished Ilya Prigogin. The literature extensively addresses the mathematical analysis of reversible Selkov equations. Deterministic reversible Selkov equations have been thoroughly investigated regarding long-term behavior of dynamical systems (attractor) by [11, 12]. Many researchers have examined the existence and robustness of random attractors for stochastic reversible Selkov equations, see [13, 14, 15, 16]. The random lattice system for Selkov was studied in [17].

For the past few years, the widespread application of lattice systems, such as in pattern formation, neural pulse propagation, and circuits, has drawn considerable attention from many researchers, for more detailed information, please refer to [18, 19, 20]. When $\epsilon_2 = 0$, $\gamma_2 = 0$, the reversible Selkov lattice systems driven by white noises have been discussed by Wang, see [21]. Transforming a system (1)-(2) with nonlinear noise interference into a deterministic reaction-diffusion equation ($\epsilon_1 = \epsilon_2 = \gamma_1 = \gamma_2 = 0$). There are numerous works about lattice systems, such as traveling wave solutions [22, 23, 24, 25], chaotic solutions [26, 27], and global attractors [28, 29]. More recently, the study of stochastic attractors in random lattice systems has been undertaken in [30, 31]. For SPDEs, articles on non-autonomous systems and autonomous systems, mainly discussed random attractors, can be referred to the following references, see, e.g., [32, 33, 34, 35, 36, 37]. Lattice systems, driven by white noise (linear, nonlinear, superlinear), have been discussed in [38, 39, 40].

1.3 Novelties, difficulties, the main features of the paper

As we are aware, the systems driven by Lévy noises in (1)-(2) are nonlinear, one fundamental yet highly limiting requirement for examining the almost sure dynamics of solutions to SPDEs, transforms it into pathwise formulations using an Ornstein-Uhlenbeck process, see [41, 42, 43, 44, 45, 46, 47, 48, 49]. However, there is no effective method to transform the Itô stochastic differential equation with nonlinear noise into the pathwise stochastic differential equation. To figure out the problem, we utilize the framework of weak mean random attractors [39, 50] to analyze the dynamical systems. This work aims to demonstrate both the presence and the uniqueness of weak pullback mean random attractors in reversible Selkov lattice systems affected by locally Lipschitz Lévy noises.

To establish the existence and uniqueness of mean random attractors, we have spent a lot of time on proving the existence and uniqueness of the solution, which is a crucial step in demonstrating the existence of attractors. Additionally, careful handling of the drift and diffusion terms is necessary, particularly within the Itô energy equations. Given the arbitrary exponent of the coupling term, new estimates will be developed to address it, as discussed in Lemma 3.2, Lemma 3.3, and Theorem 3.5. A stopping time approach will be utilized to manage the diffusion and drift terms effectively.

Once these challenges are addressed, we will be able to demonstrate the existence and uniqueness of both the solution and the mean random attractors for the problem defined in (1)-(2).

2 Preliminaries

This section provides some essential preliminaries and inequalities that will be frequently referenced in subsequent parts of the paper. Let ℓ^r be defined as $\{\xi = (\xi_i)_{i\in\mathbb{Z}} : \sum_{i\in\mathbb{Z}} |\xi_i|^r < +\infty\}$ for any $r \geq 1$. The norm in ℓ^r is indicated by $\|\cdot\|_r$. For ℓ^2, the norm and the inner product are expressed as $|\cdot|$ and $\langle\cdot,\cdot\rangle$, respectively.

Define $A, B : \ell^2 \to \ell^2$ by

$$(A\xi)_i = -\xi_{i-1} + 2\xi_i - \xi_{i+1} \quad \text{and} \quad (B\xi)_i = \xi_{i+1} - \xi_i.$$

Then we have

$$\langle A\xi, \psi\rangle = \langle B\xi, B\psi\rangle \quad \text{and} \quad \langle A\xi, \xi\rangle = \|B\psi\|^2 \geq 0 \quad \xi, \psi \in \ell^2.$$

Furthermore, for every $k \in \mathbb{N}$, define the following operators Γ, Φ, σ_k, q_k:

$$\sigma_k(\xi) = (\delta_{k,i}\tilde{\sigma}_k(\xi_i))_{i\in\mathbb{Z}}, \quad q_k(\xi, z_k) = (q_{k,i}(\xi_i, z_k))_{i\in\mathbb{Z}}, \quad \text{for all} \quad \xi \in \ell^2,$$

$$\Gamma(\xi, \psi) = (\xi_i^{2p}\psi_i)_{i\in\mathbb{Z}}, \quad \Phi(\xi) = (\xi_i^{2p+1})_{i\in\mathbb{Z}}, \quad \text{for} \quad \xi, \psi \in \ell^2.$$

The following inequalities will be utilized,

$$|\rho_1^r - \rho_2^r| \le C_r|\rho_1 - \rho_2||\rho_1^{r-1} + \rho_2^{r-1}|, \ \forall \rho_1, \rho_2 \in \mathbb{R}, \ p \ge 1,$$

$$\begin{aligned} &b_1 b_2 \rho_1^{2p+1}\rho_2 - b_2^2\rho_1^{2p+2} - b_1^2\rho_1^{2p}\rho_2^2 + b_2 b_1 \rho_1^{2p+1}\rho_2 \\ &= 2b_1 b_2 \rho_1^{2p+1}\rho_2 - \rho_1^{2p}(b_2^2\rho_1^2 + b_1^2\rho_2^2) \le 0, \ \forall \rho_1, \rho_2 \in \mathbb{R}, \ p \ge 1. \end{aligned} \tag{3}$$

According to (3) and Young's inequality, we can get

$$\begin{aligned} &\|\Gamma(\xi_1, \psi_1) - \Gamma(\xi_2, \psi_2)\|^2 \le C_1(n)\big(\|\xi_1 - \xi_2\|^2 + \|\psi_1 - \psi_2\|^2\big), \\ &|\langle \Gamma(\xi_1, \psi_1) - \Gamma(\xi_2, \psi_2), \xi_1 - \xi_2\rangle| \le C_1(n)\big(\|\xi_1 - \xi_2\|^2 + \|\psi_1 - \psi_2\|^2\big), \\ &|\langle \Gamma(\xi_1, \psi_1) - \Gamma(\xi_2, \psi_2), \psi_1 - \psi_2\rangle| \le C_1(n)\big(\|\xi_1 - \xi_2\|^2 + \|\psi_1 - \psi_2\|^2\big), \end{aligned} \tag{4}$$

and

$$\begin{aligned} &\|\Phi(\xi) - \Phi(\psi)\|^2 \le C_2(n)\|\xi - \psi\|^2, \\ &|\langle \Phi(\xi_1) - \Phi(\xi_2), \xi_1 - \xi_2\rangle| \le C_2(n)\|\xi_1 - \xi_2\|^2, \\ &|\langle \Phi(\xi_1) - \Phi(\xi_2), \psi_1 - \psi_2\rangle| \le C_2(n)\big(\|\xi_1 - \xi_2\|^2 + \|\psi_1 - \psi_2\|^2\big), \end{aligned}$$

where ξ, ψ, ξ_1, ξ_2, ψ_1, $\psi_2 \in \ell^2$ and $\|\xi\| \le n$, $\|\xi_1\| \le n$, $\|\xi_2\| \le n$, $\|\psi\| \le n$, $\|\psi_1\| \le n$, $\|\psi_2\| \le n$. For any $k \in \mathbb{N}$, there exist $\beta > 0$, $C_k(n) > 0$ such that

$$\sum_{k=1}^{\infty} \|\sigma_k(\xi_1) - \sigma_k(\xi_2)\|^2 \vee \sum_{k=1}^{\infty} \int_{|z_k|<1} \|q_k(\xi_1, z_k) - q_k(\xi_2, z_k)\|^2 v_k(dz_k) \le C_k(n)\|\xi_1 - \xi_2\|^2, \tag{5}$$

$$\sum_{k=1}^{\infty} \|\sigma_k(\xi)\|^2 \vee \sum_{k=1}^{\infty} \int_{|z_k|<1} \|q_k(\xi, z_k)\|^2 v_k(dz_k) \le 2\beta^2\|\delta\|^2(1 + \|\xi\|^2), \tag{6}$$

where $\|\delta\|^2 := \sum\limits_{k\in\mathbb{N}}\sum\limits_{i\in\mathbb{Z}} |\delta_{k,i}|^2 < \infty$. In addition,

$$\int_{\lambda}^{\lambda+T} \mathbb{E}\left(\|f_1(t)\|^2 + \|f_2(t)\|^2 + \sum_{k=1}^{\infty}\|\kappa_k(t)\|^2 + \sum_{k=1}^{\infty}\|h_k(t)\|^2\right) dt < \infty, \ \ \forall \lambda \in \mathbb{R}, \ T > 0. \tag{7}$$

To simplify the notation, we adopt an abstract framework to represent the system (1)-(2) within the space $\ell^2 \times \ell^2$ for $t \ge \lambda \in \mathbb{R}$:

$$\begin{cases} d\xi(t) = \big(- d_1 A\xi(t) - a_1\xi(t) + b_1\Gamma(\xi(t), \psi(t)) - b_2\Phi(\xi(t)) + f_1(t)\big)dt \\ \quad + \epsilon_1 \sum\limits_{k=1}^{\infty} \big(h_k(t) + \sigma_k(\xi(t))\big)dW_k(t) + \epsilon_2 \sum\limits_{k=1}^{\infty} \big(\kappa_k(t) + q_k(\xi(t-), z_k)\big)\tilde{L}_k(dt, dz_k), \\ d\psi(t) = \big(- d_2 A\psi(t) - a_2\psi(t) - b_1\Gamma(\xi(t), \psi(t)) + b_2\Phi(x(t)) + f_2(t)\big)dt \\ \quad + \gamma_1 \sum\limits_{k=1}^{\infty} \big(h_k(t) + \sigma_k(\psi(t))\big)dW_k(t) + \gamma_2 \sum\limits_{k=1}^{\infty} \big(\kappa_k(t) + q_k(\psi(t-), z_k)\big)\tilde{L}_k(dt, dz_k), \end{cases} \tag{8}$$

with initial condition

$$\xi(\lambda) = \xi_\lambda, \quad \psi(\lambda) = \psi_\lambda. \tag{9}$$

We will establish the existence and uniqueness of solutions for the system (8)-(9), where $\varphi(\lambda) = (\xi_\lambda, \psi_\lambda) \in L^2(\Omega, \ell^2 \times \ell^2)$, as defined below.

3 Existence and uniqueness of solutions

Definition 3.1. Let $\lambda \in \mathbb{R}$, $T > 0$ and $\mathcal{F}_\lambda$-measurable initial condition $\varphi(\lambda) = (\xi_\lambda, \psi_\lambda) \in L^2(\Omega, \ell^2 \times \ell^2)$, an $\ell^2 \times \ell^2$-valued *càdlàg* process $\varphi(t) = (\xi(t), \psi(t))$ is called a solution of system (8)-(9) if for any $t > \lambda$ and for almost surely $\omega \in \Omega$,

$$\begin{cases} \xi(t) = \xi_\lambda + \int_\lambda^t \big(- d_1 A\xi(s) - a_1\xi(s) + b_1\Gamma(\xi(s), \psi(s)) - b_2\Phi(\xi(s)) + f_1(s)\big) ds \\ + \epsilon_1 \sum_{k=1}^{\infty} \int_\lambda^t \big(h_k(s) + \sigma_k(\xi(s))\big) dW_k(s) + \epsilon_2 \sum_{k=1}^{\infty} \int_\lambda^t \int_{|z_k|<1} \big(\kappa_k(s) + q_k(\xi(s-), z_k)\big) \tilde{L}_k(ds, dz_k), \\ \psi(t) = \psi_\lambda + \int_\lambda^t \big(- d_2 A\psi(s) - a_2\psi(s) - b_1\Gamma(\xi(s), \psi(s)) + b_2\Phi(\xi(s)) + f_2(s)\big) ds \\ + \gamma_1 \sum_{k=1}^{\infty} \int_\lambda^t \big(h_k(s) + \sigma_k(\psi(s))\big) dW_k(s) + \gamma_2 \sum_{k=1}^{\infty} \int_\lambda^t \int_{|z_k|<1} \big(\kappa_k(s) + q_k(\psi(s-), z_k)\big) \tilde{L}_k(ds, dz_k). \end{cases}$$

It's important to note that the nonlinear terms Γ, Φ, σ_k, and q_k exhibit only local Lipschitz continuity in $\ell^2 \times \ell^2$. To establish the global existence of the solution to (8)-(9) as outlined in Definition 2.1, it is necessary to approximate these nonlinear functions with functions that are globally Lipschitz. Hence, for each $n \in \mathbb{N}$, we introduce a function $\wp_n : \mathbb{R} \to \mathbb{R}$ defined by

$$\wp_n(s) = \begin{cases} -n, & \text{for } s \in (-\infty, -n), \\ s, & \text{for } s \in [-n, n], \\ n, & \text{for } s \in (n, +\infty). \end{cases}$$

