

On the existence of flux as a function of the surface elevation for long wave solution of shallow water equations

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Abstract

The shallow water equations in mechanics of fluids, govern the motion of a shallow layer of water over a fixed impervious bed. In this paper, the bed form is assumed to be rough and horizontal, and the motion of water is assumed to be of the long wave type (Lamb [1], pp. 254-256) such that the free surface has a gradually varying propagating profile. Gravitation permits such motion but is resisted by the turbulence generated by the bed friction. A model of the governing equations based on the Reynolds averaged Navier-Stokes equations has recently been given by Bose [2], which is highly nonlinear. A heuristic approach of numerically solving the equations for the modified long waves is also presented in that article, by assuming that the horizontal flux across a section of flow is some function of the free surface elevation alone. This key assertion is analyzed in this article and proved to hold provided some boundedness criteria are satisfied by the flux gradients. The theory is apparently applicable to finding appropriate boundedness conditions on the flux of flow for numerically solving long wave equations in the case of other models for long wave propagation as well.

Key words: Shallow water equations, turbulence, horizontal flux, surface elevation

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1 Introduction

The shallow water equations describe the motion of a thin layer of fluid waves [3] over a solid bed but with a free upper surface. In such flows the vertical dimension is much smaller than a typical horizontal scale. The hydrodynamic equations for the flow are governed by the depth averaged mass and momentum equations with the stipulation that the pressure at any point is only hydrostatic. These equations have important applications in many areas, such as in

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open channel flows in hydraulics and in the case of atmospheric flows in meteorology over the globe, as also flows in river estuaries and oceans such as tsunamis. The basic difference in these applications is that while in the case of open channel flows there is only one space dimension x - along the length of the channel, in the other cases it is two in general.

The governing equations of motion are nonlinear, primarily because of the convective acceleration terms in the momentum equations. The earliest simple exposition of free surface flows was given by St. Venant for the case of one dimensional flows, which is widely treated in texts on hydraulics. The equations propounded by St. Venant takes in to account both gravitational effect and bed friction. During the same era, Boussinesq developed a nonlinear form of long wave equation, based on suitable assumption for the convective acceleration terms. In view of useful practical applications, the topic has sustained traction during contemporary times as well, with a sizeable literature. To cite a few, Garcia-Navarro et. al. [4] collate the hydrodynamic equations in conservation form for numerical treatment, neglecting viscosity; but retaining the bed friction term. Recently, Li et. al [5] have derived a highly nonlinear equation from the Euler equations of hydrodynamics, taking in to account the Coriolis force for tsunami simulation. Geyer and Quirchmayr [6] on the other hand develop a two dimensional generalisation of Boussinesq type equations for similar purpose in two dimensions. In contrast, the complicated Green and Naghdi [7] equations are based on altogether different consideration has attracted some attention, because the theory allows all wave height to length ratio (Castro-Orgaz and Cantero-Chinchilla [8], Chen et. al. [9], Cienfuegos [10]). Liu et. al. [11] on the otherhand employ homotopy perturbation method for solving the 2D shallow water equations. For numerically solving the different nonlinear equations, mostly variants of the finite volume method are used (Yang et. al. [12], Zhang et. al. [13], Cienfuegos [10], Castro Orgaz and Canterro-Chinchilla [8], Garcia-Navarro et. al. [4]) save for the one dimensional case Lidyana et. al. [14] using a finite difference method. In addition to the above references of recent times, the survey article by Delis and Nikolas [15] gives an extensive exposition of the topic citing numerous other references. Internet search engines also provide numerous other references to articles highlighting different perspectives that the equations behold.

In the simplest form of the depth averaged equations of conservation, if ζ represents the free surface elevation above a point (x, y) on the horizontal plane bed and (U, V) the depth averaged horizontal velocity at that position, then the equation of continuity for mass balance in non-dimensions is

$$\zeta_t + (\zeta U)_x + (\zeta V)_y = 0 \quad (1)$$

and the momentum equations for the balance of momenta are

$$U_t + UU_x + VU_y + \zeta_x + \tau_{0x} = 0 \quad (2)$$

$$V_t + UV_x + VV_y + \zeta_y + \tau_{0y} = 0 \quad (3)$$

(Mader [16], p. 27), where the subscripts denote differentiation with respect to the indicated variable. (τ_{0x}, τ_{0y}) on the other hand represent the components of bed resistance usually modeled by the semi-empirically discovered formulae of Chézy and Manning. The unperturbed free surface is non-dimensionally represented by the plane $\zeta = 1$. The Eqs. (2) and (3) ignore the viscosity of the fluid altogether. If that aspect is taken into consideration as the turbulence generated in the medium by the bed friction, then an elaborate analysis of the motion leads to the highly nonlinear equations (Bose [2])

