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Analytical Solutions of the D-dimensional Klein-Gordon equation with q-deformed modified Pöschl-Teller Potential

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Abstract

In this article, the D-dimensional Klein-Gordon equation within the framework of Greene-Aldrich approximations scheme for q-deformed modified Pöschl-Teller Potential is solved for s-wave and arbitrary angular momenta. The energy eigenvalues and corresponding wave functions are obtained in an exact analytical manner via the Nikiforov-Uvarov (N-U) method. Further, it is shown that in the non-relativistic limit, the energy eigenvalues reduce to that of Schrödinger equation for the potential. It is also shown that the obtained results lead to the solutions of the same problem for modified Pöschl-Teller potential for $q = 1$.

Key words: q-deformed modified Pöschl-Teller potential, Greene-Aldrich approximation, Nikiforov-Uvarov (N-U) method
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1 Introduction

Solutions of coherent and incoherent quantum mechanical wave equations with many physical potentials play an important role in determining quantum mechanical phenomena and other dynamics of quantum systems. Eigenvalues and wave functions provide important information in describing various quantum systems [1, 2, 3]. Recently, the problem of finding a solution to the multidimensional potential of the D-dimensional Klein-Gordon equation has received much attention [4, 5, 6, 7, 8, 9, 10, 11].

To explain the behavior and interaction of atoms and particles, a model that includes the concept of potential is necessary. Potentials are important to explain interactions among atomic nuclei, nuclear particles, and diatomic molecular structures. Various potentials are used in literature like pseudoharmonics [12], modified Eckart plus Hylleraas [13], Morse type [14],

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Wood-Saxon [15], Rosen-Morse [16], harmonic oscillator [17], especially at lower sizes and the method includes Nikiforov-Uvarov method [18, 19, 20], asymptotic iterative method [21], Point-Cannonical transform [22], Lie algebraic method [23], supersymmetry method [24], Laplace transform methods [25, 26], factorization method [27] and others.

My current work is to study the Klein-Gordon equation within the framework of N-U method with q-deformation correction for the Pöschl-Teller potential [28, 29, 30]. It also showed that for $q = 1$, results can be obtained by adjusting the Pöschl-Teller potential.

The q-deformed modified Pöschl-Teller potential is a compound potential widely used in molecular physics and quantum chemistry. The expression of the q-deformed Pöschl-Teller potential is as follows:

$$V(r) = -\frac{V_0}{\cosh_q^2(\alpha r)} \tag{1}$$

Here, the most affordable and powerful Nikiforov-Uvarov (N-U) method is used for D-dimensional calculations. We use the Green-Aldrich approach [31] to study the behavior of Pöschl-Teller potential in the Klein-Gordon equation and use some simple constraints to ensure that the equation can be solved by the NU method.

My work is organized as follows: Section 2 presents a brief review of the N-U approach to make it usable. In section 3, the D-dimensional Klein-Gordon equation is presented considering the Pöschl-Teller Potential as well as Greene-Aldrich approximation. In section 4, the eigenvalues of the D-dimensional Klein-Gordon equation and the corresponding wave functions are obtained by using the N-U method. The non-relativistic limit of the energy eigenvalues and corresponding wave functions are obtained in section 5. Section 6 contains the conclusions.

2 Nikiforov-Uvarov Method

The N-U method is proposed to solve second-order linear differential equations of the form :

$$\varphi''(s) + \frac{\tilde{\tau}(s)}{\sigma(s)}\varphi'(s) + \frac{\tilde{\sigma}(s)}{\sigma^2(s)}\varphi(s) = 0 \tag{2}$$

With the polynomials $\sigma(s)$, $\tilde{\sigma}(s)$ of degree atmost 2 and the polynomial $\tilde{\tau}(s)$ of degree atmost 1 . To solve equation(2) , we set

$$\varphi(s) = \Upsilon(s)p(s) \tag{3}$$

Thus equation (2) reduces to a hyper-geometric type equation of the form :

$$\sigma(s)y''(s) + \tau(s)y'(s) + \lambda y(s) = 0$$

Where $\tau(s) = \tilde{\tau}(s) + 2\pi(s)$ satisfies the condition $\tau'(s) < 0$ and $\pi(s)$ is defined as

$$\pi(s) = \frac{\sigma'(s) - \tilde{\tau}(s)}{2} \pm \sqrt{(\frac{\sigma'(s) - \tilde{\tau}(s)}{2})^2 - \tilde{\sigma}(s) + K\sigma(s)} \tag{4}$$

