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Global Attractors of the Delay 2D Navier-Stokes Equations on Unbounded Channel-Like Domains

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Abstract

This paper studies the global attractors of 2D Navier-Stokes equations with delay defined in unbounded Channel-like domains. To overcome the non-compactness of solutions, we will use the uniform tail-ends estimates of the solutions by establishing all the solutions are uniformly small.

Keywords: Navier-Stokes equations, global attractors, uniform tail-ends estimates
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1 Introduction

This paper considers the 2D Navier-Stokes equations defined on $\mathcal{O} = \mathbb{R} \times (0, d)$:

$$\begin{cases} u_t - \Delta u + (u \cdot \nabla)u = g(u_t) + \nabla p + f(x), \ \ \mathrm{div} u = 0, \\ u(t,x) = 0, \ \ t > 0, x \in \partial\mathcal{O}, \\ u(0,x) = u_0(x), \ \ x \in \mathcal{O}, \\ u(x,t) = \phi(t-\tau, x), \ \ t \in (\tau - h, \tau), x \in \mathcal{O}, \end{cases} \tag{1}$$

where u and p are the velocity and the pressure of the fluid. $f \in L^2(\mathcal{O})$ is given, g is a Lipschitz nonlinear function with delay, $r \geq 0$ is constants,

$$u_t(\theta) = u(t+\theta), \ \forall\, \theta \in (-r, 0).$$

When (1) does not contain the delay term, Wang has investigated the existence of the global attractors of the 2D Navier-Stokes equations on the Channel-like unbounded domains in [1]. In recent years, significant results have been investigated in the study of attractors for 2D Navier-Stokes with delay. Carabello has studied the attractors of 2D Navier-Stokes equations

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with delay on bounded domains in reference [2], and then proved the existence of pull-back attractors for non-autonomous delay 2D Navier-Stokes equations on unbounded domains. This paper mainly studied the asymptotic compactness of the solutions of (1) on unbounded channel-like domains $\mathcal{O}$. We use the existence, stability, and convergence of the global attractors to show the long-time asymptotic behavior of the solutions. We have known Sobolev embeddings on unbounded domains are no longer compact (see reference [3]). In unbounded domains, the main difficulty is the non-compactness of Sobolev embeddings.

In fact, we can overcome the difficulty by Rosa's idea of the Ball energy equations proposed (in reference [4, 5]), which is the method to establish asymptotic compactness of (1) solutions when $\mathcal{O}$ is unbounded in the phase space $L^2(\mathcal{O}, \mathbb{R}^2)$ (see references [2, 6, 7, 8, 9, 10]). In order to overcome the non-compactness of Sobolev embeddings in unbounded domains, we used uniform tail-ends estimates methods for the solutions proposed in reference [11] to prove the asymptotic compactness of the solution to the reaction-diffusion equations on $\mathbb{R}^n$(see references [12, 13]).

In this paper, we will derive the uniform tail-ends estimates of solutions under supposed conditions $p = 0$ and $\mathrm{div} u = 0$. We will use uniform tail-ends estimates of the scalar stream function for the Navier-Stokes equations (see [6, 12]). By uniform estimates of solutions, we will prove the asymptotic compactness of solutions of (1) defined in $\mathcal{O}$ and the existence of global attractors for (1) in H.

This paper is organized as follows, we will recall some basic concepts and results in Section 2. In Section 3, we will prove the uniform estimates of 2D Navier-Stokes equations of the solutions in H and V. We will prove the asymptotic compactness of solutions and the existence of global attractors in H.

In the paper, we denote norm $\|\cdot\|$ and inner product $(\cdot,\cdot)$ of the $L^2(\mathcal{O}, \mathbb{R}^2)$. We also denote norm $\|u\|_V = \|\nabla u\|$ for $u \in V$ of V and dual space V^* is labeled as $\langle\cdot,\cdot\rangle$ and denote the norm $(\cdot,\cdot)_{C_H}$ and the inner product $\|\cdot\|_{C_H}$ of the $C_H = (-r, 0; H)$.

2 Preliminaries

In this section, we review some basic results and knowledge.

The Poincare inequality:

$$\|\nabla u\|^2 \geq \lambda \|u\|^2, \quad \forall u \in H_0^1(\mathcal{O}, \mathbb{R}^2)\ \lambda > 0. \tag{2}$$

Let $u, v, w \in V$, we denoted

$$b(u, v, w) = \sum_{i,j=1}^{2} \int_{\mathcal{O}} u_i \frac{\partial v_j}{\partial x_i} w_j dx. \tag{3}$$

For all $u, v, w \in V$,

$$b(u, v, w) = -b(u, w, v),\ b(u, v, v) = 0$$

and

$$|b(u, v, w)| \leq \|u\|_{L^4(\mathcal{O})} \|\nabla v\| \|w\|_{L^4(\mathcal{O})} \leq c\|u\|_V \|v\|_V \|w\|_V, \tag{4}$$

with every $c > 0$.