$$U_t + UU_x + VU_y + \frac{2}{5\zeta} [\zeta^2(U^2\zeta_{xx} + V^2\zeta_{yy})]_x + \frac{7}{22\zeta} [\zeta^3(\zeta_{xx} + \zeta_{yy})]_x + \zeta_x + \tau_{0x} = 0 \quad (4)$$

$$V_t + UV_x + VV_y + \frac{2}{5\zeta} [\zeta^2(U^2\zeta_{xx} + V^2\zeta_{yy})]_y + \frac{7}{22\zeta} [\zeta^3(\zeta_{xx} + \zeta_{yy})]_y + \zeta_y + \tau_{0y} = 0 \quad (5)$$

If Eqs. (2), (3) or (4), (5) are linearised about $\zeta = 1$ and the bed resistance (τ_{0x}, τ_{0y}) neglected such that the motion is only in the x -direction, then Eqs. (1), (2), or (4) yield the well known tidal wave equations $\zeta_t + U_x = 0$, $U_t + \zeta_x = 0$, which reduce to the non-dimensional (long) wave equations $\zeta_{tt} = \zeta_{xx}$ and $U_{tt} = U_{xx}$ (Lamb [1], pp. 254-256). The solution of these two equations for waves propagating in the positive direction of the x -axis are of the form $\zeta = f(x - t)$ and $U = g(x - t)$, in which the functions $f, g : \mathcal{D}_1 \rightarrow \mathbf{R}$, where $\mathcal{D}_1 := \{(x, t) \mid |x| < \infty, t > 0\}$. These two give rise the composite function $F = g \circ f^{-1}$ defined over the same domain D_1 , as $U = g[f^{-1}(\zeta)] = F(\zeta)$, provided f^{-1} exists. Thus, we have U as a function of the local elevation ζ alone. A similar technique is used in this article to prove this type of property for the solution of the Eqs. (1), (4), and (5) when treated as quasi-linear as defined by the Eqs. (10) and (24) in the next two sections, justifying the comparatively simple numerical procedure presented in the references [2] and [3]. For proving this type of special property, the governing equations are first formulated in terms of the flux or “discharge” variables (Q, R) , where $Q = \zeta U$ and $R = \zeta V$, in place of the velocity components U and V and proceed to stipulate the conditions under which $Q = F(\zeta)$ and $R = G(\zeta)$. It is to be noted that the uniform horizontal flow solution $\zeta = 1$, $Q = Q_0$, $R = R_0$ with no bed resistance is a solution of these equations. The solution for wavy, gradually varying motion of the surface elevation ζ will therefore require conditions that restrict the flux gradients (Q_x, Q_y) and (R_x, R_y) . In order to find some rough bounds on these flux rates, the implicit function theorem (Hildebrand [17], pp. 347-348) is employed to prove the existence of the inverses of ζ , Q and R as functions of x, y, t and the fact that these functions are essentially contraction mappings of (x, y, t) (Burkill and Burkill [18], pp. 52-55) in which $|\zeta - 1|$, $|Q|$ and $|R|$ undergo only undergo restricted variation. The simpler one dimensional case of motion in the direction of the x -axis is treated first in the next section, followed by the more general two dimensional case in a later section.

2 The one-dimensional case

In the case of straight crested gradually varying long waves in the x -direction perpendicular to the crests, the governing equations of continuity and momentum are of the form

$$\zeta_t = -Q_x \quad (6)$$

$$Q_t = -\zeta\zeta_x + A_1Q_x + M_1 \quad (7)$$

where A_1 and M_1 are functions defined by

$$A_1(\zeta, Q) = -\frac{2Q}{\zeta} - \frac{4}{5}Q\zeta_{xx} \quad (8)$$

and

$$M_1(\zeta, \zeta_x, \zeta_{xx}, \zeta_{xxx}, Q) = \left(\frac{\zeta_x}{\zeta^2} - \frac{2}{5}\zeta_{xxx}\right)Q^2 - \frac{7}{22}\zeta^2(\zeta\zeta_{xxx} + 3\zeta_x\zeta_{xx}) - \frac{\tau_{0x}}{\zeta^2} \quad (9)$$