The parameter K should be so chosen that $\pi(s)$ should be an atmost 1 degree polynomial [18]. This indicates the discriminant of equation (4) should be zero, which helps us a quadratic equation in K. With the help of

$$\lambda = K + \pi'(s) = -n\tau'(s) - \frac{n(n-1)}{2}\sigma''(s) \tag{5}$$

the eigenvalues can be determined. The polynomial solution $p_n(s)$ is represented by the Rodriguez relation

$$p_n(s) = \frac{N}{\rho(s)}\left(\frac{d}{ds}\right)^n [\sigma^n(s)\rho(s)] \tag{6}$$

where N is normalization constant and $\rho(s)$ is the weight function satisfying

$$\rho(s) = \frac{1}{\sigma(s)} e^{\int \frac{\tau(s)}{\sigma(s)} ds} \tag{7}$$

And the other part of the wave function $\Upsilon(s)$ is

$$\Upsilon(s) = e^{\int \frac{\pi(s)}{\sigma(s)} ds} \tag{8}$$

3 The D-dimensional Klein-Gordon equation

The time unbiased Klein-Gordon equation in D-dimension for unit values in atomic structure ($\hbar = c = \mu = 1$), can be expressed as [32],

$$\nabla_D^2 \Psi(r,\Omega_D) + \left[\Big(E - V(r)\Big)^2 - \Big(M + S(r)\Big)^2\right]\Psi(r,\Omega_D) = 0 \tag{9}$$

where, $\mathbf{M}$, $\mathbf{E}$, $\mathbf{V}(\mathbf{r})$ and $\mathbf{S}(\mathbf{r})$ denotes the particle mass, energy, vector and scalar potentials respectively. The Laplacian operator ∇_D^2 is [33] ,

$$\nabla_D^2 = r^{1-D}\frac{\partial}{\partial r}\left(r^{D-1}\frac{\partial}{\partial r}\right) + \frac{L_D^2(\Omega_D)}{r^2} \tag{10}$$

where $L_D^2(\Omega_D)$ represents the angular momentum [34]. In addition, we know that $\frac{L_D^2(\Omega_D)}{r^2}$ is a generalization of the centrifugal barrier for the D-dimensional space and includes angular coordinates Ω_D and the eigenvalues of $L_D^2(\Omega_D)$ [33]. The operator $L_D^2(\Omega_D)$ on the unit space S^{D-1} analogously defines a 3-D angular momentum [34] as $L_D^2(\Omega_D) = -\Sigma_{i\geq j}^D (L_{ij}^2)$ where $L_{ij}^2 = x_i \frac{\partial}{\partial x_j} - x_j \frac{\partial}{\partial x_i}$ for all Cartesian component x_i of the D-dimensional vector $(x_1, x_2, \ldots, x_D)$.

To eliminate the first order derivative, the total wave function can be defined as

$$\Psi(r,\Omega_D) = r^{\frac{(D+1)}{2}} R_{nl}(r) Y_{lm}(\Omega_D) \tag{11}$$

And the eigenvalue equation is $L_2^D Y_{lm}(\Omega_D) = l(l + D - 2)Y_{lm}(\Omega_D)$ with the notation $Y_{lm}(\Omega_D)$ for generalized spherical harmonic function. In this case, the radial part of the D-dimensional Klein-Gordon equation can be written as:

$$\frac{d^2 R_{nl}(r)}{dr^2} + \left[\Big(E^2 - M^2\Big) - 2\Big(MS(r) + EV(r)\Big)\right. \\ \left. + V^2(r) - S^2(r) - \frac{(2l + D - 3)(2l + D - 1)}{4r^2}\right] R_{nl}(r) = 0 \tag{12}$$

Assuming $V(r) = S(r)$, equation (12) becomes

$$\frac{d^2 R_{nl}(r)}{dr^2} + \left[\Big(E^2 - M^2\Big) - 2V(r)(M + E) - \frac{(2l + D - 3)(2l + D - 1)}{4r^2}\right] R_{nl}(r) = 0 \tag{13}$$

The solution for the above equation with $l \neq 0$ depends on the replacement of singularity and a suitable approximation scheme. The approximation method used in this paper to manage the centrifugal terms is the Green-Aldrich approximation, defined as:

$$\frac{1}{r^2} \approx \frac{4\alpha^2 e^{-2\alpha r}}{\left(1+qe^{-2\alpha r}\right)^2} \tag{14}$$

Inserting the potential function in exponential form (i.e.

$$V(r) = -\frac{V_0}{\cosh_q^2(\alpha r)} = -\frac{4V_0 e^{-2\alpha r}}{\left(1+qe^{-2\alpha r}\right)^2}$$

by using the expression of deformed hyperbolic function $\cosh_q(\alpha r) = \frac{e^{\alpha r}+qe^{-\alpha r}}{2}$) and the modified centrifugal term as given in Eqn. (1) and Eqn. (14) respectively, Eqn. (13) reduces to

$$\frac{d^2 R_{nl}(r)}{dr^2} + \left[\left(E^2 - M^2\right) + 8(M+E)V_0 \frac{e^{-2\alpha r}}{\left(1+qe^{-2\alpha r}\right)^2} - \frac{(2l+D-3)(2l+D-1)\alpha^2 e^{-2\alpha r}}{\left(1+qe^{-2\alpha r}\right)^2}\right] R_{nl}(r) = 0 \tag{15}$$