By(3), denote a bilinear operator $B : V \times V \to V^*$, for every $u, v, w \in V$,

$$\langle B(u, v), w \rangle = b(u, v, w),$$

By (4) for all $u, v \in V$, we have

$$\|B(u, v)\|_{V^*} \leq c\|u\|_V \|v\|_V,$$

For $u \in V \cap H^2(\mathcal{O}, \mathbb{R}^2)$, we also have

$$\|B(u,u)\| \leq c\|u\|^{\frac{1}{2}}\|\nabla u\|\|\Delta u\|^{\frac{1}{2}}.$$

We suppose

$$\alpha = \lambda - L_g.$$

We will make appropriate suppose for the delay term. Let $g : (-r, 0; H) \to L^2(\mathcal{O}, \mathbb{R}^2)$ satisfy the following conditions:

(1) $g(0) = 0$;

(2) there exists $L_g > 0$, such that $\forall\, \xi, \eta \in (-r, 0; H)$

$$|g(\xi) - g(\eta)| \leq L_g\|\xi - \eta\|_{C_H}. \tag{5}$$

3 Uniform estimates of solutions of 2D Navier-Stokes equations

We will prove the uniform estimates of solutions of (1) in H and V.

Lemma 3.1. *Let (5) holds, then for every $u_0 \in H$, the solution $\widetilde{u}$ of (1) satisfies for all $t \geq 0$,*

$$\|\widetilde{u}(t)\|^2 + \frac{3}{2}\int_0^t e^{-4L_g^2(s-t)}\|\widetilde{u}(s)\|_V^2 ds \leq e^{4L_g^2 t}\|\widetilde{u}_0\|^2 + C_1,$$

and for all $t \geq r \geq 0$,

$$\frac{3}{2}\int_0^t e^{-4L_g^2(s-t)}\|\widetilde{u}(s)\|_V^2 ds \leq e^{4L_g^2 r}\|\widetilde{u}_0\|^2 + C_1(1 + t - r),$$

where $C_1 > 0$ is a constant independent of t, r or u_0.

Proof of Lemma 3.1. Suppose $t > 0$,

$$\|\partial_t \widetilde{u}(t)\|^2 + 2\|\nabla \widetilde{u}(t)\|^2 = 2(g(\widetilde{u}_t), \widetilde{u}(t)) + 2(f, \widetilde{u}(t)). \tag{6}$$

Along with (5), we obtain

$$2(g(u_t), \widetilde{u}(t)) \leq 2L_g\|\widetilde{u}_t\|_{c_H}\|\widetilde{u}\| \leq \frac{1}{2}\|\widetilde{u}_t\|_{C_H}^2 + 2L_g^2\|\widetilde{u}\|^2.$$

and by Young's inequality, we get

$$2(f, \widetilde{u}(t)) \leq \frac{\|f\|^2}{2L_g^2} + 2L_g^2\|\widetilde{u}\|^2. \tag{7}$$

By (2) and (6)-(7) for $t > 0$, we have

$$\|\partial_t \widetilde{u}(t)\|^2 + 2\|\nabla \widetilde{u}(t)\|^2 \leq \frac{1}{2}\|\widetilde{u}_t\|_{C_H} + 4L_g^2\|\widetilde{u}\|^2 + \frac{\|f\|^2}{L_g^2}. \tag{8}$$

Solve (8) to obtain for all $t \geq 0$,

$$\|\widetilde{u}(t)\|^2 + 2\int_0^t e^{-4L_g^2(s-t)}\|\nabla \widetilde{u}(s)\|^2 ds \leq e^{4L_g^2 t}\|\widetilde{u}_0\|^2 + e^{4L_g^2 t}\frac{\|f\|^2}{2L_g^2}. \tag{9}$$

Integrating (8) on (r,t) with $0 \leq r \leq t$, by (9) we have

$$2\int_r^t \|\nabla \widetilde{u}(s)\|^2 ds \leq \|\widetilde{u}(r)\|^2 + \frac{\|f\|^2}{2L_g^2}(t - r) \leq e^{4L_g^2 r}\|\widetilde{u}_0\|^2 + e^{4L_g^2 t}\frac{\|f\|^2}{2L_g^2} + \frac{\|f\|^2}{2L_g^2}(t - r).$$

This completes the proof. $\square$

We next prove the uniform estimates of solutions of (1) in V for $t > 0$ with initial data in H.