These equations follow from Eqs. (1), (4), and (5) by ignoring dependence on the y -variable and setting $Q = \zeta U$ which represents the flux in the x -direction. In the undisturbed state $\zeta(x, t) = 1$ represents the top free surface. The evolution equation (6) is linear and of first order, while Eq. (7) for the flux rate Q_t is of third order and nonlinear. As the motion is assumed to be of long wave type with gradually varying surface profiles, it is assumed that the equation is quasi-linear in the sense that

$$|M_1(\cdot)| \leq |\zeta\zeta_x|. \quad (10)$$

Obviously, the functions ζ and Q are defined over the same domain \mathcal{D}_1 , while if their inverses exist then they are defined over $\mathcal{D}'_1 := \{(\zeta, Q) \mid |\zeta - 1| < \delta_1, |Q| < Q_0\}$ for some δ_1, Q_0 . The latter space being much smaller compared to the first. This means that (ζ, Q) is a contraction mapping from \mathcal{D}_1 onto \mathcal{D}'_1 , for which the Lipschitz coercivity condition holds by the application of Lagrange's Mean Value Theorem for multi-variables (Hildebrand [17], pp. 354-355). These stipulations mean that the Jacobian J_1 of the transformation defined by

$$J_1 := \begin{vmatrix} \zeta_x & \zeta_t \\ Q_x & Q_t \end{vmatrix} \quad (11)$$

satisfies the conditions

$$0 < |J_1| < \epsilon_1 < 1 \quad (12)$$

where ϵ_1 is the Lipschitz contraction constant. The left hand inequality of Eq. (12) results from the assumption that the mappings ζ, Q are invertible. From these conditions, it follows that

Proposition 1. The quasi-linear pair of equations (6) (7) and (10), subject to the conditions (12) admits a solution of the form

$$Q = F(\zeta) \quad (13)$$

provided that the gradient of the flux Q_x is estimated as

$$\sqrt{\zeta} |\zeta_x| < |Q_x| < \frac{\sqrt{\epsilon_1}}{[(1/\zeta + 1)\{1 + 2\sqrt{\zeta} + |A_1|^2\}]^{1/4}}, \quad (x, t) \in \mathcal{D}_1 \quad (14)$$

where $\epsilon_1 \in (0, 1)$ is a Lipschitz constant.

Proof. From the left hand side of the inequality of Eq. (12), the maps ζ, Q defined over the domain \mathcal{D}_1 are invertible, since $|J_1| \neq 0$. Now,

$$J_1 = (Q_x^2 - \zeta \zeta_x^2) + \zeta_x (A_1 Q_x + M_1)$$

so that

$$|J_1| \geq |Q_x^2 - \zeta \zeta_x^2| - |\zeta_x (A_1 Q_x + M_1)| > 0 \quad (15)$$

for all possible values of A_1 and M_1 which is satisfied only if

$$|Q_x| > \sqrt{\zeta} |\zeta_x| \quad (16)$$

Eq. (16) establishes the left hand side estimate of Eq. (14). Moreover, the inverse map (x, t) defined over \mathcal{D}'_1 exists such that

$$Q = \phi(x(\zeta, Q), t(\zeta, Q)) \quad (17)$$

implicitly defining a function F such that $Q = F(\zeta)$, establishing Eq. (13).

In order to prove the right hand side estimate of Eq. (14), the determinant J_1 satisfies Hadamard's inequality for determinants (Garling [19], p. 233) as

$$\begin{aligned} |J_1| &< \sqrt{\zeta_x^2 + Q_x^2} \sqrt{\zeta_t^2 + Q_t^2} = \sqrt{\zeta_x^2 + Q_x^2} \sqrt{Q_x^2 + (-\zeta \zeta_x + A_1 Q_x + M_1)^2} \\ &\leq \sqrt{\frac{Q_x^2}{\zeta} + Q_x^2} \sqrt{Q_x^2 + (2|\zeta \zeta_x| + |A_1| |Q_x|)^2} \\ &\leq \sqrt{\frac{1}{\zeta} + 1} \sqrt{1 + (2\sqrt{\zeta} + |A_1|)^2} |Q_x|^2 < \epsilon_1 \end{aligned} \quad (18)$$

proving the right hand side estimate of Eq. (14).

