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# Global attractors of the delay 2D Navier-Stokes equations on unbounded Channel-like domains

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## Abstract

This paper studies the global attractors of 2D Navier-Stokes equations with delay defined in unbounded Channel-like domains. To overcome the non-compactness of solutions, we will use the uniform tail-ends estimates of the solutions by establishing all the solutions are uniformly small.

**Key words**: Navier-Stokes equations, global attractors, uniform tail-ends estimates
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## 1 Introduction

This paper considers the 2D Navier-Stokes equations defined on $\mathcal{O} = \mathbb{R} \times (0, d)$:

$$\begin{cases} u_t - \Delta u + (u \cdot \nabla)u = g(u_t) + \nabla p + f(x), \ \ \mathrm{div} u = 0, \\ u(t,x) = 0, \ \ t > 0, x \in \partial\mathcal{O}, \\ u(0,x) = u_0(x), \ \ x \in \mathcal{O}, \\ u(x,t) = \phi(t-\tau, x), \ \ t \in (\tau - h, \tau), x \in \mathcal{O}, \end{cases} \tag{1}$$

where $u$ and $p$ are the velocity and the pressure of the fluid. $f \in L^2(\mathcal{O})$ is given, $g$ is a Lipschitz nonlinear function with delay, $r \geq 0$ is constants,

$$u_t(\theta) = u(t+\theta), \ \forall\, \theta \in (-r, 0).$$

When (1) does not contain the delay term, Wang has investigated the existence of the global attractors of the 2D Navier-Stokes equations on the Channel-like unbounded domains in [1]. In recent years, significant results have been investigated in the study of attractors for 2D Navier-Stokes with delay. Carabello has studied the attractors of 2D Navier-Stokes equations

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with delay on bounded domains in reference [2], and then proved the existence of pull-back attractors for non-autonomous delay 2D Navier-Stokes equations on unbounded domains. This paper mainly studied the asymptotic compactness of the solutions of (1) on unbounded channel-like domains $\mathcal{O}$. We use the existence, stability, and convergence of the global attractors to show the long-time asymptotic behavior of the solutions. We have known Sobolev embeddings on unbounded domains are no longer compact (see reference [3]). In unbounded domains, the main difficulty is the non-compactness of Sobolev embeddings.

In fact, we can overcome the difficulty by Rosa's idea of the Ball energy equations proposed (in reference [4, 5]), which is the method to establish asymptotic compactness of (1) solutions when $\mathcal{O}$ is unbounded in the phase space $L^2(\mathcal{O}, \mathbb{R}^2)$ (see references [2, 6, 7, 8, 9, 10]). In order to overcome the non-compactness of Sobolev embeddings in unbounded domains, we used uniform tail-ends estimates methods for the solutions proposed in reference [11] to prove the asymptotic compactness of the solution to the reaction-diffusion equations on $\mathbb{R}^n$(see references [12, 13]).

In this paper, we will derive the uniform tail-ends estimates of solutions under supposed conditions $p = 0$ and div$u = 0$. We will use uniform tail-ends estimates of the scalar stream function for the Navier-Stokes equations (see [6, 12]). By uniform estimates of solutions, we will prove the asymptotic compactness of solutions of (1) defined in $\mathcal{O}$ and the existence of global attractors for (1) in $H$.

This paper is organized as follows, we will recall some basic concepts and results in Section 2. In Section 3, we will prove the uniform estimates of 2D Navier-Stokes equations of the solutions in $H$ and $V$. We will prove the asymptotic compactness of solutions and the existence of global attractors in $H$.

In the paper, we denote norm $\|\cdot\|$ and inner product $(\cdot,\cdot)$ of the $L^2(\mathcal{O}, \mathbb{R}^2)$. We also denote norm $\|u\|_V = \|\nabla u\|$ for $u \in V$ of $V$ and dual space $V^*$ is labeled as $\langle \cdot, \cdot \rangle$ and denote the norm $(\cdot,\cdot)_{C_H}$ and the inner product $\|\cdot\|_{C_H}$ of the $C_H = (-r, 0; H)$.

## 2 Preliminaries

In this section, we review some basic results and knowledge.

The Poincare inequality:

$$\|\nabla u\|^2 \geq \lambda \|u\|^2, \quad \forall u \in H_0^1(\mathcal{O}, \mathbb{R}^2)\ \lambda > 0. \tag{2}$$

Let $u, v, w \in V$, we denoted

$$b(u, v, w) = \sum_{i,j=1}^{2} \int_{\mathcal{O}} u_i \frac{\partial v_j}{\partial x_i} w_j dx. \tag{3}$$

For all $u, v, w \in V$,

$$b(u, v, w) = -b(u, w, v),\ b(u, v, v) = 0$$

and

$$|b(u, v, w)| \leq \|u\|_{L^4(\mathcal{O})} \|\nabla v\| \|w\|_{L^4(\mathcal{O})} \leq c\|u\|_V \|v\|_V \|w\|_V, \tag{4}$$

with every $c > 0$.