Lemma 3.2. *Let (5) holds, for every $u_0 \in H$, the solutions $\overline{u}$ of (1) satisfies for all $t \geq 1$,*

$$\|\nabla \overline{u}(t)\|^2 + \int_t^{t+1} \|\Delta \overline{u}(s)\|^2 ds + \int_t^{t+1} \|\partial_t \overline{u}(s)\|^2 ds \leq C_2,$$

where $C_2 > 0$ is a constant dependent of λ, L_g and u_0.

Proof of Lemma 3.2. By a limiting process, we will derive the uniform estimates for $t \geq 0$,

$$\|\partial_t \nabla \overline{u}(t)\|^2 + 2\|\Delta \overline{u}(t)\|^2 = 2(B(\overline{u}(t), \overline{u}(t)), \Delta \overline{u}(t)) - 2(g(\overline{u}_t(t)), \Delta \overline{u}(t)) - 2(f, \Delta \overline{u}(t)). \quad (10)$$

We first estimate the first term of (10)

$$\begin{aligned} 2(B(\overline{u}(t), \overline{u}(t)), \Delta \overline{u}(t)) \leq & \|u\|_{L^4(\mathcal{O},\mathbb{R}^2)} \|\nabla \overline{u}\|_{L^4(\mathcal{O},\mathbb{R}^4)} \|\Delta \overline{u}\| \\ \leq & c_1 \|\overline{u}\|^{\frac{1}{2}} \|\nabla \overline{u}\| \|\Delta \overline{u}\|^{\frac{3}{2}} \leq \frac{1}{2}\|\Delta \overline{u}\|^2 + c_2 \|\overline{u}\|^2 \|\nabla \overline{u}\|^4, \end{aligned}$$

where $c_2 > 0$ on $\mathcal{O}$. Combining (5) and Young's inequality, we have

$$-2(g(\overline{u}_t(t)), \delta \overline{u}(t)) - 2(f, \Delta \overline{u}(t)) \leq \frac{1}{4}\|\overline{u}_t\|^2_{C_H} + 8\|f\|^2 + 8L_g^2 \|\nabla \overline{u}\|^2.$$

It follows from (10)-(3.9) that for $t > 0$,

$$\|\partial_t \nabla \overline{u}(t)\|^2 + \frac{3}{2}\|\Delta \overline{u}(t)\|^2 \leq c_2 \|\overline{u}\|^2 \|\nabla \overline{u}\|^4 + 8\|f\|^2 + 8L_g^2 \|\nabla \overline{u}\|^2. \quad (11)$$

By Lemma 3.1 for all $t \geq 0$, we obtain

$$\|\overline{u}(t)\|^2 + \int_t^{t+1} \|\overline{u}(s)\|_V^2 ds \leq c_3,$$

with $c_3 > 0$ depends on λ, L_g, f and u_0, we have

$$\int_t^{t+1} \|\nabla \overline{u}(s)\|^2 ds \leq c_3, \quad \int_t^{t+1} \|\overline{u}(s)\|^2 \|\nabla \overline{u}(s)\|^2 ds \leq c_3^2,$$

and by Gronwall Lemma, we have

$$\|\nabla \overline{u}(t)\|^2 \leq c_4,$$

where $c_4 > 0$ depends on λ, L_g, f and u_0.

Integrating from t to $t+1$ for (11), we obtain

$$\int_t^{t+1} \|\overline{u}(s)\|^2 ds \leq c_5,$$

with $c_5 > 0$ depends on λ, L_g, f and u_0. The proof is complete. □

We have proved the following uniform estimates for the large time by Lemma 3.2.

Lemma 3.3. *Let $R > 0$, there exists $T_0 = T_0(R) > 0$ such that for all $t \geq T_0$ and $u_0 \in H$ with $\|u_0\| \leq R$, the solutions u of (1) satisfies*

$$\|\nabla u(t)\|^2 + \int_t^{t+1} \|\Delta u(s)\|^2 ds + \int_t^{t+1} \|\partial_t u(s)\|^2 \leq C_3,$$

with $C_3 > 0$ is constant dependent of λ, L_g and f.

4 Uniform tail-end estimates of the solutions

We will the asymptotic compactness of the solutions of (1) by Uniform tail-end estimates of the solutions. Furthermore, we will obtain the existence of global attractors in H.

Suppose a smooth function $\phi : \mathbb{R} \to \mathbb{R}$ for $0 \leq \phi \leq 1$ and

$$\phi(s) = 0, |s| \leq \frac{1}{2}; \quad \phi(s) = 1, |s| \leq 1.$$

If $C > 0$ and $|\phi'(s)| + |\phi''(s)| \leq C$. Let $n \in \mathbb{N}$ and $x = (x_1, x_2) \in \mathcal{O}$, let $\phi_n(x) = \phi(\frac{n_1}{k})$.

We first give the following inequalities.