3 The two dimensional case

In the case of propagation in two horizontal dimensions (x, y) , the equations of continuity and momenta as obtained by setting $Q = \zeta U$ and $R = \zeta V$ in Eqs. (1), (4), and (5), yield the equations of the form

$$\zeta_t = -Q_x - R_y \quad (19)$$

$$Q_t = -\zeta\zeta_x + A_2Q_x + B_2Q_y + C_2R_x + D_2R_y + M_2, \quad (20)$$

$$R_t = -\zeta\zeta_y + A_3Q_x + B_3Q_y + C_3R_x + D_3R_y + M_3, \quad (21)$$

where the coefficients appear in Eqs. (20), (21) are defined by the equations:

$$A_2 = -\frac{2Q}{\zeta} - \frac{4}{5}Q\zeta_{xx}, \quad (22a)$$

$$B_2 = A_3 = -\frac{R}{\zeta} \quad (22b)$$

$$C_2 = -\frac{4}{5}R\zeta_{yy} \quad (22c)$$

$$D_2 = C_3 = -\frac{Q}{\zeta} \quad (22d)$$

$$B_y = -\frac{4}{5}Q\zeta_{xx} \quad (22e)$$

$$D_3 = -\frac{2R}{\zeta} - \frac{4}{5}R\zeta_{yy} \quad (22f)$$

$$M_2(\zeta, \zeta_x, \zeta_y, \zeta_{xx}, \zeta_{yy}, \zeta_{xxx}, \zeta_{xyy}, \zeta_{yyy}) = \left(\frac{\zeta_x}{\zeta^2} - \frac{2}{5}\zeta_{xxx}\right) Q^2 + \frac{QR}{\zeta^2} \zeta_y - \frac{2}{5} \\ - \frac{2}{5}\zeta_{yyy}R^2 - \frac{7}{22}\zeta^2 [\zeta(\zeta_{xxx} + \zeta_{xyy}) + 3\zeta_x(\zeta_{xx} + \zeta_{yy})] - \frac{\tau_{0x}}{\zeta^2} \quad (22g)$$

$$M_3(\zeta, \zeta_x, \zeta_y, \zeta_{xx}, \zeta_{yy}, \zeta_{xxy}, \zeta_{yyy}) = \frac{R}{\zeta^2}(Q\zeta_x + R\zeta_y) - \frac{2}{5}(Q^2\zeta_{xxy} + R^2\zeta_{yyy}), \\ - \frac{7}{22}\zeta^2 [\zeta(\zeta_{xxy} + \zeta_{yyy}) + 3\zeta_y(\zeta_{xx} + \zeta_{yy})] - \frac{\tau_{0y}}{\zeta^2} \quad (22h)$$

(Bose [2]), where as before, $\zeta(x, y, t) = 1$ represents the free surface in the undisturbed state. Assuming quasi-linear flows in which the free surface undergoes bounded motion $|\zeta - 1| < \delta$, $|Q| < Q_0$, $|R| < R_0$, we must have the estimates

$$|M_2(\cdot)| < |\zeta\zeta_x|, \quad |M_3(\cdot)| < |\zeta\zeta_y| \quad (23)$$

in the domain of motion $\mathcal{D}_2 = \{(x, y, t) \mid |x|, |y| < \infty, t > 0\}$ which is mapped onto the contracted domain $\mathcal{D}'_2 = \{(\zeta, Q, R) \mid |\zeta - 1| < \delta_2, |Q| < Q_2, |R| < R_2\}$, provided the inverse mapping exists. Consequently by the multivariate Lagrange's Mean Value Theorem, if the Jacobian of the transformation is J_2 defined by

$$J_2 = \begin{vmatrix} \zeta_x & \zeta_y & \zeta_t \\ Q_x & Q_y & Q_t \\ R_x & R_y & R_t \end{vmatrix} \quad (24)$$

then,

$$0 < |J_2| < \epsilon_2 < 1, \quad (25)$$

where the left hand side inequality follows if the invertibility of the mapping of \mathcal{D}_2 to \mathcal{D}'_2 holds. In addition to the conditions (23) and (25), it is noted that in general the flow components (Q, R) along with their gradients (Q_x, Q_y) and (R_x, R_y) are unequal, and are predominantly in the normal directions, implying that

$$|R_x| < |Q_x|, \quad |Q_y| < |R_y|, \quad (x, y, t) \in \mathcal{D}_2 \quad (26)$$

and that

$$|R_y| \leq |Q_x|, \quad (x, y, t) \in \mathcal{D}_2 \quad (27)$$

implying that the flux gradient in the y -direction is less than that in the x -direction. Under the conditions (23), (25), (26), and (27), the following proposition holds:

Proposition 2. The quasi-linear equations (19), (20), and (21), subject to the invertibility and contraction conditions (25) as also the flux gradient conditions (26) and (27) admit an implicit solution

$$Q = F(\zeta), \quad R = G(\zeta), \quad (28)$$

provided the flux gradients are bounded as

$$0 < \left(\sqrt{\zeta} |\zeta_x|, \sqrt{\zeta} |\zeta_y| \right) < (|Q_x|, |R_y|) < \frac{\epsilon_2^{1/3}}{K^{1/3}} \quad (29)$$

where

$$K = \left(\frac{1}{\zeta} + 2 \right) \left[2 + [2\sqrt{\zeta} + |A_2| + |B_2| + |C_2| + |D_2|]^2 + [2\sqrt{\zeta} + |A_3| + |B_3| + |C_3| + |D_3|]^2 \right]^{1/2} \quad (30)$$