Then we can get

$$|\wp_n(s_1) - \wp_n(s_2)| \le |s_1 - s_2|, \ \ |\wp_n(s)| \le n, \ \ s, s_1, s_2 \in \mathbb{R}.$$

For any $n \in \mathbb{N}$ and $\xi = (\xi_i)_{i\in\mathbb{Z}} \in \ell^2$, $\psi = (\psi_i)_{i\in\mathbb{Z}} \in \ell^2$, let $\Gamma^n(\xi, \psi) = (\wp_n(\xi_i))^{2p}\wp_n(\psi_i)_{i\in\mathbb{Z}}$, $\Phi^n(\xi) = (\wp_n(\xi_i))^{2p+1}$, $\sigma_k^n(u) = (\delta_{k,i}\tilde{\sigma}_k(\wp_n(\xi_i)))_{i\in\mathbb{Z}}$, $q_k^n(\xi, z_k) = q_k(\wp_n(\xi_i), z_k)$. In conclusion, it is easy to check that Γ^n, Φ^n, σ_k^n, q_k^n satisfy (4)-(6) and the global Lipschitz condition. For each $n > 0$, we consider the following stochastic system in $\ell^2 \times \ell^2$:

$$\begin{cases} d\xi_n(t) = \big(- d_1 A\xi_n(t) - a_1\xi_n(t) + b_1\Gamma^n(\xi_n(t), \psi_n(t)) - b_2\Phi^n(\xi_n(t)) + f_1(t)\big) dt \\ + \epsilon_1 \sum_{k=1}^{\infty} \big(h_k(t) + \sigma_k^n(\xi_n(t))\big) dW_k(t) + \epsilon_2 \sum_{k=1}^{\infty} \big(\kappa_k(t) + q_k^n(\xi_n(t-), z_k)\big) \tilde{L}_k(dt, dz_k), \\ d\psi_n(t) = \big(- d_2 A\psi_n(t) - a_2\psi_n(t) - b_1\Gamma^n(\xi_n(t), \psi_n(t)) + b_2\Phi^n(\xi_n(t)) + f_2(t)\big) dt \\ + \gamma_1 \sum_{k=1}^{\infty} \big(h_k(t) + \sigma_k^n(\psi_n(t))\big) dW_k(t) + \gamma_2 \sum_{k=1}^{\infty} \big(\kappa_k(t) + q_k^n(\psi_n(t-), z_k)\big) \tilde{L}_k(dt, dz_k), \end{cases} \tag{10}$$

with initial condition

$$\xi(\lambda) = \xi_\lambda, \quad \psi(\lambda) = \psi_\lambda.$$

In the following text, we will examine the asymptotic behavior of the sequence $\{\varphi_n\}_{n=1}^\infty$, which consists of the solutions to (8)-(9) as $n \to \infty$. For each $n \in \mathbb{N}$, we define a stopping time ζ_n by

$$\begin{aligned} &\zeta_n = \inf\{t \ge 0 : \|\varphi_n(t)\| > n\}, \\ &\zeta_n = +\infty, \text{ if } \{t \ge 0 : \|\varphi_n(t)\| > n\} = \emptyset. \end{aligned} \tag{11}$$

Then $\{\varphi_n^{\zeta_n}\}_{n=1}^\infty$ is consistent in the following described manner.

Lemma 3.2. *$\varphi_{n+1}(t\wedge\zeta_n)=\varphi_n(t\wedge\zeta_n)$, $\zeta_{n+1}\geq\zeta_n$, $\mathbb{P}$-a.s., where $n\in\mathbb{N}$.*

Proof. By (10), we get

$$
\begin{aligned}
&\xi_{n+1}(t\wedge\zeta_n)-\xi_n(t\wedge\zeta_n)\\
=&\int_\lambda^{t\wedge\zeta_n}\Big(-d_1A(\xi_{n+1}(s)-\xi_n(s))-a_1(\xi_{n+1}(s)-\xi_n(s))\Big)ds\\
&-\int_\lambda^{t\wedge\zeta_n}\Big(b_2(\Phi^{n+1}(\xi_{n+1}(s))-\Phi^n(\xi_n(s)))-b_1(\Gamma^{n+1}(\xi_{n+1}(s),\psi_{n+1}(s))-\Gamma^n(\xi_n(s),\psi_n(s)))\Big)\,ds\\
&+\epsilon_1\sum_{k=1}^\infty\int_\lambda^{t\wedge\zeta_n}\Big(\sigma_k^{n+1}(\xi_{n+1}(s))-\sigma_k^n(\xi_n(s))\Big)\,dW_k(s)\\
&+\epsilon_2\sum_{k=1}^\infty\int_\lambda^{t\wedge\zeta_n}\int_{|z_k|<1}\Big(q_k^{n+1}(\xi_{n+1}(s-),z_k)-q_k^n(\xi_n(s-),z_k)\Big)\tilde{L}_k(ds,dz_k),
\end{aligned}
\tag{12}
$$

and

$$
\begin{aligned}
&\psi_{n+1}(t\wedge\zeta_n)-\psi_n(t\wedge\zeta_n)\\
=&\int_\lambda^{t\wedge\zeta_n}\Big(-d_2A(\psi_{n+1}(s)-\psi_n(s))-a_2(\psi_{n+1}(s)-\psi_n(s))\Big)ds\\
&-\int_\lambda^{t\wedge\zeta_n}\Big(-b_2(\Phi^{n+1}(\xi_{n+1}(s))-\Phi^n(\xi_n(s)))+b_1(\Gamma^{n+1}(\xi_{n+1}(s),\psi_{n+1}(s))-\Gamma^n(\xi_n(s),\psi_n(s)))\Big)\,ds\\
&+\gamma_1\sum_{k=1}^\infty\int_\lambda^{t\wedge\zeta_n}\Big(\sigma_k^{n+1}(\psi_{n+1}(s))-\sigma_k^n(\psi_n(s))\Big)\,dW_k(s)\\
&+\gamma_2\sum_{k=1}^\infty\int_\lambda^{t\wedge\zeta_n}\int_{|z_k|<1}\Big(q_k^{n+1}(\psi_{n+1}(s-),z_k)-q_k^n(\psi_n(s-),z_k)\Big)\tilde{L}_k(ds,dz_k).
\end{aligned}
\tag{13}
$$

It follows from (12)-(13) and the Itô formula that

$$
\begin{aligned}
&\|\xi_{n+1}(t\wedge\zeta_n)-\xi_n(t\wedge\zeta_n)\|^2+\int_\lambda^{t\wedge\zeta_n}\Big(2d_1\|B(\xi_{n+1}(s)-\xi_n(s))\|^2+2a_1\|\xi_{n+1}(s)-\xi_n(s)\|^2\Big)\,ds\\
=&\,2b_1\int_\lambda^{t\wedge\zeta_n}\Big\langle\Gamma^{n+1}(\xi_{n+1}(s),\psi_{n+1}(s))-\Gamma^n(\xi_n(s),\psi_n(s)),\xi_{n+1}(s)-\xi_n(s)\Big\rangle\,ds\\
&+2\epsilon_1\sum_{k=1}^\infty\int_\lambda^{t\wedge\zeta_n}\Big\langle\sigma_k^{n+1}(\xi_{n+1}(s))-\sigma_k^n(\xi_n(s)),\xi_{n+1}(s)-\xi_n(s)\Big\rangle\,dW_k(s)\\
&-2b_2\int_\lambda^{t\wedge\zeta_n}\Big\langle\Phi^{n+1}(\xi_{n+1}(s))-\Phi^n(\xi_n(s)),\xi_{n+1}(s)-\xi_n(s)\Big\rangle\,ds\\
&+\epsilon_1^2\sum_{k=1}^\infty\int_\lambda^{t\wedge\zeta_n}\|\sigma_k^{n+1}(\xi_{n+1}(s))-\sigma_k^n(\xi_n(s))\|^2\,ds\\
&+\sum_{k=1}^\infty\int_\lambda^{t\wedge\zeta_n}\int_{|z_k|<1}\Big(\|\epsilon_2(q_k^{n+1}(\xi_{n+1}(s-),z_k)-q_k^n(\xi_n(s-),z_k))+\xi_{n+1}(s-)-\xi_n(s-)\|^2\\
&\qquad\qquad-\|\xi_{n+1}(s-)-\xi_n(s-)\|^2\Big)\tilde{L}_k(ds,dz_k)\\
&+\epsilon_2^2\sum_{k=1}^\infty\int_\lambda^{t\wedge\zeta_n}\int_{|z_k|<1}\|q_k^{n+1}(\xi_{n+1}(s),z_k)-q_k^n(\xi_n(s),z_k)\|^2v_k(dz_k)\,ds,
\end{aligned}
$$

and

$$
\|\psi_{n+1}(t\wedge\zeta_n)-\psi_n(t\wedge\zeta_n)\|^2+\int_\lambda^{t\wedge\zeta_n}\Big(2d_2\|B(\psi_{n+1}(s)-\psi_n(s))\|^2+2a_2\|\psi_{n+1}(s)-\psi_n(s)\|^2\Big)\,ds
$$

$$
\begin{aligned}
&= 2b_2 \int_\lambda^{t\wedge\zeta_n} \Big\langle \Phi^{n+1}(\xi_{n+1}(s)) - \Phi^n(\xi_n(s)), \psi_{n+1}(s) - \psi_n(s) \Big\rangle \, ds \\
&\quad - 2b_1 \int_\lambda^{t\wedge\zeta_n} \Big\langle \Gamma^{n+1}(\xi_{n+1}(s), \psi_{n+1}(s)) - \Gamma^n(\xi_n(s), \psi_n(s)), \psi_{n+1}(s) - \psi_n(s) \Big\rangle \, ds \\
&\quad + 2\gamma_1 \sum_{k=1}^\infty \int_\lambda^{t\wedge\zeta_n} \Big\langle \sigma_k^{n+1}(\psi_{n+1}(s)) - \sigma_k^n(\psi_n(s)), \psi_{n+1}(s) - \psi_n(s) \Big\rangle \, dW_k(s) \\
&\quad + \gamma_1^2 \sum_{k=1}^\infty \int_\lambda^{t\wedge\zeta_n} \|\sigma_k^{n+1}(\psi_{n+1}(s)) - \sigma_k^n(\psi_n(s))\|^2 \, ds \\
&\quad + \sum_{k=1}^\infty \int_\lambda^{t\wedge\zeta_n} \int_{|z_k|<1} \Big(\|\gamma_2(q_k^{n+1}(\psi_{n+1}(s-), z_k) - q_k^n(\psi_n(s-), z_k)) + \psi_{n+1}(s-) - \psi_n(s-)\|^2 \\
&\qquad\qquad - \|\psi_{n+1}(s-) - \psi_n(s-)\|^2 \Big) \tilde{L}_k(ds, dz_k) \\
&\quad + \gamma_2^2 \sum_{k=1}^\infty \int_\lambda^{t\wedge\zeta_n} \int_{|z_k|<1} \|q_k^{n+1}(\psi_{n+1}(s), z_k) - q_k^n(\psi_n(s), z_k)\|^2 v_k(dz_k) \, ds,
\end{aligned}
$$

where $\xi_{n+1}(s) - \xi_n(s)$ and $\psi_{n+1}(s) - \psi_n(s)$ are identified with elements in the dual space of ℓ^2 by the Riesz representation theorem.