Lemma 4.1.
(a) Let $u \in H^1(\mathcal{O})$, then

$$|\|\phi_n \nabla u\| - \|\nabla(\phi_n u)\|| \leq \frac{C_4}{n}\|u\|.$$

(b) Let $u \in H^2(\mathcal{O})$, then

$$|\|\phi_n \Delta u\| - \|\Delta(\phi_n u)\|| \leq \frac{C_4}{n}\|u\|_{H^1(\mathcal{O})}.$$

(c) Let $u \in H_0^1(\mathcal{O})$, then

$$\|\phi_n \nabla u\| \geq \lambda^{\frac{1}{2}}\|\phi_n u\| - \frac{C_4}{n}\|u\|.$$

(d) If $u \in H^2(\mathcal{O}) \cap H_0^1(\mathcal{O})$, then

$$\|\phi_n \Delta u\| \geq \lambda^{\frac{1}{2}}\|\phi_n \nabla u\| - \frac{C_4}{n}\|u\|_{H^1(\mathcal{O})},$$

where $C_4 > 0$ is a constant independent of n and u.

Proof of Lemma 4.1.

(a) Note that $\nabla(\phi_n u) = u\nabla\phi_n + \phi_n \nabla u$,

$$|\|\phi_n \nabla u\| - \|\nabla(\phi_n u)\|| \leq \|\phi_n \nabla u - \nabla(\phi_n u)\| = \|u\nabla\phi_n\| \leq \frac{c}{n}\|u\|,$$

with $c > 0$ is a constant independent of n and u.
(b) Suppose $\Delta(\phi_n u) = u(\Delta\phi_n) + \phi_n(\Delta u) + 2\nabla\phi_n \cdot \nabla u$ such that

$$\|\phi_n(\Delta u) - \Delta(\phi_n u)\| \leq \|u(\Delta\phi_n)\| + 2\|\nabla\phi_n \cdot \nabla u\| \leq \frac{c}{n}(\|u\| + \|\nabla u\|).$$

(c) If $\|\nabla(\phi_n u)\| \geq \lambda^{\frac{1}{2}}\|\phi_n u\|$, by (a) we have $\|\phi_n \nabla u\| \geq \lambda^{\frac{1}{2}}\|\phi_n u\| - \frac{c}{n}\|u\|$.
(d)

$$\begin{aligned}\|\nabla(\phi_n u)\|^2 &= (\nabla(\phi_n u), \nabla(\phi_n u)) = -(\Delta(\phi_n u), \phi_n u) \\ &\leq \|\Delta(\phi_n u)\|\|\phi_n u\| \leq \lambda^{\frac{1}{2}}\|\Delta(\phi_n u)\|\|\nabla(\phi_n u)\|\end{aligned}$$

and $\|\nabla(\phi_n u)\| \leq \lambda^{-\frac{1}{2}}\|\Delta(\phi_n u)\|$. By (a) and (b) we have

$$\begin{aligned}\|\phi_n \nabla u\| &\leq \|\nabla(\phi_n u)\| + \frac{c_2}{n}\|u\| \leq \lambda^{-\frac{1}{2}}\|\Delta(\phi_n u)\| + \frac{c_2}{n}\|u\| \\ &\leq \lambda^{-\frac{1}{2}}(\|\phi_n \Delta u\| + \frac{c_3}{n}\|u\|_{H^1(\mathcal{O})}) + \frac{c_2}{n}\|u\| \leq \lambda^{-\frac{1}{2}}\|\phi_n \Delta u\| + \frac{c_4}{n}\|u\|_{H^1(\mathcal{O})},\end{aligned}$$

which shows that

$$\lambda^{\frac{1}{2}}\|\phi_n \nabla u\| \leq \|\phi_n \Delta u\| + \frac{c_4}{n}\lambda^{\frac{1}{2}}\|u\|_{H^1(\mathcal{O})}.$$

This proof is complete. □

Furthermore, we give a scalar stream function for (1). Let $u = (u_1, u_2) \in V$ and $x = (x_1, x_2) \in \bar{\mathcal{O}}$, define

$$\widehat{u}(x) = \widehat{u}(x_1, x_2) = \int_{(0,0)}^{(x_1,x_2)} -u_2 dx_1 + u_1 dx_2.$$

Since $u = 1$ on $\partial\mathcal{O}$ and $\mathrm{div}(u) = 0$ in $\mathcal{O}$.