4 The energy eigenvalues and corresponding wave functions

In order to solve Eqn. (15) by the N-U method, we need to recast it into a solvable form. To do so, I introduce a new variable $s = e^{-2\alpha r}$ and Eqn. (15) takes the form

$$\frac{d^2R(s)}{ds^2} + \frac{(1+qs)}{s(1+qs)}\frac{d}{ds} + \frac{1}{s^2(1+qs)^2}\left[q^2\epsilon^2 s^2 + 2\left(q\epsilon^2 + \beta^2 - \gamma^2\right)s + \epsilon^2\right]R(s) = 0 \tag{16}$$

Where I have used the notations

$$\epsilon^2 = \frac{E^2 - M^2}{4\alpha^2}, \quad \beta^2 = \frac{(M+E)V_0}{\alpha^2}, \quad \gamma^2 = \frac{1}{2}(2l+D-1)(2l+D-3)$$

Comparing Eqn. (16) with Eqn. (2),

$$\tilde{\tau}(s) = (1+qs), \quad \sigma(s) = s(1+qs)$$
$$\tilde{\sigma}(s) = q^2\epsilon^2 s^2 + 2\left(q\epsilon^2 + \beta^2 - \gamma^2\right)s + \epsilon^2 \tag{17}$$

Substituting them into relation (4) leads to

$$\pi(s) = \frac{qs}{2} \pm \sqrt{\left(\frac{q^2}{4} - q^2\epsilon^2 + qK\right)s^2 - [2(q\epsilon^2 + \beta^2 - \gamma^2) - K]s - \epsilon^2} \tag{18}$$

Next, we need to set the discriminant of the above equation under the square root to 0. So, one can get it easily

$$\triangle = [2(q\epsilon^2 + \beta^2 - \gamma^2) - K]^2 + 4\epsilon^2\left(\frac{q^2}{4} - q^2\epsilon^2 + qK\right) = 0 \tag{19}$$

Solving Eqn. (19) for the constant K, the double roots are obtained as $K_{1,2} = 2(\beta^2 - \gamma^2) \pm 2a\epsilon$, where $a = \sqrt{2q(\beta^2 - \gamma^2) - \frac{q^2}{4}}$. Inserting the values of K in (18), I got:

$$\pi(s) = \frac{qs}{2} \pm \begin{cases} (a-\epsilon)s - a; & for \quad K_1 = 2(\beta^2 - \gamma^2) + 2a\epsilon \\ (a+\epsilon)s - a; & for \quad K_2 = 2(\beta^2 - \gamma^2) - 2a\epsilon \end{cases} \tag{20}$$

By choosing an appropriate value for K in $\pi(s)$ which satisfies the condition $\tau'(s) < 0$, one gets $\pi(s) = -(a + \epsilon - \frac{q}{2})s) + a$ for $K_2 = 2(\beta^2 - \gamma^2) - 2a\epsilon$; giving the function:

$$\tau(s) = -2(a + \epsilon - q)s + 1 + 2a \tag{21}$$

As per Eqn. (5), the constant λ is defined as

$$\lambda = 2(\beta^2 - \gamma^2) - 2a\epsilon - (a + \epsilon - \frac{q}{2}) \tag{22}$$

Also by Eqn. (5):

$$\lambda_n = -n\tau'(s) - \frac{n(n-1)}{2}\sigma''(s) \tag{23}$$

Here,

$$\tau'(x) = -2(a + \epsilon - q) \qquad and \qquad \sigma''(s) = 2q \tag{24}$$

Carrying out some simple algebraic calculation with the equations (22), (23) and (24), we have

$$\epsilon_n = \left[\frac{q + 2\left(q(n + \frac{1}{2}) - a\right)^2}{n + \frac{1}{2} + a} \right] \tag{25}$$

Squaring both sides of Eqn. (25), we have

$$\epsilon_n^2 = \left[\frac{q + 2\left(q(n + \frac{1}{2}) - a\right)^2}{n + \frac{1}{2} + a} \right]^2 \tag{26}$$

This constitutes the energy eigenvalue equation for q-deformed modified Pöschl-Teller potential and the equation (by putting the values of notations a, ϵ, β and γ) is of the form:

$$E_{nl} = \frac{2\alpha^2}{M} \left[\frac{q + 2\left(q(n + \frac{1}{2}) - \chi\right)^2}{n + \frac{1}{2} + \chi} \right]^2 \tag{27}$$

Where, $\chi = \sqrt{q\left[\frac{4MV_0}{\alpha^2} - (2l + D - 1)(2l + D - 3)\right] - \frac{q^2}{4}}$