By(3), denote a bilinear operator $B : V \times V \to V^*$, for every $u, v, w \in V$,

$$\langle B(u, v), w \rangle = b(u, v, w),$$

By (4) for all $u, v \in V$, we have

$$\|B(u, v)\|_{V^*} \leq c\|u\|_V \|v\|_V,$$

For $u \in V \cap H^2(\mathcal{O}, \mathbb{R}^2)$, we also have

$$\|B(u,u)\| \le c\|u\|^{\frac{1}{2}}\|\nabla u\|\|\Delta u\|^{\frac{1}{2}}.$$

We suppose

$$\alpha = \lambda - L_g.$$

We will make appropriate suppose for the delay term. Let $g : (-r, 0; H) \to L^2(\mathcal{O}, \mathbb{R}^2)$ satisfy the following conditions:

(1) $g(0) = 0$;

(2) there exists $L_g > 0$, such that $\forall\, \xi, \eta \in (-r, 0; H)$

$$|g(\xi) - g(\eta)| \le L_g \|\xi - \eta\|_{C_H}. \tag{5}$$

## 3 Uniform estimates of solutions of 2D Navier-Stokes equations

We will prove the uniform estimates of solutions of (1) in $H$ and $V$.

**Lemma 3.1.** *Let (5) holds, then for every $u_0 \in H$, the solution $\tilde{u}$ of (1) satisfies for all $t \ge 0$,*

$$\|\tilde{u}(t)\|^2 + \frac{3}{2}\int_0^t e^{-4L_g^2(s-t)}\|\tilde{u}(s)\|_V^2 ds \le e^{4L_g^2 t}\|\tilde{u}_0\|^2 + C_1,$$

*and for all $t \ge r \ge 0$,*

$$\frac{3}{2}\int_0^t e^{-4L_g^2(s-t)}\|\tilde{u}(s)\|_V^2 ds \le e^{4L_g^2 r}\|\tilde{u}_0\|^2 + C_1(1 + t - r),$$

*where $C_1 > 0$ is a constant independent of $t, r$ or $u_0$.*

*Proof of Lemma 3.1.* Suppose $t > 0$,

$$\|\partial_t \tilde{u}(t)\|^2 + 2\|\nabla \tilde{u}(t)\|^2 = 2(g(\tilde{u}_t), \tilde{u}(t)) + 2(f, \tilde{u}(t)). \tag{6}$$

Along with (5), we obtain

$$2(g(u_t), \tilde{u}(t)) \le 2L_g\|\tilde{u}_t\|_{c_H}\|\tilde{u}\| \le \frac{1}{2}\|\tilde{u}_t\|_{C_H}^2 + 2L_g^2\|\tilde{u}\|^2.$$

and by Young's inequality, we get

$$2(f, \tilde{u}(t)) \le \frac{\|f\|^2}{2L_g^2} + 2L_g^2\|\tilde{u}\|^2. \tag{7}$$

By (2) and (6)-(7) for $t > 0$, we have

$$\|\partial_t \tilde{u}(t)\|^2 + 2\|\nabla \tilde{u}(t)\|^2 \le \frac{1}{2}\|\tilde{u}_t\|_{C_H} + 4L_g^2\|\tilde{u}\|^2 + \frac{\|f\|^2}{L_g^2}. \tag{8}$$

Solve (8) to obtain for all $t \ge 0$,

$$\|\tilde{u}(t)\|^2 + 2\int_0^t e^{-4L_g^2(s-t)}\|\nabla \tilde{u}(s)\|^2 ds \le e^{4L_g^2 t}\|\tilde{u}_0\|^2 + e^{4L_g^2 t}\frac{\|f\|^2}{2L_g^2}. \tag{9}$$

Integrating (8) on (r,t) with $0 \le r \le t$, by (9) we have

$$2\int_r^t \|\nabla \tilde{u}(s)\|^2 ds \le \|\tilde{u}(r)\|^2 + \frac{\|f\|^2}{2L_g^2}(t - r) \le e^{4L_g^2 r}\|\tilde{u}_0\|^2 + e^{4L_g^2 t}\frac{\|f\|^2}{2L_g^2} + \frac{\|f\|^2}{2L_g^2}(t - r).$$

This completes the proof. □

We next prove the uniform estimates of solutions of (1) in $V$ for $t > 0$ with initial data in $H$.

**Lemma 3.2.** *Let (5) holds, for every $u_0 \in H$, the solutions $\overline{u}$ of (1) satisfies for all $t \geq 1$,*

$$\|\nabla\overline{u}(t)\|^2 + \int_t^{t+1} \|\Delta\overline{u}(s)\|^2 ds + \int_t^{t+1} \|\partial_t\overline{u}(s)\|^2 ds \leq C_2,$$

*where $C_2 > 0$ is a constant dependent of $\lambda, L_g$ and $u_0$.*

*Proof of Lemma 3.2.* By a limiting process, we will derive the uniform estimates for $t \geq 0$,

$$\|\partial_t\nabla\overline{u}(t)\|^2 + 2\|\Delta\overline{u}(t)\|^2 = 2(B(\overline{u}(t),\overline{u}(t)),\Delta\overline{u}(t)) - 2(g(\overline{u}_t(t)),\Delta\overline{u}(t)) - 2(f,\Delta\overline{u}(t)). \quad (10)$$