Since $\|\xi_n(s)\| \le n$, $\|\psi_n(s)\| \le n$ for all $s \in [\lambda, \zeta_n)$, we have

$$
\begin{aligned}
&\Gamma^{n+1}(\xi_n(s), \psi_n(s)) = \Gamma^n(\xi_n(s), \psi_n(s)), \;\; \Phi^{n+1}(\xi_n(s)) = \Phi^n(\xi_n(s)), \\
&\sigma_k^{n+1}(\xi_n(s)) = \sigma_k^n(\xi_n(s)), \;\; q_k^{n+1}(\xi_n(s), z_k) = q_k^n(\xi_n(s), z_k).
\end{aligned} \tag{14}
$$

By (4)-(6) and (12)-(14), there exists $C_1(n+1) > 0$ such that

$$
\begin{aligned}
&\|\xi_{n+1}(t\wedge\zeta_n) - \xi_n(t\wedge\zeta_n)\|^2 + \|\psi_{n+1}(t\wedge\zeta_n) - \psi_n(t\wedge\zeta_n)\|^2 \\
\le& C_1(n+1) \int_\lambda^{t\wedge\zeta_n} \Big(\|\xi_{n+1}(s) - \xi_n(s)\|^2 + \|\psi_{n+1}(s) - \psi_n(s)\|^2 \Big) \, ds \\
&+ 2\epsilon_1 \sum_{k=1}^\infty \int_\lambda^{t\wedge\zeta_n} \Big\langle \sigma_k^{n+1}(\xi_{n+1}(s)) - \sigma_k^n(\xi_n(s)), \xi_{n+1}(s) - \xi_n(s) \Big\rangle \, dW_k(s) \\
&+ 2\gamma_1 \sum_{k=1}^\infty \int_\lambda^{t\wedge\zeta_n} \Big\langle \sigma_k^{n+1}(\psi_{n+1}(s) - \sigma_k^n(\psi_n(s)), \psi_{n+1}(s) - \psi_n(s) \Big\rangle \, dW_k(s) \\
&+ \sum_{k=1}^\infty \int_\lambda^{t\wedge\zeta_n} \int_{|z_k|<1} \Big(\|\epsilon_2(q_k^{n+1}(\xi_{n+1}(s-), z_k) - q_k^n(\xi_n(s-), z_k)) + \xi_{n+1}(s-) - \xi_n(s-)\|^2 \\
&\qquad\qquad - \|\xi_{n+1}(s-) - \xi_n(s-)\|^2 \Big) \tilde{L}_k(ds, dy_k) \\
&+ \sum_{k=1}^\infty \int_\lambda^{t\wedge\zeta_n} \int_{|z_k|<1} \Big(\|\gamma_2(q_k^{n+1}(\psi_{n+1}(s-), z_k) - q_k^n(\psi_n(s-), z_k)) + \psi_{n+1}(s-) - \psi_n(s-)\|^2 \\
&\qquad\qquad - \|\psi_{n+1}(s-) - \psi_n(s-)\|^2 \Big) \tilde{L}_k(ds, dz_k).
\end{aligned} \tag{15}
$$

From now on, we utilize $\mathbb{E}$ to denote the expectation of a random variable. By (15), we can obtain

$$
\begin{aligned}
&\mathbb{E}\Big[\sup_{\lambda\le s\le t} \Big(\|\xi_{n+1}(s\wedge\zeta_n) - \xi_n(s\wedge\zeta_n)\|^2 + \|\psi_{n+1}(s\wedge\zeta_n) - \psi_n(s\wedge\zeta_n)\|^2 \Big) \Big] \\
\le& C_1(n+1) \int_\lambda^t \mathbb{E}\Big[\sup_{\lambda\le s\le r} \Big(\|\xi_{n+1}(s\wedge\zeta_n) - \xi_n(s\wedge\zeta_n)\|^2 + \|\psi_{n+1}(s\wedge\zeta_n) - \psi_n(s\wedge\zeta_n)\|^2 \Big) \Big] \, dr
\end{aligned}
$$

$$
\begin{aligned}
&+2\epsilon_1\mathbb{E}\Big[\sup_{\lambda\le s\le t\wedge\zeta_n}\Big|\sum_{k=1}^{\infty}\int_\lambda^s\langle\sigma_k^{n+1}(\xi_{n+1}(r))-\sigma_k^n(\xi_n(r)),\xi_{n+1}(r)-\xi_n(r)\rangle\,dW_k(r)\Big|\Big]\\
&+2\gamma_1\mathbb{E}\Big[\sup_{\lambda\le s\le t\wedge\zeta_n}\Big|\sum_{k=1}^{\infty}\int_\lambda^s\langle\sigma_k^{n+1}(\psi_{n+1}(r))-\sigma_k^n(\psi_n(r)),\psi_{n+1}(r)-\psi_n(r)\rangle\,dW_k(r)\Big|\Big]\\
&+\mathbb{E}\Big[\sup_{\lambda\le s\le t\wedge\zeta_n}\Big|\sum_{k=1}^{\infty}\int_\lambda^s\int_{|z_k|<1}\big(\|\epsilon_2(q_k^{n+1}(\xi_{n+1}(r-),z_k)-q_k^n(\xi_n(r-),z_k))+\xi_{n+1}(r-)-\xi_n(r-)\|^2\\
&\qquad\qquad-\|\xi_{n+1}(r-)-\xi_n(r-)\|^2\big)\tilde{L}_k(dr,dz_k)\Big|\Big]\\
&+\mathbb{E}\Big[\sup_{\lambda\le s\le t\wedge\zeta_n}\Big|\sum_{k=1}^{\infty}\int_\lambda^s\int_{|z_k|<1}\big(\|\gamma_2(q_k^{n+1}(\psi_{n+1}(r-),z_k)-q_k^n(\psi_n(r-),z_k))+\psi_{n+1}(r-)-\psi_n(r-)\|^2\\
&\qquad\qquad-\|\psi_{n+1}(r-)-\psi_n(r-)\|^2\big)\tilde{L}_k(dr,dz_k)\Big|\Big]\\
:=&I_0+I_1+I_2+I_3+I_4. \qquad (16)
\end{aligned}
$$

By (5) and the BDG inequality, there exists $C_2(n+1)>0$ such that

$$
\begin{aligned}
I_1\le&\,2\epsilon_1C_{BDG}\mathbb{E}\Big[\Big(\int_\lambda^{t\wedge\zeta_n}\big(\|\xi_{n+1}(r)-\xi_n(r)\|^2\sum_{k=1}^{\infty}\|\sigma_k^{n+1}(\xi_{n+1}(r))-\sigma_k^{n+1}(\xi_n(r))\|^2\big)\,dr\Big)^{\frac12}\Big]\\
\le&\,2\epsilon_1C_{BDG}\mathbb{E}\Big[\sup_{\lambda\le r\le t}\|\xi_{n+1}^{\zeta_n}(r)-\xi_n^{\zeta_n}(r)\|\Big(\int_\lambda^{t\wedge\zeta_n}\sum_{k=1}^{\infty}\|\sigma_k^{n+1}(\xi_{n+1}(r))-\sigma_k^{n+1}(\xi_n(r))\|^2\,dr\Big)^{\frac12}\Big]\\
\le&\,\frac14\mathbb{E}\Big[\sup_{\lambda\le s\le t}\|\xi_{n+1}^{\zeta_n}(s)-\xi_n^{\zeta_n}(s)\|^2\Big]+4\epsilon_1^2C_{BDG}^2C_2(n+1)\mathbb{E}\Big[\int_\lambda^{t\wedge\zeta_n}\|\xi_{n+1}(r)-\xi_n(r)\|^2\,dr\Big]\\
\le&\,\frac14\mathbb{E}\Big[\sup_{\lambda\le s\le t}\|\xi_{n+1}^{\zeta_n}(s)-\xi_n^{\zeta_n}(s)\|^2\Big]+4\epsilon_1^2C_{BDG}^2C_2(n+1)\int_\lambda^t\mathbb{E}\Big[\sup_{\lambda\le s\le r}\|\xi_{n+1}^{\zeta_n}(s)-\xi_n^{\zeta_n}(s)\|^2\Big]\,dr,
\end{aligned}
$$

and

$$
\begin{aligned}
I_2\le&\frac14\mathbb{E}\Big[\sup_{\lambda\le s\le t}\|\psi_{n+1}^{\zeta_n}(s)-\psi_n^{\zeta_n}(s)\|^2\Big]\\
&+4\gamma_1^2C_{BDG}^2C_2(n+1)\int_\lambda^t\mathbb{E}\Big[\sup_{\lambda\le s\le r}\|\psi_{n+1}^{\zeta_n}(s)-\psi_n^{\zeta_n}(s)\|^2\Big]\,dr.
\end{aligned}
$$

By the BDG-type inequality with respect to the Poisson random measures, see e.g., [2, Lemma 8.22], we can use (5) and (14) to deduce that

$$
\begin{aligned}
I_3\le&\,\epsilon_2^2\mathbb{E}\Big[\sup_{\lambda\le s\le t\wedge\zeta_n}\Big|\sum_{k=1}^{\infty}\int_\lambda^s\int_{|z_k|<1}\|q_k^{n+1}(\xi_{n+1}(r-),z_k)-q_k^n(\xi_n(r-),z_k)\|^2\tilde{L}_k(dr,dz_k)\Big|\Big]\\
&+2\epsilon_2\mathbb{E}\Big[\sup_{\lambda\le s\le t\wedge\zeta_n}\Big|\sum_{k=1}^{\infty}\int_\lambda^s\int_{|z_k|<1}\langle q_k^{n+1}(\xi_{n+1}(r-),z_k)-q_k^n(\xi_n(r-),z_k),\\
&\qquad\qquad\xi_{n+1}(r-)-\xi_n(r-)\rangle\tilde{L}_k(dr,dz_k)\Big|\Big]\\
\le&\,\epsilon_2^2C_1\mathbb{E}\Big[\int_\lambda^{t\wedge\zeta_n}\int_{|z_k|<1}\sum_{k=1}^{\infty}\|q_k^{n+1}(\xi_{n+1}(r),z_k)-q_k^n(\xi_n(r),z_k)\|v_k(dz_k)\,dr\Big]\\
&+2\epsilon_2C_{BDG}\mathbb{E}\Big[\int_\lambda^{t\wedge\zeta_n}\int_{|z_k|<1}\sum_{k=1}^{\infty}\|q_k^{n+1}(\xi_{n+1}(r),z_k)-q_k^{n+1}(\xi_n(r),z_k)\|^2
\end{aligned}
$$

$$
\begin{aligned}
&\qquad\qquad\qquad\qquad \|\xi_{n+1}(r) - \xi_n(r)\|^2 v_k(dz_k)\, ds\Big]^{\frac{1}{2}} \\
&\leq \frac{1}{4}\mathbb{E}\Big[\sup_{\lambda\leq s\leq t} \|\xi_{n+1}(s\wedge\zeta_n) - \xi_n(s\wedge\zeta_n)\|^2\Big] \\
&\quad + \epsilon_2^2(4C_{BDG}^2 + C_1)\mathbb{E}\Big[\int_\lambda^{t\wedge\zeta_n}\int_{|z_k|<1}\sum_{k=1}^{\infty} \|q_k^{n+1}(\xi_{n+1}(r), z_k) - q_k^{n+1}(\xi_n(r), z_k)\|^2\, dr\Big] \\
&\leq \frac{1}{4}\mathbb{E}\Big[\sup_{\lambda\leq s\leq t} \|\xi_{n+1}(s\wedge\zeta_n) - \xi_n(s\wedge\zeta_n)\|^2\Big] \\
&\quad + \epsilon_2^2(4C_{BDG}^2 + C_1)C_2(n+1)\int_\lambda^t \mathbb{E}\Big[\sup_{\lambda\leq s\leq r} \|\xi_{n+1}(s\wedge\zeta_n) - \xi_n(s\wedge\zeta_n)\|^2\, dr\Big].
\end{aligned}
$$

Similarly,

$$
\begin{aligned}
I_4 &\leq \frac{1}{4}\mathbb{E}\Big[\sup_{\lambda\leq s\leq t} \|\psi_{n+1}(s\wedge\zeta_n) - \psi_n(s\wedge\zeta_n)\|^2\Big] \\
&\quad + \gamma_2^2(4C_{BDG}^2 + C_1)C_2(n+1)\int_\lambda^t \mathbb{E}\Big[\sup_{\lambda\leq s\leq r} \|\psi_{n+1}(s\wedge\zeta_n) - \psi_n(s\wedge\zeta_n)\|^2\, dr\Big],
\end{aligned}
\tag{17}
$$

where $C_{BDG} > 0$ is a constant resulting from the BDG-type inequality.

Form (16)-(17), there exists $C_3(n+1) > 0$ such that

$$
\begin{aligned}
&\mathbb{E}\Big[\sup_{\lambda\leq s\leq t} \|\varphi_{n+1}(s\wedge\zeta_n)\|^2 - \|\varphi_n(s\wedge\zeta_n)\|^2\Big] \\
\leq &C_3(n+1)\int_\lambda^t \mathbb{E}\Big[\sup_{\lambda\leq s\leq r} \|\varphi_{n+1}(s\wedge\zeta_n)\|^2 - \|\varphi_n(s\wedge\zeta_n)\|^2\Big]\, dr,
\end{aligned}
\tag{18}
$$

where

$$
C_3(n+1) = 2\Big(C_1(n+1) + C_2(n+1)\big(4C_{BDG}^2(\epsilon_1^2 + \gamma_1^2) + (4C_{BDG}^2 + C_1)(\epsilon_2^2 + \gamma_2^2)\big)\Big).
$$

Applying Gronwall's lemma to (18), we obtain

$$
\mathbb{E}\Big[\sup_{\lambda\leq s\leq t} \|\varphi_{n+1}(s\wedge\zeta_n)\|^2 - \|\varphi_n(s\wedge\zeta_n)\|^2\Big] = 0, \quad \forall t \geq \lambda,
$$

therefore, $\varphi_{n+1}(t\wedge\zeta_n) = \varphi_n(t\wedge\zeta_n)$, $\mathbb{P}$-a.s., for all $t \geq \lambda$. In a conclusion, by (11), we find that $\zeta_{n+1} \geq \zeta_n$ a.s. Then we finish the proof. □

By $\zeta_{n+1} \geq \zeta_n$ a.s., we can define a stopping time ζ by

$$
\zeta = \lim_{n\to\infty} \zeta_n = \sup_{n\in\mathbb{N}} \zeta_n. \tag{19}
$$

Next, we will show $\zeta = \infty$ a.s. To achieve this, we require the specified uniform estimates.