Then, we get

$$\partial_{x_1}\widehat{u} = -u_2, \quad \partial_{x_2}\widehat{u} = u_1, \tag{12}$$

and

$$\widehat{u}|_{\partial\mathcal{O}} = 0, \ \ \nabla\widehat{u}|_{\partial\mathcal{O}} = 0.$$

If T is the curl operator that has

$$Tu = \partial_{x_2}u_1 - \partial_{x_1}u_2, \ \forall u = (u_1, u_2). \tag{13}$$

By (12) and (13), we have

$$\partial_t\Delta\widehat{u} = \Delta^2\widehat{u} + \widehat{B}(\widehat{u}, \widehat{u}) + \widehat{g}(u_t) + \widehat{f}, \tag{14}$$

where

$$\widehat{g}(u_t) = T(g(u_t)), \ \ \widehat{f} = Tf, \ \ \widehat{B}(\widehat{u}, \widehat{u}) = \partial_{x_2}((\partial_{x_1}\widehat{u})\Delta\widehat{u}) - \partial_{x_2}((\partial_{x_2}\widehat{u})\Delta\widehat{u}).$$

Let $n \in \mathbb{N}$, we denote $\mathcal{O}_n = (-n, n) \times (0, d)$. By (14), we prove the uniform tail-end estimates of the solutions of (1) on $\mathcal{O} \setminus \mathcal{O}_n$ as below.

Lemma 4.2. *Let $u_0 \in H$ and $\varepsilon > 0$, there exists $\mathcal{N} = \mathcal{N}(\lambda, L_g, f, u_0, \varepsilon) \geq 1$ such that the solution u of (1) satisfies, for all $n \geq N$ and $t \geq 1$,*

$$\int_{\mathcal{O}\setminus\mathcal{O}_k} |u(t,x)|^2 dx \leq \varepsilon.$$

Proof of Lemma 4.2. We prove the uniform estimates of the solutions by a limiting process. By (14) we get

$$\begin{aligned} -\frac{d}{dt}(\Delta\widehat{u}, \phi_n^2\widehat{u}) &= -(\Delta^2\widehat{u}, \phi_n^2\widehat{u}) - (\widehat{B}(\widehat{u}, \widehat{u}), \phi_n^2\widehat{u}) - (\widehat{g}(u_t), \phi_n^2\widehat{u}) - (\widehat{f}, \phi_n^2\widehat{u}) \\ &= J_i(i = 1, 2, 3). \end{aligned} \tag{15}$$

For the left-hand side of (15) we get

$$\begin{aligned} -\frac{d}{dt}(\Delta\widehat{u}, \phi_n^2\widehat{u}) &= \frac{1}{2}\frac{d}{dt}\int_{\mathcal{O}} \phi_n^2|\nabla\widehat{u}|^2 dx + \int_{\mathcal{O}} (\nabla\widehat{u}_t, \nabla\phi_n^2)\widehat{u}dx \\ &\geq \frac{1}{2}\frac{d}{dt}\int_{\mathcal{O}} \phi_n^2|\nabla\widehat{u}|^2 dx - \frac{c_1}{n}\|\nabla\widehat{u}_t\|\|\widehat{u}\|, \end{aligned}$$

where $c_1 > 0$ is independent of n. For the J_1 of (15) we obtain

$$\begin{aligned} J_1 = -(\Delta^2\widehat{u}, \phi_n^2\widehat{u}) &= -\int_{\mathcal{O}} \Delta\widehat{u}(\phi_n^2\Delta\widehat{u} + (\Delta\phi_n^2)\widehat{u} + 2\nabla\phi_n^2 \cdot \nabla\widehat{u})dx \\ &\leq -\|\phi_n\Delta\widehat{u}\|^2 + \frac{c_2}{n^2}\|\widehat{u}\|\|\Delta\widehat{u}\| + \frac{c_2}{n}\|\nabla\widehat{u}\|\|\Delta\widehat{u}\|. \end{aligned} \tag{16}$$

By Lemma4.1 we suppose exists $c_3 > 0$ independent of k such that

$$\lambda^{\frac{1}{2}}\|\phi_n\nabla\widehat{u}\| \leq \|\phi_n\Delta\widehat{u}\| + \frac{c_3}{n}\|\widehat{u}\|_{H^1(\mathcal{O})}$$

and hence by Young's inequality, we have

$$\begin{aligned}\lambda\|\phi_n\nabla\widehat{u}\|^2 \le& \|\phi_n\Delta\widehat{u}\|^2+\frac{2c_3}{n}\|\phi_n\Delta\widehat{u}\|\|\widehat{u}\|_{H^1\mathcal{O}}+\frac{c_3^2}{n^2}\|\widehat{u}\|_{H^1\mathcal{O}}\\ \le& \|\phi_n\Delta\widehat{u}\|^2+\frac{\alpha}{4\lambda-\alpha}\|\phi_n\Delta\widehat{u}\|^2+\frac{(4\lambda-\alpha)c_3^2}{\alpha n^2}\|\widehat{u}\|^2_{H^1(\mathcal{O})}+\frac{c_3^2}{n^2}\|\widehat{u}\|^2_{H^1(\mathcal{O})}\\ =&\frac{4\lambda}{4\lambda-\alpha}\|\phi_n\Delta\widehat{u}\|^2+\frac{4\lambda c_3^2}{\alpha n^2}\|\widehat{u}\|^2_{H^1(\mathcal{O})}.\end{aligned}$$

we have

$$-\|\phi_n\Delta\widehat{u}\|^2\le(\frac{1}{4}\alpha-\lambda)\|\phi_n\nabla\widehat{u}\|^2+\frac{(4\lambda-\alpha)c_3^2}{\alpha n^2}\|\widehat{u}\|^2_{H^1(\mathcal{O})},$$