We first estimate the first term of (10)

$$\begin{aligned} 2(B(\overline{u}(t),\overline{u}(t)),\Delta\overline{u}(t)) \leq & \|u\|_{L^4(\mathcal{O},\mathbb{R}^2)}\|\nabla\overline{u}\|_{L^4(\mathcal{O},\mathbb{R}^4)}\|\Delta\overline{u}\| \\ \leq & c_1\|\overline{u}\|^{\frac{1}{2}}\|\nabla\overline{u}\|\|\Delta\overline{u}\|^{\frac{3}{2}} \leq \frac{1}{2}\|\Delta\overline{u}\|^2 + c_2\|\overline{u}\|^2\|\nabla\overline{u}\|^4, \end{aligned}$$

where $c_2 > 0$ on $\mathcal{O}$. Combining (5) and Young's inequality, we have

$$-2(g(\overline{u}_t(t)),\delta\overline{u}(t)) - 2(f,\Delta\overline{u}(t)) \leq \frac{1}{4}\|\overline{u}_t\|^2_{C_H} + 8\|f\|^2 + 8L_g^2\|\nabla\overline{u}\|^2.$$

It follows from (10)-(3.9) that for $t > 0$,

$$\|\partial_t\nabla\overline{u}(t)\|^2 + \frac{3}{2}\|\Delta\overline{u}(t)\|^2 \leq c_2\|\overline{u}\|^2\|\nabla\overline{u}\|^4 + 8\|f\|^2 + 8L_g^2\|\nabla\overline{u}\|^2. \quad (11)$$

By Lemma 3.1 for all $t \geq 0$, we obtain

$$\|\overline{u}(t)\|^2 + \int_t^{t+1} \|\overline{u}(s)\|^2_V ds \leq c_3,$$

with $c_3 > 0$ depends on $\lambda, L_g, f$ and $u_0$, we have

$$\int_t^{t+1} \|\nabla\overline{u}(s)\|^2 ds \leq c_3, \quad \int_t^{t+1} \|\overline{u}(s)\|^2\|\nabla\overline{u}(s)\|^2 ds \leq c_3^2,$$

and by Gronwall Lemma, we have

$$\|\nabla\overline{u}(t)\|^2 \leq c_4,$$

where $c_4 > 0$ depends on $\lambda, L_g, f$ and $u_0$.

Integrating from $t$ to $t+1$ for (11), we obtain

$$\int_t^{t+1} \|\overline{u}(s)\|^2 ds \leq c_5,$$

with $c_5 > 0$ depends on $\lambda, L_g, f$ and $u_0$. The proof is complete. □

We have proved the following uniform estimates for the large time by Lemma 3.2.

**Lemma 3.3.** *Let $R > 0$, there exists $T_0 = T_0(R) > 0$ such that for all $t \geq T_0$ and $u_0 \in H$ with $\|u_0\| \leq R$, the solutions $u$ of (1) satisfies*

$$\|\nabla u(t)\|^2 + \int_t^{t+1} \|\Delta u(s)\|^2 ds + \int_t^{t+1} \|\partial_t u(s)\|^2 \leq C_3,$$

*with $C_3 > 0$ is constant dependent of $\lambda, L_g$ and $f$.*

## 4 Uniform tail-end estimates of the solutions

We will the asymptotic compactness of the solutions of (1) by Uniform tail-end estimates of the solutions. Furthermore, we will obtain the existence of global attractors in $H$.

Suppose a smooth function $\phi : \mathbb{R} \to \mathbb{R}$ for $0 \le \phi \le 1$ and

$$\phi(s) = 0, |s| \le \frac{1}{2}; \quad \phi(s) = 1, |s| \le 1.$$

If $C > 0$ and $|\phi'(s)| + |\phi''(s)| \le C$. Let $n \in \mathbb{N}$ and $x = (x_1, x_2) \in \mathcal{O}$, let $\phi_n(x) = \phi(\frac{n_1}{k})$.

We first give the following inequalities.

**Lemma 4.1.**
*(a) Let $u \in H^1(\mathcal{O})$, then*

$$|\|\phi_n \nabla u\| - \|\nabla(\phi_n u)\|| \le \frac{C_4}{n}\|u\|.$$

*(b) Let $u \in H^2(\mathcal{O})$, then*

$$|\|\phi_n \Delta u\| - \|\Delta(\phi_n u)\|| \le \frac{C_4}{n}\|u\|_{H^1(\mathcal{O})}.$$

*(c) Let $u \in H_0^1(\mathcal{O})$, then*

$$\|\phi_n \nabla u\| \ge \lambda^{\frac{1}{2}}\|\phi_n u\| - \frac{C_4}{n}\|u\|.$$

*(d) If $u \in H^2(\mathcal{O}) \cap H_0^1(\mathcal{O})$, then*

$$\|\phi_n \Delta u\| \ge \lambda^{\frac{1}{2}}\|\phi_n \nabla u\| - \frac{C_4}{n}\|u\|_{H^1(\mathcal{O})},$$