Lemma 3.3. *The φ_n given by (10) satisfies*

$$
\mathbb{E}\Big[\sup_{\lambda\leq s\leq \lambda+T} \|\varphi_n(s)\|_{\ell^2\times\ell^2}^2\Big] \leq M_1 e^{MT} \quad \forall\, T > 0,
$$

where

$$
M_1 = 2\mathbb{E}(b_2\|\xi_\lambda\|^2 + b_1\|\psi_\lambda\|^2) + M_2\mathbb{E}\int_\lambda^t \Big(\|f_1(s)\|^2 + \|f_2(s)\|^2 + \|h_k(s)\|^2 + \|\kappa_k(s)\|^2\Big) + M_3 T.
$$

Proof. By (10) and the Itô formula, we can obtain

$$\begin{aligned}
&b_2\|\xi_n(t)\|^2+\int_\lambda^t\Big(2d_1b_2\|B\xi_n(s)\|^2+2a_1b_2\|\xi_n(s)\|^2+2b_2^2\langle\Phi^n(\xi_n(s)),\xi_n(s)\rangle\Big)\,ds\\
=&b_2\|\xi_\lambda\|^2+2b_2\int_\lambda^t\langle f_1(s),\xi_n(s)\rangle\,ds+2b_1b_2\int_\lambda^t\langle\Gamma^n(\xi_n(s),\psi_n(s)),\xi_n(s)\rangle\,ds\\
&+b_2\epsilon_1^2\sum_{k=1}^\infty\int_\lambda^t\|h_k(s)+\sigma_k^n(\xi_n(s))\|^2\,ds+2b_2\epsilon_1\sum_{k=1}^\infty\int_\lambda^t\langle h_k(s)+\sigma_k^n(\xi_n(s)),\xi_n(s)\rangle\,dW_k(s)\\
&+b_2\sum_{k=1}^\infty\int_\lambda^t\int_{|z_k|<1}\Big(\|\xi_n(s-)+\epsilon_2(q_k^n(\xi_n(s-),z_k)+\kappa_k(s))\|^2-\|\xi_n(s-)\|^2\Big)\tilde{L}_k(ds,dz_k)\\
&+b_2\epsilon_2^2\sum_{k=1}^\infty\int_\lambda^t\int_{|z_k|<1}\|q_k^n(\xi_n(s),z_k)+\kappa_k(s)\|^2v_k(dz_k)\,ds,
\end{aligned}\tag{20}$$

and

$$\begin{aligned}
&b_1\|\psi_n(t)\|^2+\int_\lambda^t\Big(2d_2b_1\|B\psi_n(s)\|^2+2a_2b_1\|\psi_n(s)\|^2+2b_1^2\langle\Gamma^n(\xi_n(s),\psi_n(s)),\psi_n(s)\rangle\Big)ds\\
=&b_1\|\psi_\lambda\|^2+2b_1\int_\lambda^t(f_2(s),\psi_n(s))ds+2b_2b_1\int_\lambda^t\langle\Phi^n(\xi_n(s)),\psi_n(s)\rangle\,ds\\
&+b_1\gamma_1^2\sum_{k=1}^\infty\int_\lambda^t\|h_k(s)+\sigma_k^n(\psi_n(s))\|^2ds+2b_1\gamma_1\sum_{k=1}^\infty\int_\lambda^t\langle h_k(s)+\sigma_k^n(\psi_n(s)),\psi_n(s)\rangle dW_k(s)\\
&+b_1\sum_{k=1}^\infty\int_\lambda^t\int_{|z_k|<1}\Big(\|\psi_n(s-)+\gamma_2(q_k^n(\psi_n(s-),z_k)+\kappa_k(s))\|^2-\|\psi_n(s-)\|^2\Big)\tilde{L}_k(ds,dz_k)\\
&+b_1\gamma_2^2\sum_{k=1}^\infty\int_\lambda^t\int_{|z_k|<1}\|q_k^n(\psi_n(s),z_k)+\kappa_k(s)\|^2v_k(dz_k)\,ds.
\end{aligned}\tag{21}$$

By (3) and (20)-(21), we have

$$\begin{aligned}
&\mathbb{E}\Big[\sup_{\lambda\le r\le t}\Big(b_2\|\xi_n(r)\|^2+b_1\|\psi_n(r)\|^2\Big)\Big]\\
\le&\mathbb{E}\left[b_2\|\xi_\lambda\|^2+b_1\|\psi_\lambda\|^2\right]+a_1^{-1}b_2\int_\lambda^t\mathbb{E}[\|f_1(s)\|^2]ds+a_2^{-1}b_1\int_\lambda^t\mathbb{E}[\|f_2(s)\|^2]ds\\
&+b_2\epsilon_1^2\sum_{k=1}^\infty\int_\lambda^t\mathbb{E}\left[\|h_k(s)+\sigma_k^n(\xi_n(s))\|^2\right]ds+b_1\gamma_1^2\sum_{k=1}^\infty\int_\lambda^t\mathbb{E}\left[\|h_k(s)+\sigma_k^n(\psi_n(s))\|^2\right]ds\\
&+\underbrace{2b_2\epsilon_1\mathbb{E}\Big[\sup_{\lambda\le r\le t}\Big|\sum_{k=1}^\infty\int_\lambda^r(h_k(s)+\sigma_k^n(\xi_n(s)))(\xi_n(s))dW_k(s)\Big|\Big]}_{I_5}\\
&+\underbrace{2b_1\gamma_1\mathbb{E}\Big[\sup_{\lambda\le r\le t}\Big|\sum_{k=1}^\infty\int_\lambda^r(h_k(s)+\sigma_k^n(\psi_n(s)))(\psi_n(s))dW_k(s)\Big|\Big]}_{I_6}\\
&+\underbrace{b_2\mathbb{E}\Big[\sup_{\lambda\le r\le t}\Big|\sum_{k=1}^\infty\int_\lambda^r\int_{|z_k|<1}(\|\xi_n(s-)+\epsilon_2(q_k^n(\xi_n(s-),z_k)+\kappa_k(s))\|^2-\|\xi_n(s-)\|^2)\tilde{L}_k(ds,dz_k)\Big|\Big]}_{I_7}\\
&+\underbrace{b_2\epsilon_2^2\mathbb{E}\Big[\sum_{k=1}^\infty\int_\lambda^t\int_{|z_k|<1}\|q_k^n(\xi_n(s),z_k)+\kappa_k(s)\|^2v_k(dz_k)\,ds\Big]}_{I_8}
\end{aligned}$$

$$+\underbrace{b_1\mathbb{E}\left[\sup_{\lambda\le r\le t}\left|\sum_{k=1}^{\infty}\int_{\lambda}^{r}\int_{|z_k|<1}\left(\|\psi_n(s-)+\gamma_2(q_k^n(\psi_n(s-),z_k)+\kappa_k(s))\|^2-\|\psi_n(s-)\|^2\right)\tilde{L}_k(ds,dz_k)\right|\right]}_{I_9}$$

$$+\underbrace{b_1\gamma_2^2\mathbb{E}\left[\sum_{k=1}^{\infty}\int_{\lambda}^{t}\int_{|z_k|<1}\|q_k^n(\psi_n(s),z_k)+\kappa_k(s)\|^2 v_k(dz_k)\,ds\right]}_{I_{10}}. \tag{22}$$

For the terms I_5 and I_6 in (22), by the BDG inequality, we can obtain

$$\begin{aligned}I_5 &\le 2b_2\epsilon_1 C_{BDG}\mathbb{E}\left[\int_{\lambda}^{t}\sum_{k=1}^{\infty}\|h_k(s)+\sigma_k^n(\xi_n(s))\|^2\|\xi_n(s)\|^2\,ds\right]^{\frac{1}{2}}\\ &\le \frac{1}{4}\mathbb{E}\big[\sup_{\lambda\le s\le t}\|b_2\xi_n(s)\|^2\big]+4b_2\epsilon_1^2C_{BDG}^2\sum_{k=1}^{\infty}\int_{\lambda}^{t}\mathbb{E}\big[\|h_k(s)+\sigma_k^n(\xi_n(s))\|^2\big]\,ds,\end{aligned} \tag{23}$$

and

$$I_6 \le \frac{1}{4}\mathbb{E}\big[\sup_{\lambda\le s\le t}\|b_1\psi_n(s)\|^2\big]+4b_1\gamma_1^2C_{BDG}^2\sum_{k=1}^{\infty}\int_{\lambda}^{t}\mathbb{E}\big[\|h_k(s)+\sigma_k^n(\psi_n(s))\|^2\big]\,ds. \tag{24}$$

By (23)-(24), we get

$$\begin{aligned}&4b_2\epsilon_1^2C_{BDG}^2\sum_{k=1}^{\infty}\int_{\lambda}^{t}\mathbb{E}\big[\|h_k(s)+\sigma_k^n(\xi_n(s))\|^2\big]\,ds\\ &\le 8b_2\epsilon_1^2C_{BDG}^2\sum_{k=1}^{\infty}\int_{\lambda}^{t}\mathbb{E}\left[\|h_k(s)\|^2\right]ds+16b_2\epsilon_1^2C_{BDG}^2\beta^2\|\delta\|^2\int_{\lambda}^{t}\mathbb{E}\left[\|\xi_n(s)\|^2\right]ds\\ &\quad+16b_2\epsilon_1^2C_{BDG}^2\beta^2\|\delta\|^2(t-\lambda),\end{aligned}$$

and

$$\begin{aligned}&4b_1\gamma_1^2C_{BDG}^2\sum_{k=1}^{\infty}\int_{\lambda}^{t}\mathbb{E}\big[\|h_k(s)+\sigma_k^n(\psi_n(s))\|^2\big]\,ds\\ &\le 8b_1\epsilon_1^2C_{BDG}^2\sum_{k=1}^{\infty}\int_{\lambda}^{t}\mathbb{E}\left[\|h_k(s)\|^2\right]ds+16b_1\gamma_1^2C_{BDG}^2\beta^2\|\delta\|^2\int_{\lambda}^{t}\mathbb{E}\left[\|\psi_n(s)\|^2\right]ds\\ &\quad+16b_1\gamma_1^2C_{BDG}^2\beta^2\|\delta\|^2(t-\lambda).\end{aligned}$$