By (16) and (17) we have

$$-(\Delta^2\widehat{u},\phi_n^2\widehat{u})\le(-\frac{1}{4}\alpha-\lambda)\|\phi_n\nabla\widehat{u}\|^2+\frac{c_4}{n}\|\Delta\widehat{u}\|^2,$$

where $c_4>0$ is independent of n.

For the J_2 of (15), we obtain

$$\begin{aligned}J_2=-(\widehat{B}(\widehat{u},\widehat{u}),\phi_n^2\widehat{u})=&-\int_{\mathcal{O}}\widehat{u}(\Delta\widehat{u})(\partial_{x_2}\widehat{u})(\partial_{x_1}\phi_n^2)dx\\ \le&\frac{c_5}{n}\|\Delta\widehat{u}\|\|\widehat{u}\|_{L^4_(\mathcal{O})}\|\partial_{x_2}\widehat{u}\|_{L^4(\mathcal{O})}\\ \le&\frac{c_6}{n}\|\widehat{u}\|^{\frac{3}{2}}\|\nabla\widehat{u}\|^{\frac{3}{2}}\le\frac{c_7}{n}(\|\Delta\widehat{u}\|^2+\|\nabla\widehat{u}\|^6),\end{aligned}$$

where $c_7>0$ is independent of n.

Suppose $g(u_t)=(g_1(u_t),g_2(u_t))$ and $f=(f_1,f_2)$. Then for J_3 of (15), by (5), (12) we have

$$\begin{aligned}-(\widehat{g}(u_t),\phi_n^2\widehat{u})-(\widehat{f},\phi_n^2\widehat{u})=&\int_{\wr}(-g_2(u_t),g_1(u_t))\cdot\nabla(\phi_n^2\widehat{u})dx+\int_{\mathcal{O}}(-f_2,f_1)\cdot\nabla(\phi_n^2\widehat{u})dx\\ \le&\int_{\mathcal{O}}\phi_n^2|\nabla\widehat{u}|(|g(u_t)|_{C_H}+|f|)dx+\int_{\mathcal{O}}|\widehat{u}|(|g(u_t)|+|f|)|\nabla\phi_n^2|dx\\ \le& L_g\int_{\mathcal{O}}\phi_n^2|\nabla\widehat{u}|^2_{C_H}dx+\int_{\mathcal{O}}\phi_n^2|\nabla\widehat{u}||f|dx+\int_{\mathcal{O}}|\widehat{u}|(L_g|\nabla\widehat{u}|+|f|)|\nabla\psi_n^2|dx\\ \le&(L_g+\frac{1}{4}\alpha)\int_{\mathcal{O}}\phi_n^2|\nabla\widehat{u}|^2_{C_H}dx+\frac{1}{\alpha}\int_{\mathcal{O}}\phi_n^2|f|^2dx+\frac{c_8}{n}(\|\nabla\widehat{u}\|^2+\|f\|^2),\end{aligned}\tag{17}$$

where $c_8>0$ is independent of n. which along with (15)-(17) we have

$$\frac{d}{dt}\|\phi_n\nabla\widehat{u}\|^2+\alpha\|\phi_n\nabla\widehat{u}\|^2\le\frac{c_9}{n}(\|\nabla\widehat{u}_t\|^2_{C_H}+\|\nabla\widehat{u}\|^6+\|\Delta\widehat{u}\|^2+\|f\|^2)+c_9\|\phi_nf\|^2,\tag{18}$$

with $c_9>0$ is independent of n.

combining (12) and (18), we have

$$\|\partial_t\phi_nu\|^2+\alpha\|\phi_nu\|^2\le\frac{c_9}{n}(\|u_t\|^2_{C_H}+\|u\|^6+2\|\nabla u\|^2+\|f\|^2)+c_9\|\phi_nf\|^2.\tag{19}$$

Integrating (19) on $(1,t)$ for $t\ge1$ to have

$$\begin{aligned}\|\phi_nu(t)\|^2\le& e^{\alpha(1-t)}\|\phi_nu(1)\|^2\\ &+\frac{c_9}{n}\int_1^te^{\alpha(s-t)}(\|u_t(s)\|^2_{C_H}+\|u(s)\|^6+2\|\nabla u(s)\|^2+\|f\|^2)ds+c_9\alpha^{-1}\|\phi_nf\|^2.\\ &=J_i(i=1,2,3)\end{aligned}\tag{20}$$