*where $C_4 > 0$ is a constant independent of $n$ and $u$.*

*Proof of Lemma 4.1.*

(a) Note that $\nabla(\phi_n u) = u\nabla\phi_n + \phi_n \nabla u$,

$$|\|\phi_n \nabla u\| - \|\nabla(\phi_n u)\|| \le \|\phi_n \nabla u - \nabla(\phi_n u)\| = \|u\nabla\phi_n\| \le \frac{c}{n}\|u\|,$$

with $c > 0$ is a constant independent of $n$ and $u$.
(b) Suppose $\Delta(\phi_n u) = u(\Delta\phi_n) + \phi_n(\Delta u) + 2\nabla\phi_n \cdot \nabla u$ such that

$$\|\phi_n(\Delta u) - \Delta(\phi_n u)\| \le \|u(\Delta\phi_n)\| + 2\|\nabla\phi_n \cdot \nabla u\| \le \frac{c}{n}(\|u\| + \|\nabla u\|).$$

(c) If $\|\nabla(\phi_n u)\| \ge \lambda^{\frac{1}{2}}\|\phi_n u\|$, by (a) we have $\|\phi_n \nabla u\| \ge \lambda^{\frac{1}{2}}\|\phi_n u\| - \frac{c}{n}\|u\|$.
(d)

$$\begin{aligned}\|\nabla(\phi_n u)\|^2 &= (\nabla(\phi_n u), \nabla(\phi_n u)) = -(\Delta(\phi_n u), \phi_n u)\\ &\le \|\Delta(\phi_n u)\|\|\phi_n u\| \le \lambda^{\frac{1}{2}}\|\Delta(\phi_n u)\|\|\nabla(\phi_n u)\|\end{aligned}$$

and $\|\nabla(\phi_n u)\| \le \lambda^{-\frac{1}{2}}\|\Delta(\phi_n u)\|$. By (a) and (b) we have

$$\|\phi_n \nabla u\| \le \|\nabla(\phi_n u)\| + \frac{c_2}{n}\|u\| \le \lambda^{-\frac{1}{2}}\|\Delta(\phi_n u)\| + \frac{c_2}{n}\|u\|$$
$$\le \lambda^{-\frac{1}{2}}(\|\phi_n \Delta u\| + \frac{c_3}{n}\|u\|_{H^1(\mathcal{O})}) + \frac{c_2}{n}\|u\| \le \lambda^{-\frac{1}{2}}\|\phi_n \Delta u\| + \frac{c_4}{n}\|u\|_{H^1(\mathcal{O})},$$

which shows that

$$\lambda^{\frac{1}{2}}\|\phi_n \nabla u\| \le \|\phi_n \Delta u\| + \frac{c_4}{n}\lambda^{\frac{1}{2}}\|u\|_{H^1(\mathcal{O})}.$$

This proof is complete. □

Furthermore, we give a scalar stream function for (1). Let $u = (u_1, u_2) \in V$ and $x = (x_1, x_2) \in \bar{\mathcal{O}}$, define

$$\widehat{u}(x) = \widehat{u}(x_1, x_2) = \int_{(0,0)}^{(x_1,x_2)} -u_2 dx_1 + u_1 dx_2.$$

Since $u = 1$ on $\partial\mathcal{O}$ and $\operatorname{div}(u) = 0$ in $\mathcal{O}$.

Then, we get

$$\partial_{x_1}\widehat{u} = -u_2, \quad \partial_{x_2}\widehat{u} = u_1, \tag{12}$$

and

$$\widehat{u}|_{\partial\mathcal{O}} = 0, \ \ \nabla\widehat{u}|_{\partial\mathcal{O}} = 0.$$

If $T$ is the curl operator that has

$$Tu = \partial_{x_2}u_1 - \partial_{x_1}u_2, \ \forall u = (u_1, u_2). \tag{13}$$

By (12) and (13), we have

$$\partial_t\Delta\widehat{u} = \Delta^2\widehat{u} + \widehat{B}(\widehat{u}, \widehat{u}) + \widehat{g}(u_t) + \widehat{f}, \tag{14}$$

where

$$\widehat{g}(u_t) = T(g(u_t)), \ \ \widehat{f} = Tf, \ \ \widehat{B}(\widehat{u}, \widehat{u}) = \partial_{x_2}((\partial_{x_1}\widehat{u})\Delta\widehat{u}) - \partial_{x_2}((\partial_{x_2}\widehat{u})\Delta\widehat{u}).$$

Let $n \in \mathbb{N}$, we denote $\mathcal{O}_n = (-n, n) \times (0, d)$. By (14), we prove the uniform tail-end estimates of the solutions of (1) on $\mathcal{O} \setminus \mathcal{O}_n$ as below.

**Lemma 4.2.** *Let $u_0 \in H$ and $\varepsilon > 0$, there exists $\mathcal{N} = \mathcal{N}(\lambda, L_g, f, u_0, \varepsilon) \geq 1$ such that the solution $u$ of* (1) *satisfies, for all $n \geq N$ and $t \geq 1$,*

$$\int_{\mathcal{O}\setminus\mathcal{O}_k} |u(t,x)|^2 dx \leq \varepsilon.$$

*Proof of Lemma 4.2.* We prove the uniform estimates of the solutions by a limiting process. By (14) we get

$$\begin{aligned} -\frac{d}{dt}(\Delta\widehat{u}, \phi_n^2\widehat{u}) &= -(\Delta^2\widehat{u}, \phi_n^2\widehat{u}) - (\widehat{B}(\widehat{u}, \widehat{u}), \phi_n^2\widehat{u}) - (\widehat{g}(u_t), \phi_n^2\widehat{u}) - (\widehat{f}, \phi_n^2\widehat{u}) \\ &= J_i(i = 1, 2, 3). \end{aligned} \tag{15}$$