For the terms I_7 and I_9 in (22), we have

$$\begin{aligned}I_7 &\le b_2\epsilon_2^2\mathbb{E}\left[\sup_{\lambda\le r\le t}\Big|\sum_{k=1}^{\infty}\int_{\lambda}^{r}\int_{|z_k|<1}\|q_k^n(\xi_n(s-),z_k)+\kappa_k(s)\|^2\tilde{L}_k(ds,dz_k)\Big|\right]\\ &\quad+2b_2\epsilon_2\mathbb{E}\left[\sup_{\lambda\le r\le t}\left|\sum_{k=1}^{\infty}\int_{\lambda}^{r}\int_{|z_k|<1}\langle\xi_n(s-),q_k^n(\xi_n(s-),z_k)+\kappa_k(s)\rangle\tilde{L}_k(ds,dz_k)\right|\right]\\ &\le C_2b_2\mathbb{E}\left[\int_{\lambda}^{t}\int_{|z_k|<1}\sum_{k=1}^{\infty}\|q_k^n(\xi_n(s),z_k)+\kappa_k(s)\|^2v_k(dz_k)ds\right]\\ &\quad+2C_{BDG}b_2\mathbb{E}\left[\left(\int_{\lambda}^{t}\int_{|z_k|<1}\sum_{k=1}^{\infty}\|\xi_n(s)\|^2\|q_k^n(\xi_n(s),z_k)+\kappa_k(s)\|^2v_k(dz_k)ds\right)^{\frac{1}{2}}\right]\\ &\le 2C_2b_2\int_{\lambda}^{t}2\beta^2\|\delta\|^2\mathbb{E}[\|\xi_n(s)\|^2]\,ds+4C_2b_2\beta^2\|\delta\|^2(t-\lambda)+2C_2b_2\sum_{k=1}^{\infty}\int_{\lambda}^{t}\mathbb{E}[\|\kappa_k(s)\|^2]\,ds\end{aligned}$$

$$
\begin{aligned}
&+\frac{1}{4}b_2\mathbb{E}[\sup_{\lambda\le s\le t}\|\xi_n(s)\|^2]+4C_{BDG}^2b_2\mathbb{E}\Big[\int_\lambda^t\int_{|z_k|<1}\sum_{k=1}^\infty\|q_k^n(\xi_n(s),z_k)+\kappa_k(s)\|^2v_k(dz_k)ds\Big]\\
&\le 2C_2b_2\int_\lambda^t 2\beta^2\|\delta\|^2\mathbb{E}[\|\xi_n(s)\|^2]\,ds+4C_2b_2\beta^2\|\delta\|^2(t-\lambda)+2C_2b_2\sum_{k=1}^\infty\int_\lambda^t\mathbb{E}[\|\kappa_k(s)\|^2]\,ds\\
&+\frac{1}{4}b_2\mathbb{E}[\sup_{\lambda\le s\le t}\|\xi_n(s)\|^2]+8C_{BDG}^2b_2\int_\lambda^t 2\beta^2\|\delta\|^2\mathbb{E}[\|\xi_n(s)\|^2]\,ds+16C_{BDG}^2b_2\beta^2\|\delta\|^2(t-\lambda)\\
&+8C_{BDG}^2b_2\sum_{k=1}^\infty\int_\lambda^t\mathbb{E}[\|\kappa_k(s)\|^2]\,ds,
\end{aligned}
$$

and

$$
\begin{aligned}
I_9&\le 2C_2b_1\int_\lambda^t 2\beta^2\|\delta\|^2\mathbb{E}[\|\psi_n(s)\|^2]\,ds+4C_2b_1\beta^2\|\delta\|^2(t-\lambda)+2C_2b_1\sum_{k=1}^\infty\int_\lambda^t\mathbb{E}[\|\kappa_k(s)\|^2]\,ds\\
&+\frac{1}{4}b_1\mathbb{E}[\sup_{\lambda\le s\le t}\|\psi_n(s)\|^2]+8C_{BDG}^2b_1\int_\lambda^t 2\beta^2\|\delta\|^2\mathbb{E}[\|\psi_n(s)\|^2]\,ds+16C_{BDG}^2b_1\beta^2\|\delta\|^2(t-\lambda)\\
&+8C_{BDG}^2b_1\sum_{k=1}^\infty\int_\lambda^t\mathbb{E}[\|\kappa_k(s)\|^2]\,ds.
\end{aligned}
$$

For the terms I_8 and I_{10} in (22), we have

$$
I_8\le b_2\epsilon_2^2 4\beta^2\|\delta\|^2\int_\lambda^t\mathbb{E}[\|\xi_n(s)\|^2]\,ds+4\beta^2\|\delta\|^2(t-\lambda)b_2\epsilon_2^2+2b_2\epsilon_2^2\sum_{k=1}^\infty\int_\lambda^t\mathbb{E}[\|\kappa_k(s)\|^2]\,ds, \quad (25)
$$

$$
I_{10}\le b_1\gamma_2^2 4\beta^2\|\delta\|^2\int_\lambda^t\mathbb{E}[\|\psi_n(s)\|^2]\,ds+4\beta^2\|\delta\|^2(t-\lambda)b_1\gamma_2^2+2b_1\gamma_2^2\sum_{k=1}^\infty\int_\lambda^t\mathbb{E}[\|\kappa_k(s)\|^2]\,ds. \quad (26)
$$

By (22)-(26), we find that there exist $M>0$, $M_2>0$, $M_3>0$ independent of ξ_λ, ψ_λ, T, n such that for all $t\in[\lambda,T]$,

$$
\begin{aligned}
&\mathbb{E}\Big[\sup_{\lambda\le r\le t}\Big(b_2\|\xi_n(r)\|^2+b_1\|\psi_n(r)\|^2\Big)\Big]\\
\le& 2\mathbb{E}\Big[b_2\|\xi_\lambda\|^2+b_1\|\psi_\lambda\|^2\Big]+M_2\mathbb{E}\Big[\int_\lambda^t\Big(\|f_1(s)\|^2+\|f_2(s)\|^2+\sum_{k=1}^\infty\|h_k(s)\|^2+\sum_{k=1}^\infty\|\kappa_k(s)\|^2\Big)ds\Big]\\
&+M_3T+M\int_\lambda^t\mathbb{E}\Big[\sup_{\lambda\le r\le s}\Big(b_2\|\xi_n(r)\|^2+b_1\|\psi_n(r)\|^2\Big)\Big]ds\\
\le& M_1+M\int_\lambda^t\mathbb{E}\Big[\sup_{\lambda\le r\le s}\Big(b_2\|\xi_n(r)\|^2+b_1\|\psi_n(r)\|^2\Big)\Big]ds, \qquad (27)
\end{aligned}
$$

where

$$
\begin{aligned}
M_1=&M_2\int_\lambda^t\mathbb{E}\Big[\|f_1(s)\|^2+\|f_2(s)\|^2+\sum_{k=1}^\infty\|h_k(s)\|^2+\sum_{k=1}^\infty\|\kappa_k(s)\|^2\Big]ds+M_3T\\
&+2\mathbb{E}[b_2\|\xi_\lambda\|^2+b_1\|\psi_\lambda\|^2].
\end{aligned}
$$

Note that M_1 is a finite number. It follows from (27) and Gronwall's lemma that for all $t\in[\lambda,T]$,

$$
\mathbb{E}\Big[\sup_{\lambda\le r\le t}\Big(b_2\|\xi_n(r)\|^2+b_1\|\psi_n(r)\|^2\Big)\Big]\le M_1e^{MT},
$$

which completes the proof. □

Lemma 3.4. *ζ_n and ζ are given by (11) and (19), respectively. Then we have a.s.,*

$$\zeta = \lim_{n\to\infty} \zeta_n = \infty.$$

Proof. For $T > \lambda$, according to Lemma 3.3 and the Chebyshev inequality, there exists $C = C(\varphi_\lambda, \lambda, T)$ such that

$$\mathbb{P}\{\lambda \le \zeta_n < T\} \le \mathbb{P}\Big\{ \sup_{\lambda\le t\le T} \|\varphi_n(s) \ge n\| \Big\} \le \frac{1}{n^2}\mathbb{E}\Big(\sup_{\lambda\le t\le T} \|\varphi_n\|^2 \Big) \le \frac{C}{n^2}. \tag{28}$$

By (28), we can obtain

$$\mathbb{P}\Big(\bigcap_{m=1}^{\infty} \bigcup_{n=m}^{\infty} \{\zeta_n < T\} \Big) = 0.$$

We get $\hat{\Omega} \in \Omega$, $\mathbb{P}(\hat{\Omega}) = 0$, $\zeta(\omega) = \infty$, $\omega \in \Omega \setminus \hat{\Omega}$. $\square$

Theorem 3.5. *Assume that (4)-(6) and (7) hold, if $\varphi_\lambda \in L^2(\Omega, \ell^2 \times \ell^2)$, then (9)-(8) has a unique solution φ which satisfies that for every $T > \lambda$,*

$$\mathbb{E}\Big[\sup_{\lambda\le t\le T} \|\varphi(s)\|^2 \Big] \le M_1 e^{MT},$$

where M_1, M are defined in Lemma 3.3.

Proof. Let φ_n be the solution of (10). By Lemmas (3.4) and (3.2), there exists a subset $\hat{\Omega}$ of Ω with $\mathbb{P}(\hat{\Omega}) = 0$ such that for all $n \ge n_0$,

$$\varphi_n(t \wedge \zeta_{n_0}) = \varphi_{n_0}(t \wedge \zeta_{n_0}) \ \text{ and } \lim_{n\to\infty} \zeta_n = \infty \ \text{ on } \hat{\Omega}. \tag{29}$$

Define $\varphi : [\lambda, \infty] \times \Omega \to \ell^2 \times \ell^2$ by

$$\varphi(t,\omega) = \begin{cases} \varphi_n(t,\omega) & \text{if } \omega \in \hat{\Omega}, \quad t \in [\lambda, \zeta_n(\omega)]; \\ \varphi_0(\omega) & \text{if } \ \omega \in \Omega \backslash \hat{\Omega}, \quad t \ge \lambda. \end{cases}$$

By (29), for every $\omega \in \hat{\Omega}$ and $t \ge 0$, there exists $N_0 = N_0(t,\omega) \ge 1$ such that for $N \ge N_0$,

$$\zeta_n(\omega) > t \quad \text{and} \quad \varphi_n(t,\omega) = \varphi_{n_0}(t,\omega), \tag{30}$$

We know φ is well-defined in (30). Furthermore, as φ_n is a continuous $\ell^2 \times \ell^2$-valued process, according to (30), we conclude that φ is almost surely continuous with regard to t in $\ell^2 \times \ell^2$. It's worth noting that (30) implies that for any fixed $t \ge 0$,

$$\lim_{n\to\infty} \varphi_n(t,\omega) = \varphi(t,\omega), \qquad \omega \in \hat{\Omega}. \tag{31}$$

φ_n is $\mathcal{F}_t$-adapted process, we know that φ is $\mathcal{F}_t$-adapted process by (31). According to (29)-(30), Lemma 3.3 and Fatou's lemma, we obtain

$$\mathbb{E}\Big[\sup_{\lambda\le t\le T} \|\varphi\|^2 \Big] \le M_1 e^{MT}.$$

Then we demonstrate that φ is a solution of problem (8)-(9). Since φ_n is given by (10), we can obtain

$$\begin{cases} \xi(t\wedge\zeta_n) = \xi_\lambda + \int_\lambda^{t\wedge\zeta_n} \big(-d_1 A\xi_n(s) - a_1\xi_n(s) + b_1\Gamma(\xi_n(s),\psi_n(s)) - b_2\Phi(\xi_n(s)) + f_1(s)\big)ds \\ \qquad + \epsilon_1 \sum_{k=1}^{\infty}\int_\lambda^{t\wedge\zeta_n} \big(h_k(s) + \sigma_k^n(\xi_n(s))\big)dW_k(s) \\ \qquad + \epsilon_2 \sum_{k=1}^{\infty}\int_\lambda^{t\wedge\zeta_n}\int_{|z_k|<1} \big(\kappa_k(s) + q_k^n(\xi_n(s-),z_k)\big)\tilde{L}_k(ds,dz_k), \\ \psi(t\wedge\zeta_n) = \psi_\lambda + \int_\lambda^{t\wedge\zeta_n} \big(-d_2 A\psi_n(s) - a_2\psi_n(s) - b_1\Gamma(\xi_n(s),\psi_n(s)) + b_2\Phi(\xi_n(s)) + f_2(s)\big)ds \\ \qquad + \gamma_1 \sum_{k=1}^{\infty}\int_\lambda^{t\wedge\lambda_n} \big(h_k(s) + \sigma_k^n(\psi_n(s))\big)dW_k(s) \\ \qquad + \gamma_2 \sum_{k=1}^{\infty}\int_\lambda^{t\wedge\zeta_n}\int_{|z_k|<1} \big(\kappa_k(s) + q_k^n(\psi_n(s-),z_k)\big)\tilde{L}_k(ds,dz_k). \end{cases} \tag{32}$$

By (30), we see that $\xi_n(t\wedge\zeta_n) = \xi(t\wedge\zeta_n)$, $\psi_n(t\wedge\zeta_n) = \psi(t\wedge\zeta_n)$ $\mathbb{P}$-a.s. Then by the definition of Γ^n, Φ^n, σ_k^n, q_k^n, for $s\in[\lambda,\zeta_n]$, we get

$$\Gamma^n(\xi_n(s),\psi_n(s)) = \Gamma(\xi(s),\psi(s)),\ \Phi^n(\xi_n(s)) = \Phi(\xi(s)),$$

$$\sigma_k^n(\xi_n(s)) = \sigma_k(\xi(s)),\ q_k^n(\xi_n(s),z_k) = q_k(\xi_n(s),z_k).$$