For the J_1 of (20), we have for $t \geq 1$,

$$e^{\alpha(1-t)}\|\phi_n u(1)\|^2 \leq \int_{\{(x_1,x_2)\in\mathcal{O}:|x_1|\geq\frac{1}{2}n\}} |u(1,x)|^2dx \to 0, \ \ n \to \infty.$$

For the J_2 of (20), by Lemma3.1 and Lemma3.2 we have for $t \geq 1$,

$$\begin{aligned}
&\frac{c_9}{n}\int_1^t e^{\alpha(s-t)}(\|u_t(s)\|^2_{C_H} + \|u(s)\|^6 + 2\|\nabla u(s)\|^2 + \|f\|^2) \\
\leq&\frac{c_9}{n}(c_{10} + \|f\|)\int_1^t e^{\alpha(s-t)}ds + \frac{c_9}{n}\int_1^t e^{\alpha(s-t)}\|u_t(s)\|^2_{C_H}ds \\
\leq&\frac{c_9}{n}(c_{10} + \|f\|) + \frac{c_9}{n}(\int_1^2 e^{\alpha(s-t)}\|u_t(s)\|^2_{C_H}ds + \cdots + \int_{[t]}^t e^{\alpha(s-t)}\|u_t(s)\|^2_{C_H}ds) \\
\leq&\frac{c_9}{n}(c_{10} + \|f\|) + \frac{c_9}{n}(e^{\alpha(2-t)}\int_1^2 \|u_t(s)\|^2_{C_H}ds + \cdots \\
&+ e^{\alpha([t]-t)}\int_{[t]-1}^{[t]} \|u_t(s)\|^2_{C_H}ds + \int_{[t]}^t \|u_t(s)\|^2_{C_H}ds) \\
\leq&\frac{c_{11}}{n} + \frac{c_{11}}{n}(e^{\alpha(2-t)} + \cdots + e^{\alpha([t]-t)}) \leq \frac{c_{11}}{n} + \frac{c_{11}}{n}e^{\alpha([t]-t)}\Sigma_{j=0}^{\infty}r^{-\alpha j} \\
\leq&\frac{c_{11}}{n} + \frac{c_{11}}{n}(1-e^{-\alpha})^{-1} \to 0, \ \ n \to \infty,
\end{aligned}$$

with $c_{11} > 0$ dependent of λ, L_g, f and u_0.

If $f \in H$, for the J_3 of (20), we get

$$c_9\alpha^{-1}\|\phi_n f\|^2 = c_9\alpha^{-1}\int_{\{(x_1,x_2)\in\mathcal{O}:|x_1|\geq\frac{1}{2}n\}} |f(x)|^2dx \to 0, \ \ n \to \infty. \tag{21}$$

From (20)-(21) for every $\varepsilon > 0$, then exists $N = N(\lambda, L_g, f, u_0, \varepsilon) \geq 1$, such that for all $t \geq 1$ and $n \geq N$,

$$\int_{\{(x_1,x_2)\in\mathcal{O}:|x_1|\geq n\}} |u(t,x)|^2dx \leq \|\phi_n u(t)\|^2 \leq \varepsilon.$$

□

Next, we will consider the uniform tail-ends estimates of the solutions under bounded initial data in H.

Lemma 4.3. *Let $R > 0$ and $\varepsilon > 0$, there exists $T_1 = T_1(\lambda, L_g, f, R, \varepsilon) > 0$ and $N = N(\lambda, L_g, f, \varepsilon) \geq 1$, such that if $u_0 \in H$ with $\|u_0\| \leq R$, then the solution u of (1) satisfies, for all $t \geq T_1$ and $n \geq N$,*

$$\int_{\mathcal{O}\backslash\mathcal{O}_n} |u(t,x)|^2dx \leq \varepsilon.$$

Proof of Lemma 4.3. Let $u_0 \in H$ with $\|u_0\| \leq R$. By Lemma 3.3, we have that there exists $T_0 = T_0(R) > 0$ such that for all $t \geq T_0$,

$$\|\nabla u(t)\|^2 + \int_t^{t+1} \|\partial_t u(s)\|^2ds \leq C_1, \tag{22}$$

where $C_1 > 0$ dependent of λ, L_g and f.