For the left-hand side of (15) we get

$$\begin{aligned} -\frac{d}{dt}(\Delta\widehat{u}, \phi_n^2\widehat{u}) &= \frac{1}{2}\frac{d}{dt}\int_{\mathcal{O}} \phi_n^2|\nabla\widehat{u}|^2 dx + \int_{\mathcal{O}} (\nabla\widehat{u}_t, \nabla\phi_n^2)\widehat{u}dx \\ &\geq \frac{1}{2}\frac{d}{dt}\int_{\mathcal{O}} \phi_n^2|\nabla\widehat{u}|^2 dx - \frac{c_1}{n}\|\nabla\widehat{u}_t\|\|\widehat{u}\|, \end{aligned}$$

where $c_1 > 0$ is independent of $n$. For the $J_1$ of (15) we obtain

$$\begin{aligned} J_1 = -(\Delta^2\widehat{u}, \phi_n^2\widehat{u}) &= -\int_{\mathcal{O}} \Delta\widehat{u}(\phi_n^2\Delta\widehat{u} + (\Delta\phi_n^2)\widehat{u} + 2\nabla\phi_n^2 \cdot \nabla\widehat{u})dx \\ &\leq -\|\phi_n\Delta\widehat{u}\|^2 + \frac{c_2}{n^2}\|\widehat{u}\|\|\Delta\widehat{u}\| + \frac{c_2}{n}\|\nabla\widehat{u}\|\|\Delta\widehat{u}\|. \end{aligned} \tag{16}$$

By Lemma4.1 we suppose exists $c_3 > 0$ independent of $k$ such that

$$\lambda^{\frac{1}{2}}\|\phi_n\nabla\widehat{u}\| \leq \|\phi_n\Delta\widehat{u}\| + \frac{c_3}{n}\|\widehat{u}\|_{H^1(\mathcal{O})}$$

and hence by Young's inequality, we have

$$\begin{aligned}\lambda\|\phi_n\nabla\widehat{u}\|^2 \leq&\|\phi_n\Delta\widehat{u}\|^2+\frac{2c_3}{n}\|\phi_n\Delta\widehat{u}\|\|\widehat{u}\|_{H^1\mathcal{O}}+\frac{c_3^2}{n^2}\|\widehat{u}\|_{H^1\mathcal{O}}\\ \leq&\|\phi_n\Delta\widehat{u}\|^2+\frac{\alpha}{4\lambda-\alpha}\|\phi_n\Delta\widehat{u}\|^2+\frac{(4\lambda-\alpha)c_3^2}{\alpha n^2}\|\widehat{u}\|^2_{H^1(\mathcal{O})}+\frac{c_3^2}{n^2}\|\widehat{u}\|^2_{H^1(\mathcal{O})}\\ =&\frac{4\lambda}{4\lambda-\alpha}\|\phi_n\Delta\widehat{u}\|^2+\frac{4\lambda c_3^2}{\alpha n^2}\|\widehat{u}\|^2_{H^1(\mathcal{O})}.\end{aligned}$$

we have

$$-\|\phi_n\Delta\widehat{u}\|^2\leq(\frac{1}{4}\alpha-\lambda)\|\phi_n\nabla\widehat{u}\|^2+\frac{(4\lambda-\alpha)c_3^2}{\alpha n^2}\|\widehat{u}\|^2_{H^1(\mathcal{O})},$$

By (16) and (17) we have

$$-(\Delta^2\widehat{u},\phi_n^2\widehat{u})\leq(-\frac{1}{4}\alpha-\lambda)\|\phi_n\nabla\widehat{u}\|^2+\frac{c_4}{n}\|\Delta\widehat{u}\|^2,$$

where $c_4>0$ is independent of $n$.

For the $J_2$ of (15), we obtain

$$\begin{aligned}J_2=-(\widehat{B}(\widehat{u},\widehat{u}),\phi_n^2\widehat{u})=&-\int_{\mathcal{O}}\widehat{u}(\Delta\widehat{u})(\partial_{x_2}\widehat{u})(\partial_{x_1}\phi_n^2)dx\\ \leq&\frac{c_5}{n}\|\Delta\widehat{u}\|\|\widehat{u}\|_{L^4(\mathcal{O})}\|\partial_{x_2}\widehat{u}\|_{L^4(\mathcal{O})}\\ \leq&\frac{c_6}{n}\|\widehat{u}\|^{\frac{3}{2}}\|\nabla\widehat{u}\|^{\frac{3}{2}}\leq\frac{c_7}{n}(\|\Delta\widehat{u}\|^2+\|\nabla\widehat{u}\|^6),\end{aligned}$$

where $c_7>0$ is independent of $n$.