Then by (32), we obtain

$$\begin{cases} \xi(t\wedge\zeta_n) = \xi_\lambda + \int_\lambda^{t\wedge\zeta_n} \big(-d_1 A\xi(s) - a_1\xi(s) + b_1\Gamma(\xi(s),\psi(s)) - b_2\Phi(\xi(s)) + f_1(s)\big)ds \\ \qquad + \epsilon_1 \sum_{k=1}^{\infty}\int_\lambda^{t\wedge\zeta_n} \big(h_k(s) + \sigma_k(\xi(s))\big)dW_k(s) \\ \qquad + \epsilon_2 \sum_{k=1}^{\infty}\int_\lambda^{t\wedge\zeta_n}\int_{|z_k|<1} \big(\kappa_k(s) + q_k(\xi(s-),z_k)\big)\tilde{L}_k(ds,dz_k), \\ \psi(t\wedge\zeta_n) = \psi_\lambda + \int_\lambda^{t\wedge\zeta_n} \big(-d_2 A\psi(s) - a_2\psi(s) - b_1\Gamma(\xi(s),\psi(s)) + b_2\Phi(\xi(s)) + f_2(s)\big)ds \\ \qquad + \gamma_1 \sum_{k=1}^{\infty}\int_\lambda^{t\wedge\zeta_n} \big(h_k(s) + \sigma_k^n(\psi(s))\big)dW_k(s) \\ \qquad + \gamma_2 \sum_{k=1}^{\infty}\int_\lambda^{t\wedge\zeta_n}\int_{|z_k|<1} \big(\kappa_k(s) + q_k^n(\psi(s-),z_k)\big)\tilde{L}_k(ds,dz_k). \end{cases} \tag{33}$$

Since $\lim_{n\to\infty} \zeta_n = \infty$ a.s., we can get from (33) that

$$
\begin{cases}
\xi(t) = \xi_\lambda + \int_\lambda^{t\wedge\zeta_n} \big(- d_1 A\xi(s) - a_1\xi(s) + b_1\Gamma(\xi(s),\psi(s)) - b_2\Phi(\xi(s)) + f_1(s)\big)ds \\
\qquad\qquad + \epsilon_1 \sum_{k=1}^{\infty} \int_\lambda^{t\wedge\zeta_n} \big(h_k(s) + \sigma_k(\xi(s))\big)dW_k(s) \\
\qquad\qquad + \epsilon_2 \sum_{k=1}^{\infty} \int_\lambda^{t\wedge\zeta_n} \int_{|z_k|<1} \big(\kappa_k(s) + q_k(\xi(s-),z_k)\big)\tilde{L}_k(ds,dz_k), \\
\psi(t) = \psi_\lambda + \int_\lambda^{t\wedge\zeta_n} \big(- d_2 A\psi(s) - a_2\psi(s) - b_1\Gamma(\xi(s),\psi(s)) + b_2\Phi(\xi(s)) + f_2(s)\big)ds \\
\qquad\qquad + \gamma_1 \sum_{k=1}^{\infty} \int_\lambda^{t\wedge\zeta_n} \big(h_k(s) + \sigma_k^n(\psi(s))\big)dW_k(s) \\
\qquad\qquad + \gamma_2 \sum_{k=1}^{\infty} \int_\lambda^{t\wedge\zeta_n} \int_{|z_k|<1} \big(\kappa_k(s) + q_k^n(\psi(s-),z_k)\big)\tilde{L}_k(ds,dz_k).
\end{cases}
$$

Then φ is a solution of problem (8)-(9).

Next, we prove the uniqueness of the solutions. Assume that φ_1 and φ_2 are two solutions of (8)-(9). Then we will show that for every $T > 0$,

$$\mathbb{P}\big(\|\varphi_1(t) - \varphi_2(t)\|\big) = 0, \text{ for all } t \in [\lambda, \lambda + T]\big) = 1.$$

For every $n \in \mathbb{N}$, we define a stopping time $\widehat{T}_n$ by

$$\widehat{T}_n = \inf\{t \geq \lambda : \|\varphi_1(t)\|_{\ell^2\times\ell^2} > n \text{ or } \|\varphi_2\|_{\ell^2\times\ell^2 n} > n\} \wedge T.$$

Let us define $\varphi(t \wedge \widehat{T}_n) = \varphi^{\widehat{T}_n}(t)$. By (8)-(9) and the Itô formula, we obtain

$$
\begin{aligned}
&\|\xi_1(t\wedge\widehat{T}_n) - \xi_2(t\wedge\widehat{T}_n)\|^2 + \int_\lambda^{t\wedge\widehat{T}_n} \Big(2d_1\|B(\xi_1(s) - \xi_2(s))\|^2 + 2a_1\|\xi_1(s) - \xi_2(s)\|^2\Big)\, ds \\
&+ 2b_2 \int_\lambda^{t\wedge\widehat{T}_n} \big\langle \Phi(\xi_1(s)) - \Phi(\xi_2(s)), \xi_1(s) - \xi_2(s)\big\rangle\, ds \\
=&\|\xi_1(\lambda) - \xi_2(\lambda)\|^2 + 2b_1 \int_\lambda^{t\wedge\widehat{T}_n} \big\langle \Gamma(\xi_1(s),\psi_1(s)) - \Gamma(\xi_2(s),\psi_2(s)), \xi_1(s) - \xi_2(s)\big\rangle\, ds \\
&+ 2\epsilon_1 \sum_{k=1}^{\infty} \int_\lambda^{t\wedge\widehat{T}_n} \big\langle \sigma_k(\xi_1(s)) - \sigma_k(\xi_2(s)), \xi_1(s) - \xi_2(s)\big\rangle\, dW_k(s) \\
&+ \epsilon_1^2 \sum_{k=1}^{\infty} \int_\lambda^{t\wedge\widehat{T}_n} \|\sigma_k(\xi_1(s)) - \sigma_k(\xi_2(s))\|^2\, ds \\
&+ \sum_{k=1}^{\infty} \int_\lambda^{t\wedge\widehat{T}_n} \int_{|z_k|<1} \Big(\|\epsilon_2(q_k(\xi_1(s-),z_k) - q_k(\xi_2(s-),z_k)) + \xi_1(s-) - \xi_2(s-)\|^2 \\
&- \|\xi_1(s-) - \xi_2(s-)\|^2\Big)\tilde{L}_k(ds,dz_k) \\
&+ \epsilon_2^2 \sum_{k=1}^{\infty} \int_\lambda^{t\wedge\widehat{T}_n} \int_{|z_k|<1} \|q_k(\xi_1(s),z_k) - q_k(\xi_2(s),z_k)\|^2 v_k(dz_k)\, ds,
\end{aligned} \tag{34}
$$

and

$$\|\psi_1(t\wedge\widehat{T}_n) - \psi_2(t\wedge\widehat{T}_n)\|^2 + \int_\lambda^{t\wedge\widehat{T}_n} (2d_2\|B(\psi_1(s) - \psi_2(s))\|^2 + 2a_2\|\psi_1(s) - \psi_2(s)\|^2)\, ds$$

$$
\begin{aligned}
&+2b_1\int_\lambda^{t\wedge\widehat{T}_n}\big\langle\Gamma(\xi_1(s),\psi_1(s))-\Gamma(\xi_2(s),\psi_2(s)),\psi_1(s)-\psi_2(s)\big\rangle\,ds\\
=&\|\psi_1(\lambda)-\psi_2(\lambda)\|^2+2b_2\int_\lambda^{t\wedge\widehat{T}_n}\big\langle\Phi(\xi_1(s))-\Phi^n(\xi_2(s)),\psi_1(s)-\psi_2(s)\big\rangle\,ds\\
&+2\gamma_1\sum_{k=1}^\infty\int_\lambda^{t\wedge\widehat{T}_n}\big\langle\sigma_k(\psi_1(s))-\sigma_k(\psi_2(s)),\psi_1(s)-\psi_2(s)\big\rangle\,dW_k(s)\\
&+\gamma_1^2\sum_{k=1}^\infty\int_\lambda^{t\wedge\widehat{T}_n}\|\sigma_k(\psi_1(s))-\sigma_k(\psi_2(s))\|^2\,ds\\
&+\sum_{k=1}^\infty\int_\lambda^{t\wedge\widehat{T}_n}\int_{|z_k|<1}\Big(\|\gamma_2(q_k(\psi_1(s-),z_k)-q_k(\psi_2(s-),z_k))+\psi_1(s-)-\psi_2(s-)\|^2\\
&-\|\psi_1(s-)-\psi_2(s-)\|^2\Big)\tilde{L}_k(ds,dz_k)\\
&+\gamma_1^2\sum_{k=1}^\infty\int_\lambda^{t\wedge\widehat{T}_n}\int_{|z_k|<1}\|q_k(\psi_1(s),z_k)-q_k(\psi_2(s),z_k)\|^2v_k(dz_k)\,ds.
\end{aligned}\tag{35}
$$

By (34)-(35), we get

$$
\begin{aligned}
&\mathbb{E}\Big[\sup_{\lambda\le s\le t}\Big(\|\xi_1(s\wedge\widehat{T}_n)-\xi_2(s\wedge\widehat{T}_n)\|^2+\|\psi_1(s\wedge\widehat{T}_n)-\psi_2(s\wedge\widehat{T}_n)\|^2\Big)\Big]\\
\le&C_4(n+1)\int_\lambda^t\mathbb{E}\Big[\sup_{\lambda\le s\le r}\Big(\|\xi_1(s\wedge\widehat{T}_n)-\xi_2(s\wedge\widehat{T}_n)\|^2+\|\psi_1(s\wedge\widehat{T}_n)-\psi_2(s\wedge\widehat{T}_n)\|^2\Big)\,dr\Big]\\
&+2\epsilon_1\mathbb{E}\Big[\sup_{\lambda\le s\le t\wedge\widehat{T}_n}\Big|\sum_{k=1}^\infty\int_\lambda^s\langle\sigma_k(\xi_1(r))-\sigma_k(\xi_2(r)),\xi_1(r)-\xi_2(r)\rangle\,dW_k(r)\Big|\Big]\\
&+2\gamma_1\mathbb{E}\Big[\sup_{\lambda\le s\le t\wedge\widehat{T}_n}\Big|\sum_{k=1}^\infty\int_\lambda^s\langle\sigma_k(\psi_1(r))-\sigma_k(\psi_2(r)),\psi_1(r)-\psi_2(r)\rangle\,dW_k(r)\Big|\Big]\\
&+\mathbb{E}\Big[\sup_{\lambda\le s\le t\wedge\widehat{T}_n}\Big|\sum_{k=1}^\infty\int_\lambda^s\int_{|z_k|<1}(\|\epsilon_2(q_k(\xi_1(r-),z_k)-q_k(\xi_2(r-),z_k))+\xi_1(r-)-\xi_2(r-)\|^2\\
&-\|\xi_1(r-)-\xi_2(r-)\|^2)\tilde{L}_k(dr,dz_k)\Big|\Big]\\
&+\mathbb{E}\Big[\sup_{\lambda\le s\le t\wedge\widehat{T}_n}\Big|\sum_{k=1}^\infty\int_\lambda^s\int_{|z_k|<1}(\|\gamma_2(q_k(\psi_1(r-),z_k)-q_k(\psi_2(r-),z_k))+\psi_1(r-)-\psi_2(r-)\|^2\\
&-\|\psi_1(r-)-\psi_2(r-)\|^2)\tilde{L}_k(dr,dz_k)\Big|\Big]\\
:=&I_{11}+I_{12}+I_{13}+I_{14}+I_{15}.
\end{aligned}\tag{36}
$$

Similar to Lemma 3.2, we could use Young's inequality and the BDG inequality to get

$$
\begin{aligned}
&I_{12}\le\frac{1}{4}\mathbb{E}\Big[\sup_{\lambda\le s\le t}\|\xi_1^{\widehat{T}_n}(s)-\xi_2^{\widehat{T}_n}(s)\|^2\Big]+4\epsilon_1^2C_{BDG}^2c_2^2\mathbb{E}\Big[\int_\lambda^t\|\xi_1^{\widehat{T}_n}(s)-\xi_2^{\widehat{T}_n}(s)\|^2\Big]\,dr,\\
&I_{13}\le\frac{1}{4}\mathbb{E}\Big[\sup_{\lambda\le s\le t}\|\psi_1^{\widehat{T}_n}(s)-\psi_2^{\widehat{T}_n}(s)\|^2]+4\gamma_1^2C_{BDG}^2c_2^2\int_\lambda^t\mathbb{E}\Big[\sup_{\lambda\le s\le r}\|\psi_1^{\widehat{T}_n}(s)-\psi_2^{\widehat{T}_n}(s)\|^2\Big]\,dr,\\
&I_{14}\le\frac{1}{4}\mathbb{E}\Big[\sup_{\lambda\le s\le t}\|\xi_1^{\widehat{T}_n}(s)-\xi_2^{\widehat{T}_n}(s)\|^2\Big]
\end{aligned}\tag{37}
$$

$$
\begin{aligned}
&+\epsilon_2^2(4C_{BDG}^2+C_b)C_5(n+1)\int_\lambda^t \mathbb{E}\Big[\sup_{\lambda\le s\le r}\|\xi_1^{\widehat{T}_n}(s)-\xi_2^{\widehat{T}_n}(s)\|^2\Big]dr,\\
I_{15}\le{}&\frac{1}{4}\mathbb{E}\Big[\sup_{\lambda\le s\le t}\|\psi_1^{\widehat{T}_n}(s)-\psi_2^{\widehat{T}_n}(s)\|^2\Big]\\
&+\gamma_2^2(4C_{BDG}^2+C_b)C_5(n+1)\int_\lambda^t \mathbb{E}\Big[\sup_{\lambda\le s\le r}\|\psi_1^{\widehat{T}_n}(s)-\psi_2^{\widehat{T}_n}(s)\|^2\Big]dr.
\end{aligned}
$$