Using (7) I got, $\rho(s) = s^{2\epsilon}(1 + qs)^{2a}$ and inserting this into (6)

$$p_n(s) = \frac{N}{s^{2\epsilon}(1+qs)^{2a}} \left(\frac{d}{ds}\right)^n [s^n(1+qs)^n s^{2\epsilon}(1+qs)^{2a}]$$
$$= \frac{N}{s^{2\epsilon}(1+qs)^{2a}} P_n^{(2\epsilon,2a)}(1+2qs) \tag{28}$$

where $P_n^{(2\epsilon,2a)}(1 + 2qs)$ is Jacobi polynomial [35, 36] and N is normalization constant. And from (8),

$$\varphi(s) = s^{\epsilon}(1 + qs)^{(\frac{1}{2} + a)} \tag{29}$$

Therefore, the wave function is:

$$R(s) = Ns^{\epsilon}(1 + qs)^{(a + \frac{1}{2})} P_n^{(2\epsilon,2a)}(1 + 2qs) \tag{30}$$

4.1 For $q = 1$

For $q = 1$, the q-deformed modified Pöschl-Teller potential converted to modified Pöschl-Teller potential of the form: $V(r) = -\frac{V_0}{\cosh^2(\alpha r)} = -\frac{4V_0 e^{-2\alpha r}}{\left(1+e^{-2\alpha r}\right)^2}$ and the eigenenergy takes the form:

$$E_{nl} = \frac{2\alpha^2}{M}\left[\frac{1+2\left(n+\frac{1}{2}-\chi\right)^2}{n+\frac{1}{2}+\kappa}\right]^2 \tag{31}$$

Where, $\kappa = \sqrt{\left[\frac{4MV_0}{\alpha^2} - (2l+D-1)(2l+D-3)\right] - \frac{1}{4}}$

And the corresponding wave function is given by

$$R(s) = B_n s^{\epsilon}(1+s)^{(a+\frac{1}{2})} P_n^{(2\epsilon,2a)}(1+2s) \tag{32}$$

5 Non-relativistic limit

The Schrodinger equation represents the non-relativistic spinless particle while the Klein-Gordon equation represents a particle with spin-zero. Actually, for lower values of $S(r)$ and $V(r)$ with respect to the rest mass, the transformation $E + \mu c^2 \longrightarrow 2\mu c^2$ and $E - \mu c^2 \longrightarrow E$ are suitable to find non-relativistic energies. Applying this in Eqn. (26), we have

$$E_{nl} = \frac{\alpha^2}{\mu c^2}\left[\frac{q+2\left(q(n+\frac{1}{2})-\zeta\right)^2}{n+\frac{1}{2}+\zeta}\right]^2 \tag{33}$$

Where $\zeta = \sqrt{q\left[\frac{8\mu c^2 V_0}{\alpha^2} - (2l+D-1)(2l+D-3)\right] - \frac{q^2}{4}}$ and the corresponding wave function is obtained as follows

$$R(s) = B_n s^{\epsilon}(1+qs)^{(\zeta+\frac{1}{2})} P_n^{(2\epsilon,2\zeta)}(1+2qs) \tag{34}$$

5.1 For $q = 1$

For $q = 1$, the modified q-deformed Pöschl-Teller potential converted to modified Pöschl-Teller potential of the form: $V(r) = -\frac{V_0}{\cosh^2(\alpha r)} = -\frac{4V_0 e^{-2\alpha r}}{\left(1+e^{-2\alpha r}\right)^2}$ and the eigenenergy and takes the form:

$$E_{nl} = \frac{\alpha^2}{\mu c^2}\left[\frac{1+2\left(n+\frac{1}{2}-\varsigma\right)^2}{n+\frac{1}{2}+\varsigma}\right]^2 \tag{35}$$

Where $\varsigma = \sqrt{\left[\frac{8\mu c^2 V_0}{\alpha^2} - (2l+D-1)(2l+D-3)\right] - \frac{1}{4}}$ The corresponding wave function is given by

$$R(s) = B_n s^{\epsilon}(1+s)^{(\varsigma+\frac{1}{2})} P_n^{(2\epsilon,2\varsigma)}(1+2s) \tag{36}$$

6 Conclusion

In this article, the solutions of the D-dimensional Klein-Gordon equation with equal scalar and vector potentials for the q-deformed modified Pöschl-Teller potential applying N-U method upon application of Greene-Aldrich approximation to the centrifugal term. The relativistic and non-relativistic eigenvalues of the energy were obtained, and the corresponding wave function was expressed in terms of the Jacobi polynomials. It is also shown that the outputs are similar for Pöschl-Teller potential for $q = 1$ in both cases of relativistic and non-relativistic phenomena.

7 Declarations

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Competing Interests

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Ethical Approval

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