Suppose $g(u_t)=(g_1(u_t),g_2(u_t))$ and $f=(f_1,f_2)$. Then for $J_3$ of (15), by (5), (12) we have

$$\begin{aligned}-(\widehat{g}(u_t),\phi_n^2\widehat{u})-(\widehat{f},\phi_n^2\widehat{u})=&\int_{\wr}(-g_2(u_t),g_1(u_t))\cdot\nabla(\phi_n^2\widehat{u})dx+\int_{\mathcal{O}}(-f_2,f_1)\cdot\nabla(\phi_n^2\widehat{u})dx\\ \leq&\int_{\mathcal{O}}\phi_n^2|\nabla\widehat{u}|(|g(u_t)|_{C_H}+|f|)dx+\int_{\mathcal{O}}|\widehat{u}|(|g(u_t)|+|f|)|\nabla\phi_n^2|dx\\ \leq&L_g\int_{\mathcal{O}}\phi_n^2|\nabla\widehat{u}|^2_{C_H}dx+\int_{\mathcal{O}}\phi_n^2|\nabla\widehat{u}||f|dx+\int_{\mathcal{O}}|\widehat{u}|(L_g|\nabla\widehat{u}|+|f|)|\nabla\psi_n^2|dx\\ \leq&(L_g+\frac{1}{4}\alpha)\int_{\mathcal{O}}\phi_n^2|\nabla\widehat{u}|^2_{C_H}dx+\frac{1}{\alpha}\int_{\mathcal{O}}\phi_n^2|f|^2dx+\frac{c_8}{n}(\|\nabla\widehat{u}\|^2+\|f\|^2),\end{aligned}\tag{17}$$

where $c_8>0$ is independent of $n$. which along with (15)-(17) we have

$$\frac{d}{dt}\|\phi_n\nabla\widehat{u}\|^2+\alpha\|\phi_n\nabla\widehat{u}\|^2\leq\frac{c_9}{n}(\|\nabla\widehat{u}_t\|^2_{C_H}+\|\nabla\widehat{u}\|^6+\|\Delta\widehat{u}\|^2+\|f\|^2)+c_9\|\phi_nf\|^2,\tag{18}$$

with $c_9>0$ is independent of $n$.

combining (12) and (18), we have

$$\|\partial_t\phi_nu\|^2+\alpha\|\phi_nu\|^2\leq\frac{c_9}{n}(\|u_t\|^2_{C_H}+\|u\|^6+2\|\nabla u\|^2+\|f\|^2)+c_9\|\phi_nf\|^2.\tag{19}$$

Integrating (19) on $(1,t)$ for $t\geq 1$ to have

$$\begin{aligned}\|\phi_nu(t)\|^2\leq& e^{\alpha(1-t)}\|\phi_nu(1)\|^2\\ &+\frac{c_9}{n}\int_1^t e^{\alpha(s-t)}(\|u_t(s)\|^2_{C_H}+\|u(s)\|^6+2\|\nabla u(s)\|^2+\|f\|^2)ds+c_9\alpha^{-1}\|\phi_nf\|^2.\\ &=J_i(i=1,2,3)\end{aligned}\tag{20}$$

For the $J_1$ of (20), we have for $t \geq 1$,

$$e^{\alpha(1-t)}\|\phi_n u(1)\|^2 \leq \int_{\{(x_1,x_2)\in\mathcal{O}:|x_1|\geq\frac{1}{2}n\}}|u(1,x)|^2dx \to 0, \ \ n\to\infty.$$

For the $J_2$ of (20), by Lemma3.1 and Lemma3.2 we have for $t \geq 1$,

$$\begin{aligned}
&\frac{c_9}{n}\int_1^t e^{\alpha(s-t)}(\|u_t(s)\|^2_{C_H}+\|u(s)\|^6+2\|\nabla u(s)\|^2+\|f\|^2)\\
\leq&\frac{c_9}{n}(c_{10}+\|f\|)\int_1^t e^{\alpha(s-t)}ds+\frac{c_9}{n}\int_1^t e^{\alpha(s-t)}\|u_t(s)\|^2_{C_H}ds\\
\leq&\frac{c_9}{n}(c_{10}+\|f\|)+\frac{c_9}{n}(\int_1^2 e^{\alpha(s-t)}\|u_t(s)\|^2_{C_H}ds+\cdots+\int_{[t]}^t e^{\alpha(s-t)}\|u_t(s)\|^2_{C_H}ds)\\
\leq&\frac{c_9}{n}(c_{10}+\|f\|)+\frac{c_9}{n}(e^{\alpha(2-t)}\int_1^2\|u_t(s)\|^2_{C_H}ds+\cdots\\
&+e^{\alpha([t]-t)}\int_{[t]-1}^{[t]}\|u_t(s)\|^2_{C_H}ds+\int_{[t]}^t\|u_t(s)\|^2_{C_H}ds)\\
\leq&\frac{c_{11}}{n}+\frac{c_{11}}{n}(e^{\alpha(2-t)}+\cdots+e^{\alpha([t]-t)})\leq\frac{c_{11}}{n}+\frac{c_{11}}{n}e^{\alpha([t]-t)}\Sigma_{j=0}^{\infty}r^{-\alpha j}\\
\leq&\frac{c_{11}}{n}+\frac{c_{11}}{n}(1-e^{-\alpha})^{-1}\to 0, \ \ n\to\infty,
\end{aligned}$$

with $c_{11} > 0$ dependent of $\lambda, L_g, f$ and $u_0$.