By (36)-(37), we obtain

$$
\begin{aligned}
&\mathbb{E}\Big[\sup_{\lambda\le s\le t}\|\varphi_1^{\widehat{T}_n}(s)-\varphi_2^{\widehat{T}_n}(s)\|^2\Big]\\
\le{}&2\mathbb{E}\Big[\|\varphi_1(\lambda)-\varphi_2(\lambda)\|^2\Big]+C_6(n+1)\int_\lambda^t \mathbb{E}\Big[\sup_{\lambda\le s\le r}\|\varphi_1^{\widehat{T}_n}-\varphi_2^{\widehat{T}_n}\|^2\Big]dr,
\end{aligned}
$$

where

$$
C_6(n+1)=2\Big(C_4(n+1)+C_5(n+1)\Big(4C_{BDG}^2(\epsilon_1^2+\gamma_1^2)+(4C_{BDG}^2+C_b)(\epsilon_2^2+\gamma_2^2)\Big)\Big).
$$

Applying Gronwall's lemma, for all $t\in[\lambda,\lambda+T]$, we get

$$
\mathbb{E}\Big[\sup_{\lambda\le s\le t}\|\varphi_1^{\widehat{T}_n}(s)-\varphi_2^{\widehat{T}_n}(s)\|^2\Big]\le 2e^{C_6(n+1)T}\mathbb{E}\Big[\|\varphi_1(\lambda)-\varphi_2(\lambda)\|^2\Big].
$$

It then follows from $\varphi_1(\lambda)=\varphi_2(\lambda)$ in $L^2(\Omega,\ell^2\times\ell^2)$ that

$$
\mathbb{E}\Big[\|\varphi_1^{\widehat{T}_n}(t)-\varphi_2^{\widehat{T}_n}(t)\|^2\Big]=0.
$$

This implies that $\|\varphi_1(t\wedge\widehat{T}_n)-\varphi_2(t\wedge\widehat{T}_n)\|_{\ell^2\times\ell^2}=0$ for all $t\in[\lambda,\lambda+T]$ $\mathbb{P}$-a.s. Let $n\to\infty$, by the continuity of φ_1 and φ_2, we obtain that $\|\varphi_1(t)-\varphi_2(t)\|_{\ell^2\times\ell^2}=0$ for all $t\in[\lambda,\lambda+T]$ $\mathbb{P}$-a.s. This implies $\mathbb{P}(\varphi_1(t)=\varphi_2(t),$ for all $t\ge\lambda)=1$. This ensures the uniqueness of the solutions. □

4 Weak pullback mean random attractors

The goal of this segment is to establish the existence and uniqueness of weak pullback mean random attractors for problem (1)-(2).

assuming that the coefficient $\|\delta\|^2$ is enough small,

$$
\|\delta\|^2\le\frac{\varsigma}{8\beta^2c_0}, \tag{38}
$$

where $\varsigma=a_1\wedge a_2$, $c_0=b_2\epsilon_1^2+b_2\epsilon_2^2+b_1\gamma_1^2+b_1\gamma_2^2$. For every $t>0$ and $\lambda\in\mathbb{R}$, we define a mapping $\Psi(t,\lambda):L^2(\Omega,\mathcal{F}_\lambda;\ell^2\times\ell^2)\to L^2(\Omega,\mathcal{F}_{\lambda+t};\ell^2\times\ell^2)$ by

$$
\Psi(t,\lambda)(\varphi_\lambda)=\varphi(\lambda+t,\lambda,\varphi_\lambda),\ \ \forall\varphi_\lambda\in L^2(\Omega,\mathcal{F}_\lambda;\ell^2\times\ell^2),
$$

where $\varphi(t+\lambda,\lambda,\varphi_\lambda)$ is the solution of (8)-(9) at time λ with initial value φ_λ. For any $t,s\ge0$ and $\lambda\in\mathbb{R}$,

$$
\Psi(t+s,\lambda)=\Psi(t,s+\lambda)\circ\Psi(s,\lambda),
$$

Ψ is a mean random dynamical system related to (8)-(9) on

$$L^2(\Omega, \mathcal{F}; \ell^2 \times \ell^2) \text{ over } (\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\in\mathbb{R}}, \mathbb{P}).$$

Let $\mathcal{D} = \{C = \{C(\lambda) \subseteq L^2(\Omega, \mathcal{F}_\lambda, \ell^2 \times \ell^2) : C(\lambda) \neq \emptyset \text{ is bounded}, \lambda \in \mathbb{R}\}\}$ be a collection of all families of nonempty bounded sets satisfying :

$$\lim_{\lambda\to-\infty} e^{\varsigma\lambda}\|C(\lambda)\|^2_{L^2(\Omega,\mathcal{F}_\lambda,\ell^2\times\ell^2)} = 0,$$

where $\|C(\lambda)\|^2_{L^2(\Omega,\mathcal{F}_\lambda;\ell^2\times\ell^2)} = \sup_{\varphi\in C(\lambda)} \|\varphi\|^2_{L^2(\Omega,\mathcal{F}_\lambda;\ell^2\times\ell^2)}$.

Next, we are going to demonstrate that Ψ has a unique weak $\mathcal{D}$-pullback mean random attractors in the space $L^2(\Omega, \ell^2 \times \ell^2)$. We further assume

$$\int_{-\infty}^{\lambda} e^{\varsigma s}\mathbb{E}\left[\|f_1(s)\|^2 + \|f_2(s)\|^2 + \sum_{k=1}^{\infty}\|h_k(s)\|^2 + \sum_{k=1}^{\infty}\|\kappa_k^2(s)\|^2\right] ds < \infty, \quad \forall\, \lambda \in \mathbb{R}. \tag{39}$$

In the following, we present the uniform estimates on the solutions of (8)-(9) in $L^2(\Omega, \ell^2 \times \ell^2)$.

Lemma 4.1. *For every $\lambda \in \mathbb{R}$ and $C = \{C(t)\}_{t\in\mathbb{R}} \in \mathcal{D}$, there exists $T = T(\lambda, C) > 0$ such that for all $t \geq T$, the solution φ of (8)-(9) satisfies*

$$\begin{aligned}&\mathbb{E}\left[\|\varphi(\lambda, \lambda - t, \varphi_\lambda)\|^2\right]\\ \leq& M_4 \int_{-\infty}^{\lambda} e^{\varsigma(s-\lambda)}\mathbb{E}\left[\|f_1(s)\|^2 + \|f_2(s)\|^2 + \sum_{k=1}^{\infty}\|h_k(s)\|^2 + \sum_{k=1}^{\infty}\|\kappa_k(s)\|^2\right] ds + M_5,\end{aligned} \tag{40}$$

where $\varphi_\lambda \in C(\lambda - t)$, $M_4 = (\frac{2}{\varsigma} + 2b_2\epsilon_1^2 + 2b_1\gamma_1^2 + 2b_1\gamma_2^2 + 2b_2\epsilon_2^2)$ and $M_5 = e^{-\varsigma t}\mathbb{E}\left[b_2\|\xi_\lambda\|^2 + b_1\|\psi_\lambda\|^2\right] + \frac{1}{2}$.

Proof. By (8) and the Itô formula, we obtain

$$\begin{aligned}&d\left[b_2\|\xi(s)\|^2\right] + \left[2d_1b_2\|B\xi(s)\|^2 + 2a_1b_2\|\xi(s)\|^2 + 2b_2^2\langle\Phi(\xi(s)), \psi(s)\rangle\right] ds\\ =&2b_2\langle f_1(s), \xi(s)\rangle\, ds + 2b_1b_2\langle\Gamma(\xi(s), \psi(s)), \xi(s)\rangle\, ds\\ &+ b_2\epsilon_1^2\sum_{k=1}^{\infty}\|h_k(s) + \sigma_k(\xi(s))\|^2\, ds + 2b_2\epsilon_1\sum_{k=1}^{\infty}\langle h_k(s) + \sigma_k(\xi(s)), \xi(s)\rangle\, dW_k(s)\\ &+ b_2\sum_{k=1}^{\infty}\int_{|z_k|<1}\left(\|\xi(s-) + \epsilon_2(q_k(\xi(s-), z_k) + \kappa_k(s))\|^2 - \|\xi(s-)\|^2\right)\tilde{L}_k(ds, dz_k)\\ &+ b_2\epsilon_2^2\sum_{k=1}^{\infty}\int_{|z_k|<1}\|q_k(\xi(s), z_k) + \kappa_k(s)\|^2 v_k(dz_k)\, ds,\end{aligned} \tag{41}$$

and

$$\begin{aligned}&d\left[b_1\|\psi(s)\|^2\right] + \left[2d_2b_1\|B\psi(s)\|^2 + 2a_2b_1\|\psi(s)\|^2 + 2b_1^2\left(\Gamma(\xi(s), \psi(s)), \psi(s)\right)\right] ds\\ =&2b_1\langle f_2(s), \psi_n(s)\rangle\, ds + 2b_2b_1\langle\Phi^n(\xi_n(s)), \psi_n(s)\rangle\, ds\\ &+ b_1\gamma_1^2\sum_{k=1}^{\infty}\|h_k(s) + \sigma_k(\psi(s))\|^2\, ds + 2b_1\gamma_1\sum_{k=1}^{\infty}\langle h_k(s) + \sigma_k(\psi(s)), \psi(s)\rangle dW_k(s)\end{aligned}$$

$$+ b_1 \sum_{k=1}^{\infty} \int_{|z_k|<1} \Big(\|\psi(s-) + \gamma_2(q_k(\psi(s-), z_k) + \kappa_k(s))\|^2 - \|\psi(s-)\|^2 \Big) \tilde{L}_k(ds, dz_k)$$
$$+ b_1 \gamma_2^2 \sum_{k=1}^{\infty} \int_{|z_k|<1} \|q_k(\psi(s), z_k) + \kappa_k(s)\|^2 v_k(dz_k)\, ds. \tag{42}$$

Taking expectation of (41)+(42), we get

$$\begin{aligned}
\frac{d}{ds}\mathbb{E}\Big[\|b_2\xi(s)\|^2 + \|b_1\psi(s)\|^2\Big] &= \mathbb{E}\Big[-2d_1b_2\|B\xi(s)\|^2 - 2a_1b_2\|\xi(s)\|^2 - 2b_2^2\langle \Phi(\xi(s)), \xi(s)\rangle \Big] \\
&+ \mathbb{E}\Big[-2d_2b_1\|B\psi(s)\|^2 - 2a_2b_1\|\psi(s)\|^2 - 2b_1^2(\Gamma(\xi(s), \psi(s)), \psi(s)) \Big] \\
&+ 2b_2\mathbb{E}\Big[\langle f_1(s), \xi(s)\rangle\Big] + 2b_1b_2\mathbb{E}\Big[\langle \Gamma(\xi(s), \psi(s)), \xi(s)\rangle\Big] \\
&+ b_2\epsilon_1^2 \sum_{k=1}^{\infty} \mathbb{E}\Big[\|h_k(s) + \sigma_k(\xi(s))\|^2\Big] + b_1\gamma_1^2 \sum_{k=1}^{\infty} \mathbb{E}\Big[\|h_k(s) + \sigma_k(\psi(s))\|^2\Big] \\
&+ 2b_1\mathbb{E}\Big[\langle f_2(s), \psi_n(s)\rangle\Big] + 2b_2b_1\mathbb{E}\Big[\langle \Phi(\xi(s)), \psi(s)\rangle\Big]\, ds \\
&+ b_1\gamma_2^2 \sum_{k=1}^{\infty} \int_{|z_k|<1} \mathbb{E}\Big[\|q_k(\psi(s), z_k) + \kappa_k(s)\|^2 v_k(dz_k)\Big] \\
&+ b_2\epsilon_2^2 \sum_{k=1}^{\infty} \int_{|z_k|<1} \mathbb{E}\Big[\|q_k(\xi(s), z_k) + \kappa_k(s)\|^2 v_k(dz_k)\Big].
\end{aligned} \tag{43}$$