Integrating (19) on (T_0, t) for $t \geq T_0$, we obtain

$$\begin{aligned}
\|\phi_n u(u)\|^2 \leq& e^{\alpha(T_0-t)}\|\phi_n u(T_0)\|^2 \\
&+ \frac{c_9}{n}\int_{T_0}^t e^{\alpha(s-t)}(\|u_t(s)\|^2_{C_H} + \|u(s)\|^6 + 2\|\nabla u(s)\|^2 + \|f\|^2)ds + c_9\alpha^{-1}\|\phi_n f\|^2 \\
=& I_1 + I_2.
\end{aligned} \tag{23}$$

For I_1 of (23), by Lemma3.1 we have get

$$e^{\alpha(T_0-t)}\|\phi_n u(T_0)\|^2 \leq e^{\alpha(T_0-t)}\|u(T_0)\|^2 \leq e^{\alpha(T_0-t)}(e^{-\frac{1}{2}\alpha T_0}R^2 + C_2),$$

where $C_2 > 0$ dependent of λ, L_g and f. If $\varepsilon > 0$, there exists $\mathcal{T} = \mathcal{T}(\lambda, L_g, f, R, \varepsilon) \geq T_0$ such that for all $t \geq T_1$,

$$e^{\alpha(T_0-t)}\|\phi_n u(T_0)\|^2 \leq \frac{1}{4}\varepsilon.$$

For I_2 of (23), by (22) we obtain

$$\begin{aligned}
&\frac{c_9}{n}\int_{T_0}^{t} e^{\alpha(s-t)}(\|u_t(s)\|^2_{C_H} + \|u(s)\|^6 + 2\|\nabla u(s)\|^2 + \|f\|^2) \\
\leq&\frac{C_3}{n} + \frac{C_3}{n}(e^{\alpha(T_0+1-t)}\int_{T_0}^{T_0+1}\|u_t(s)\|^2_{C_H}ds + \cdots \\
&+ e^{\alpha([t]-t)}\int_{[t]-1}^{[t]}\|u_t(s)\|^2_{C_H}ds + \int_{[t]}^{t}\|u_t(s)\|^2_{C_H}ds) \\
\leq&\frac{C_3}{n} + \frac{C_1C_3}{n} + \frac{C_1C_3}{n}(1-e^{-\alpha})^{-1} \to 0, \quad n \to \infty,
\end{aligned} \tag{24}$$

where $C_3 > 0$ dependent of λ, L_g and f. Combining (23)-(24) and (21), we have that there exists $N = N(\lambda, L_g, \varepsilon) \geq 1$ such that for all $t \geq T_1$ and $n \geq N$,

$$\int_{\{(x_1,x_2)\in\mathcal{O}:|x_1|\geq n\}} |u(t,x)|^2 dx \leq \|\phi_n u(t)\|^2 \leq \varepsilon.$$

□

By Lemma 4.3, we derive the asymptotic compactness of the solutions in H.

Lemma 4.4. *Let (5) hold, suppose $\{u_{0,n}\}_{n=1}^{\infty}$ is bounded in H and $t_n \to \infty$, then $\{S(t_n)u_{0,n}\}_{n=1}^{\infty}$ is precompact in H.*

Proof of Lemma 4.4. If $\{u_{0,n}\}_{n=1}^{\infty}$ is bounded in H, by Lemma 4.3 for every $\varepsilon > 0$, there exist $N_1 = N_1(\varepsilon) \geq 1$ and $\mathcal{N} = \mathcal{N}(\varepsilon) \geq 1$ such that for all $n \geq N_1$, we have

$$\|S(t_n)u_{0,n}\|_{L^2(\mathcal{O}\setminus\mathcal{O}_N)} < \frac{\varepsilon}{4}.$$

By Lemma 3.3 we have that there exists $N_2 = N_2(\varepsilon) \geq N_1$ such that $\{S(t_n)u_{0,n}\}_{n=N_2}^{\infty}$ is bounded in V. Since the compactness of the embedding $H^1(\mathcal{O}_N) \hookrightarrow L^2(\mathcal{O}_N)$ we infer that $\{S(t_n)u_{0,n}\}_{n=1}^{\infty}$ has a finite cover of radius $\frac{1}{4}$ in $L^2(\mathcal{O}_N)$, and by (4.23) show that $\{S(t_n)u_{0,n}\}_{n=1}^{\infty}$ has a finite cover of radius ε in H □

By Lemma 3.3 and 4.4, we prove the existence of global attractors of (1) follows.

Theorem 4.5. *Let (5) holds, then system (1) has a unique global attractor in H.*

5 Conclusion

We establish the uniform estimates of 2D Navier-Stokes equations of the solutions in H and V to obtain uniform estimates of the solutions. Furthermore, We proved the asymptotic compactness and the existence of global attractors in H by the uniform tail-end estimates. Finally, we proved the existence of the global attractors of delay 2D Naiver-stokes equations on unbounded Channel-like domains.

6 Declarations

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Authors's Contributions

Conceptualization, Z. Zhang; methodology, Z. Zhang and X. Yao; investigation, Z. Zhang; writing—original draft preparation, Z. Zhang; writing—review and editing, X. Yao.

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