If $f \in H$, for the $J_3$ of (20), we get

$$c_9\alpha^{-1}\|\phi_n f\|^2 = c_9\alpha^{-1}\int_{\{(x_1,x_2)\in\mathcal{O}:|x_1|\geq\frac{1}{2}n\}}|f(x)|^2dx\to 0, \ \ n\to\infty. \tag{21}$$

From (20)-(21) for every $\varepsilon > 0$, then exists $N = N(\lambda, L_g, f, u_0, \varepsilon) \geq 1$, such that for all $t \geq 1$ and $n \geq N$,

$$\int_{\{(x_1,x_2)\in\mathcal{O}:|x_1|\geq n\}}|u(t,x)|^2dx \leq \|\phi_n u(t)\|^2 \leq \varepsilon.$$

□

Next, we will consider the uniform tail-ends estimates of the solutions under bounded initial data in $H$.

**Lemma 4.3.** *Let $R > 0$ and $\varepsilon > 0$, there exists $T_1 = T_1(\lambda, L_g, f, R, \varepsilon) > 0$ and $N = N(\lambda, L_g, f, \varepsilon) \geq 1$, such that if $u_0 \in H$ with $\|u_0\| \leq R$, then the solution $u$ of (1) satisfies, for all $t \geq T_1$ and $n \geq N$,*

$$\int_{\mathcal{O}\setminus\mathcal{O}_n}|u(t,x)|^2dx \leq \varepsilon.$$

*Proof of Lemma 4.3.* Let $u_0 \in H$ with $\|u_0\| \leq R$. By Lemma 3.3, we have that there exists $T_0 = T_0(R) > 0$ such that for all $t \geq T_0$,

$$\|\nabla u(t)\|^2+\int_t^{t+1}\|\partial_t u(s)\|^2ds \leq C_1, \tag{22}$$

where $C_1 > 0$ dependent of $\lambda, L_g$ and $f$.

Integrating (19) on $(T_0, t)$ for $t \geq T_0$, we obtain

$$\begin{aligned}
\|\phi_n u(u)\|^2 \leq& e^{\alpha(T_0-t)}\|\phi_n u(T_0)\|^2\\
&+\frac{c_9}{n}\int_{T_0}^t e^{\alpha(s-t)}(\|u_t(s)\|^2_{C_H}+\|u(s)\|^6+2\|\nabla u(s)\|^2+\|f\|^2)ds+c_9\alpha^{-1}\|\phi_n f\|^2\\
=& I_1+I_2.
\end{aligned} \tag{23}$$

For $I_1$ of (23), by Lemma3.1 we have get

$$e^{\alpha(T_0-t)}\|\phi_n u(T_0)\|^2 \le e^{\alpha(T_0-t)}\|u(T_0)\|^2 \le e^{\alpha(T_0-t)}(e^{-\frac{1}{2}\alpha T_0}R^2 + C_2),$$

where $C_2 > 0$ dependent of $\lambda, L_g$ and $f$. If $\varepsilon > 0$, there exists $\mathcal{T} = \mathcal{T}(\lambda, L_g, f, R, \varepsilon) \ge T_0$ such that for all $t \ge T_1$,

$$e^{\alpha(T_0-t)}\|\phi_n u(T_0)\|^2 \le \frac{1}{4}\varepsilon.$$

For $I_2$ of (23), by (22) we obtain

$$\begin{aligned}
&\frac{c_9}{n}\int_{T_0}^{t} e^{\alpha(s-t)}(\|u_t(s)\|^2_{C_H} + \|u(s)\|^6 + 2\|\nabla u(s)\|^2 + \|f\|^2)\\
\le&\frac{C_3}{n} + \frac{C_3}{n}(e^{\alpha(T_0+1-t)}\int_{T_0}^{T_0+1}\|u_t(s)\|^2_{C_H}ds + \cdots\\
&+ e^{\alpha([t]-t)}\int_{[t]-1}^{[t]}\|u_t(s)\|^2_{C_H}ds + \int_{[t]}^{t}\|u_t(s)\|^2_{C_H}ds)\\
\le&\frac{C_3}{n} + \frac{C_1C_3}{n} + \frac{C_1C_3}{n}(1-e^{-\alpha})^{-1} \to 0, \quad n \to \infty,
\end{aligned} \tag{24}$$

where $C_3 > 0$ dependent of $\lambda, L_g$ and $f$. Combining (23)-(24) and (21), we have that there exists $N = N(\lambda, L_g, \varepsilon) \ge 1$ such that for all $t \ge T_1$ and $n \ge N$,

$$\int_{\{(x_1,x_2)\in\mathcal{O}:|x_1|\ge n\}} |u(t,x)|^2 dx \le \|\phi_n u(t)\|^2 \le \varepsilon.$$

□

By Lemma 4.3, we derive the asymptotic compactness of the solutions in $H$.