By (43), we have

$$\begin{aligned}
&\frac{d}{ds}\mathbb{E}\Big[\|b_2\xi(s)\|^2 + \|b_1\psi(s)\|^2\Big] \\
\le & -2\varsigma\mathbb{E}\Big[b_2\|\xi(s)\|^2 + b_1\|\psi(s)\|^2\Big] + b_2\epsilon_1^2 \sum_{k=1}^{\infty} \mathbb{E}\Big[\|h_k(s) + \sigma_k(\xi(s))\|^2\Big] \\
& + b_1\gamma_1^2 \sum_{k=1}^{\infty} \mathbb{E}\Big[\|h_k(s) + \sigma_k(\psi(s))\|^2\Big] + b_1\gamma_2^2 \sum_{k=1}^{\infty} \int_{|z_k|<1} \mathbb{E}\Big[\|q_k(\psi(s), z_k) + \kappa_k(s)\|^2 v_k(dz_k)\Big] \\
& + b_2\epsilon_2^2 \sum_{k=1}^{\infty} \int_{|z_k|<1} \mathbb{E}\Big[\|q_k(\xi(s), z_k) + \kappa_k(s)\|^2 v_k(dz_k)\Big] + 2b_2\mathbb{E}\Big[\langle f_1(s), \xi(s)\rangle\Big] \\
& + 2b_1\mathbb{E}\Big[\langle f_2(s), \psi_n(s)\rangle\Big],
\end{aligned} \tag{44}$$

where $\varsigma = (a_1 \wedge a_2) > 0$. By (44), we get

$$2b_2\mathbb{E}\Big[\langle f_1(s), \xi(s)\rangle\Big] \le \frac{\varsigma}{2}\mathbb{E}\Big[b_2\|\xi(s)\|^2\Big] + \frac{2}{\varsigma}b_2\mathbb{E}\Big[\|f_1(s)\|^2\Big],$$

and

$$2b_1\mathbb{E}\Big[\langle f_2(s), \psi(s)\rangle\Big] \le \frac{\varsigma}{2}\mathbb{E}\Big[b_1\|\psi(s)\|^2\Big]\, ds + \frac{2}{\varsigma}b_1\mathbb{E}\Big[\|f_2(s)\|^2\Big].$$

By (44), we get

$$\begin{aligned}
& b_2\epsilon_1^2 \sum_{k=1}^{\infty} \mathbb{E}\Big(\|h_k(s) + \sigma_k(\xi(s))\|^2\Big) \\
\le & 2b_2\epsilon_1^2 \sum_{k=1}^{\infty} \mathbb{E}\Big[\|h_k(s)\|^2\Big] + 4b_2\epsilon_1^2\beta^2\|\delta\|^2 + 4b_2\epsilon_1^2\beta^2\|\delta\|^2\mathbb{E}\Big[\|\xi(s)\|^2\Big],
\end{aligned}$$

and

$$
\begin{aligned}
& b_1\gamma_1^2\sum_{k=1}^{\infty}\mathbb{E}\left[\|h_k(s)+\sigma_k(\psi(s))\|^2\right] \\
\leq & 2b_1\gamma_1^2\sum_{k=1}^{\infty}\mathbb{E}\left[\|h_k(s)\|^2\right]+4b_1\gamma_1^2\beta^2\|\delta\|^2+4b_1\gamma_1^2\beta^2\|\delta\|^2\mathbb{E}\left[\|\psi(s)\|^2\right].
\end{aligned}
$$

By (44), we get

$$
\begin{aligned}
& b_2\epsilon_2^2\sum_{k=1}^{\infty}\int_{|z_k|<1}\mathbb{E}\left[\|q_k(\xi(s),z_k)+\kappa_k(s)\|^2 v_k(dz_k)\right] \\
\leq & 2b_2\epsilon_2^2\sum_{k=1}^{\infty}\mathbb{E}\left[\|q_k(\xi(s),z_k)\|^2\right]v_k(dz_k)+2b_2\epsilon_2^2\sum_{k=1}^{\infty}\mathbb{E}\left[\|\kappa_k(s)\|^2\right] \\
\leq & 4b_2\epsilon_2^2\beta^2\|\delta\|^2+4b_2\epsilon_2^2\beta^2\|\delta\|^2\mathbb{E}\left[\|\xi(s)\|^2\right]+2b_2\epsilon_2^2\sum_{k=1}^{\infty}\mathbb{E}\left[\|\kappa_k(s)\|^2\right],
\end{aligned}
$$

and

$$
\begin{aligned}
& b_1\gamma_2^2\sum_{k=1}^{\infty}\int_{|z_k|<1}\mathbb{E}\left[\|q_k(\psi(s),z_k)+\kappa_k(s)\|^2 v_k(dz_k)\right] \\
\leq & 4b_1\gamma_2^2\beta^2\|\delta\|^2+4b_1\gamma_2^2\beta^2\|\delta\|^2\mathbb{E}\left[\|\psi(s)\|^2\right]+2b_1\gamma_2^2\sum_{k=1}^{\infty}\mathbb{E}\left[\|\kappa_k(s)\|^2\right].
\end{aligned} \tag{45}
$$

From (44)-(45), we have

$$
\begin{aligned}
& \frac{d}{ds}\mathbb{E}\left[b_1\|\psi(s)\|^2+b_2\|\xi(s)\|^2\right] \\
\leq & -2\varsigma\mathbb{E}\left(b_2\|\xi(s)\|^2+b_1\|\psi(s)\|^2\right)+\left(4b_2\epsilon_1^2+4b_1\gamma_1^2+4b_2\epsilon_2^2+4b_1\gamma_2^2\right)\beta^2\|\delta\|^2 \\
& +M_4\mathbb{E}\Big[\sum_{k=1}^{\infty}\|\kappa_k(s)\|^2+\sum_{k=1}^{\infty}\|h_k(s)\|^2+b_1\|f_2(s)\|^2+b_2\|f_1(s)\|^2\Big] \\
& +(\frac{\varsigma}{2}+4\beta^2\|\delta\|^2(b_2\epsilon_1^2+b_2\epsilon_2^2+b_1\gamma_1^2+b_1\gamma_2^2))\mathbb{E}\left[b_2\|\xi(s)\|^2+b_1\|\psi(s)\|^2\right].
\end{aligned} \tag{46}
$$

where $M_4=(\frac{2}{\varsigma}+2b_2\epsilon_1^2+2b_1\gamma_1^2+2b_1\gamma_2^2+2b_2\epsilon_2^2)$.

By (38) and (46), we have

$$
\begin{aligned}
& \frac{d}{ds}\mathbb{E}\left[b_1\|\psi(s)\|^2+b_2\|\xi(s)\|^2\right]+\varsigma\mathbb{E}\left[b_2\|\xi(s)\|^2+b_1\|\psi(s)\|^2\right] \\
\leq & M_4\mathbb{E}\Big[\sum_{k=1}^{\infty}\|\kappa_k(s)\|^2+\sum_{k=1}^{\infty}\|h_k(s)\|^2+b_1\|f_2(s)\|^2+b_2\|f_1(s)\|^2\Big] \\
& +\left(4b_2\epsilon_1^2+4b_1\gamma_1^2+4b_2\epsilon_2^2+4b_1\gamma_2^2\right)\beta^2\|\delta\|^2.
\end{aligned} \tag{47}
$$

Multiplying (47) by $e^{\varsigma t}$, and then integrating on $(\lambda-t,\lambda)$, we have

$$
\begin{aligned}
& \mathbb{E}\left[b_1\|\psi(\lambda,\lambda-t,\psi_\lambda)\|^2+b_2\|\xi(\lambda,\lambda-t,\xi_\lambda)\|^2\right] \\
\leq & M_4\int_{\lambda-t}^{\lambda}e^{\varsigma(s-\lambda)}\mathbb{E}\Big[\sum_{k=1}^{\infty}\|\kappa_k(s)\|^2+\sum_{k=1}^{\infty}\|h_k(s)\|^2+b_1\|f_2(s)\|^2+b_2\|f_1(s)\|^2\Big]\,ds
\end{aligned}
$$

$$
\begin{aligned}
& e^{-\varsigma t}\mathbb{E}\left[b_2\|\xi_\lambda|^2+b_1\|\psi_\lambda\|^2\right]+\left(4b_2\epsilon_1^2+4b_1\gamma_1^2+4b_2\epsilon_2^2+4b_1\gamma_2^2\right)\frac{\beta^2\|\delta\|^2}{\varsigma}\\
\leq & M_4\int_{\lambda-t}^{\lambda} e^{\varsigma(s-\lambda)}\mathbb{E}\Big[\sum_{k=1}^{\infty}\|\kappa_k(s)\|^2+\sum_{k=1}^{\infty}\|h_k(s)\|^2+b_1\|f_2(s)\|^2+b_2\|f_1(s)\|^2\Big]\,ds\\
& +e^{-\varsigma t}\mathbb{E}\left[b_2\|\xi_\lambda\|^2+b_1\|\psi_\lambda\|^2\right]+\frac{1}{2}\\
\leq & M_4\int_{-\infty}^{\lambda} e^{\varsigma(s-\lambda)}\mathbb{E}\left[\|f_1(s)\|^2+\|f_2(s)\|^2+\sum_{k=1}^{\infty}\|h_k(s)\|^2+\sum_{k=1}^{\infty}\|\kappa_k(s)\|^2\right]ds+M_5.
\end{aligned}
\tag{48}
$$

where $M_5=e^{-\varsigma t}\mathbb{E}\left[b_2\|\xi_\lambda\|^2+b_1\|\psi_\lambda\|^2\right]+\frac{1}{2}$.

Since $\varphi_\lambda\in C(\lambda-t)$, by (48), we get (40). □

Next, we elaborate the weak $\mathcal{D}$-pullback mean random attractors of dynamical system (Ψ).

Theorem 4.2. *Assume that* (38)-(39) *and* (5)-(6) *hold. Then the mean random dynamical system Ψ for the stochastic system* (8)-(9) *possesses a unique weak $\mathcal{D}$-pullback mean random attractor $\mathcal{B}=\{\mathcal{B}(\lambda):\lambda\in\mathbb{R}\}\in\mathcal{D}$ in $L^2(\Omega,\mathcal{F};\ell^2\times\ell^2)$, that is:*
(1) $\mathcal{B}(\lambda)$ is a weakly compact subset of $L^2(\Omega,\mathcal{F}_\lambda;\ell^2\times\ell^2)$ for every $\lambda\in\mathbb{R}$.
(2) $\mathcal{B}$ is a $\mathcal{D}$-pullback weakly attracting set of Ψ.
(3) $\mathcal{B}$ is the minimal element of $\mathcal{D}$ with properties (1) and (2); that is, if $C=\{C(\lambda):\lambda\in\mathbb{R}\}\in\mathcal{D}$ is a $\mathcal{D}$-pullback weakly compact weakly attracting set of Ψ, then $\mathcal{B}\subseteq C(\lambda)$ for all $\lambda\in\mathbb{R}$.

Proof. For every $\lambda\in\mathbb{R}$, we denote by

$$
\mathscr{K}=\{\mathscr{K}(\lambda):\lambda\in\mathbb{R}\}\ \text{ with }\ \mathscr{K}(\lambda):=\{\varphi\in L^2(\Omega,\mathcal{F}_\lambda,\ell^2\times^2):\mathbb{E}(\|\varphi\|^2)\leq\mathscr{R}(\lambda)\},
$$

where

$$
\mathscr{R}(\lambda)=M_4\int_{-\infty}^{\lambda}e^{\varsigma(s-\lambda)}\mathbb{E}\left(\|f_1(s)\|^2+\|f_2(s)\|^2+\sum_{k=1}^{\infty}\|h_k(s)\|^2+\sum_{k=1}^{\infty}\|\kappa_k(s)\|^2\right)ds+M_5.
$$

Since $\mathscr{K}(\lambda)$ is weakly compact in $L^2(\Omega,\mathcal{F};\ell^2\times\ell^2)$, According to (39), we obtain

$$
\lim_{\lambda\to-\infty}e^{\varsigma\lambda}\|\mathscr{K}(\lambda)\|^2_{L^2(\Omega,\mathcal{F}_\lambda;\ell^2\times\ell^2)}=\lim_{\lambda\to-\infty}e^{\varsigma\lambda}\mathscr{R}(\lambda)=0.
$$

This suggests that $\mathscr{K}\in\mathcal{D}$. Moreover, we deduce from Lemma 4.1 that $\mathscr{K}$ is a $\mathcal{D}$-pullback absorbing set for Ψ in $L^2(\Omega,\mathcal{F};\ell^2\times\ell^2)$. By [39, Theorem 2.13], we get the existence as well as uniqueness of weak $\mathcal{D}$-pullback mean random attractor $\mathcal{B}\in\mathcal{D}$ of Ψ. □

5 Declarations

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Competing Interests

The authors declare that they have no competing interests.

Ethical Approval

Not applicable.

Author's Contributions

The final version of the manuscript was read and approved by Guofu Li, who also prepared the original draft and conducted the review and editing.

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Not applicable.

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