**Lemma 4.4.** *Let (5) hold, suppose $\{u_{0,n}\}_{n=1}^{\infty}$ is bounded in $H$ and $t_n \to \infty$, then $\{S(t_n)u_{0,n}\}_{n=1}^{\infty}$ is precompact in $H$.*

*Proof of Lemma 4.4.* If $\{u_{0,n}\}_{n=1}^{\infty}$ is bounded in $H$, by Lemma 4.3 for every $\varepsilon > 0$, there exist $N_1 = N_1(\varepsilon) \ge 1$ and $\mathcal{N} = \mathcal{N}(\varepsilon) \ge 1$ such that for all $n \ge N_1$, we have

$$\|S(t_n)u_{0,n}\|_{L^2(\mathcal{O}\setminus\mathcal{O}_N)} < \frac{\varepsilon}{4}.$$

By Lemma 3.3 we have that there exists $N_2 = N_2(\varepsilon) \ge N_1$ such that $\{S(t_n)u_{0,n}\}_{n=N_2}^{\infty}$ is bounded in $V$. Since the compactness of the embedding $H^1(\mathcal{O}_N) \hookrightarrow L^2(\mathcal{O}_N)$ we infer that $\{S(t_n)u_{0,n}\}_{n=1}^{\infty}$ has a finite cover of radius $\frac{1}{4}$ in $L^2(\mathcal{O}_N)$, and by (4.23) show that $\{S(t_n)u_{0,n}\}_{n=1}^{\infty}$ has a finite cover of radius $\varepsilon$ in $H$ □

By Lemma 3.3 and 4.4, we prove the existence of global attractors of (1) follows.

**Theorem 4.5.** *Let (5) holds, then system (1) has a unique global attractor in $H$.*

## 5 Conclusion

We establish the uniform estimates of 2D Navier-Stokes equations of the solutions in $H$ and $V$ to obtain uniform estimates of the solutions. Furthermore, We proved the asymptotic compactness and the existence of global attractors in $H$ by the uniform tail-end estimates. Finally, we proved the existence of the global attractors of delay 2D Naiver-stokes equations on unbounded Channel-like domains.

## 6 Declarations

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### Authors's Contributions

Conceptualization, Z. Zhang; methodology, Z. Zhang and X. Yao; investigation, Z. Zhang; writing—original draft preparation, Z. Zhang; writing—review and editing, X. Yao.

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## References

[1] Wang B, *Uniform tail-ends estimates of the Navier-Stokes equations on unbounded channel-like domains*, Proceedings of the American Mathematical Society, 151(2023), no. 11, pp. 4841-4853, DOI org/10.1090/proc/16539.

[2] Caraballo T, Kukaszewioz G, Real J, *Pullback attractors for non-autonomous 2D-Navier-Stokes equations in some unbounded domains*, Comptes Rendus Mathematique, 342(2006), no. 4, pp. 263-268. DOI 10.1016/j.crma.2005.12.015.

[3] Temam R, *Navier-Stokes Equations: Theory and Numerical Analysis*, North-Holland Publishing Co. Amsterdam-New York-Oxford, 1977.

[4] Rosa R, *The global attractor for the 2D Navier-Stokes flow on some unbounded domains*, Nonlinear Anal. 32(1998), no. 1, pp. 71-85, DOI 10.1016/S0362-546X(97)00453-7.

[5] Ball, John, *Global attractors for damped semilinear wave equations*, Discrete and Continuous Dynamical Systems, 2003, DOI 10.3934/DCDS.2004.10.31.

[6] Zdzisaw Brzeniak, Li Y, *Asymptotic compactness and absorbing sets for 2D stochastic Navier-Stokes equations on some unbounded domains*, Transactions of the American Mathematical Society, 385(2006), no. 12, pp. 5587-5629, DOI 10.2307/20161558.

[7] Lukaszewicz G, Sadowski W, *Uniform attractor for 2D magneto-micropolar fluid flow in some unbounded domains*, Zeitschrift für angewandte Mathematik und Physik ZAMP, 55(2004), no. 2, pp. 247-257, DOI 10.1007/s00033-003-1127-7.

[8] Moise I, Rosa R, Wang X, *Attractors for noncompact nonautonomous systems via energy equations*, Discrete Contin. Dyn Syst. 10(2004), no. 2, pp. 247-257, DOI 10.3934/dcds.2004.10.473.

[9] Pedro Marín-Rubio, José Real, *Attractors for 2D-Navier-Stokes equations with delays on some unbounded domains*, Nonlinear Analysis. 67(2007), no. 10, pp. 2784-2799, DOI 10.1016/j.na.2006.09.035.

[10] Wang B, *Periodic random attractors for stochastic Navier-Stokes equations on unbounded domains*, Electronic Journal of Differential Equations, 2012(2012), no. 59, DOI 10.1016/j.jmaa.2011.11.022.

[11] Wang B, *Attractors for reaction-diffusion equations in unbounded domains*, Physica D Nonlinear Phenomena, 128(1999), no. 1, pp. 41-52, DOI 10.1016/S0167-2789(98)00304-2.

[12] Bates P W, Lu K, Wang B, *Random attractors for stochastic reaction-diffusion equations on unbounded domains*, Journal of Differential Equations, 246(2009), no. 2, pp. 845-869, DOI 10.1016/j.jde.2008.05.017.

[13] Wang X, Lu K, Wang B, *Wong-Zakai approximations and attractors for stochastic reaction-diffusion equations on unbounded domains*, Journal of Differential Equations, 2017:S0022039617304850, DOI 10.1016/j.jde.2017.09.006.