Proof. From the left hand side inequality of Eq. (25), the functions (ζ, Q) have an inverse, since $J_2 \neq 0$. Now,

$$|J_2| = \left| - \left[R_y(Q_x^2 - \zeta \zeta_x^2) + Q_x(R_y^2 - \zeta \zeta_y^2) \right] + \dots \right| \quad (31)$$

Hence,

$$|J_2| \geq \left| R_y(Q_x^2 - \zeta \zeta_x^2) + Q_x(R_y^2 - \zeta \zeta_y^2) \right| - \left| \dots \right| \quad (32)$$

The right hand side of Eq. (32) is strictly positive if $Q_x^2 > \zeta \zeta_x^2$ and $R_y^2 > \zeta \zeta_y^2$. This establishes the left hand side estimates of Eq. (29) for the existence of the inverse mappings:

$$\left. \begin{aligned} Q &= \psi \left(x(\zeta, Q, R), y(\zeta, Q, R), t(\zeta, Q, R) \right) \\ R &= \chi \left(x(\zeta, Q, R), y(\zeta, Q, R), t(\zeta, Q, R) \right) \end{aligned} \right\} \quad (33)$$

that implicitly define Q and R as functions of ζ as postulated in Eq. (28). For establishing the right hand side estimate, Hadamard's inequality (Garling [19], p.233) in this case becomes

$$\begin{aligned} |J_2| &< \sqrt{\zeta_x^2 + Q_x^2 + R_x^2} \sqrt{\zeta_y^2 + Q_y^2 + R_y^2} \sqrt{\zeta_t^2 + Q_t^2 + R_t^2} \\ &\leq \sqrt{Q_x^2/\zeta + Q_x^2 + R_x^2} \sqrt{Q_y^2/\zeta + Q_y^2 + R_y^2} \sqrt{\zeta_t^2 + Q_t^2 + R_t^2} \end{aligned}$$

$$\begin{aligned} &\leq \sqrt{1/\zeta + 2} |Q_x| \sqrt{1/\zeta + 2} |Q_x| \sqrt{\zeta_t^2 + Q_t^2 + R_t^2} \\ &= (1/\zeta + 2) |Q_x|^2 \sqrt{\zeta_t^2 + Q_t^2 + R_t^2} \end{aligned} \quad (34)$$

using Eqs. (27) and (28). Now,

$$\begin{aligned} \zeta_t^2 + Q_t^2 + R_t^2 &= |\zeta_t|^2 + |Q_t|^2 + |R_t|^2 \\ &\leq (|Q_x| + |R_y|)^2 \\ &\quad + \left[|\zeta\zeta_x| + |A_2| |Q_x| + |B_2| |Q_y| + |C_2| |R_x| + |D_2| |R_y| + |M_2| \right]^2 \\ &\quad + \left[|\zeta\zeta_y| + |A_3| |Q_x| + |B_3| |Q_y| + |C_3| |R_x| + |D_3| |R_y| + |M_3| V \right]^2 \\ &\leq |Q_x|^2 \left\{ 2 + \left[2\sqrt{\zeta} + |A_2| + |B_2| + |C_2| + |D_2| \right]^2 + \left[2\sqrt{\zeta} + |A_3| + |B_3| + |C_3| \right. \right. \\ &\quad \left. \left. + |D_3| \right]^2 \right\} \end{aligned} \quad (35)$$

using Eqs. (26), (27) and (23). Hence, from Eqs. (34), (35)

$$|J_2| < |Q_x|^3 \times K < \epsilon_2 \quad (36)$$

$$\text{if } |Q_x| < \epsilon^{1/3} / K^{1/3} \quad (37)$$

which establishes the right hand side estimate of Eq. (29).

4 Conclusions and Remarks

The shallow water equations of hydrodynamics governing the wavy flow in a body of water of shallow depth over a rough bed is of wide application in the study of flows in open channels and oceans as also in the atmosphere surrounding the globe with suitable inclusion of the Coriolis force and the sphericity of the earth. In the case of plane bed, the equations are succinctly derived in Mader [16]. Because of the nonlinearity of the equations, several numerical schemes have been developed by different authors. References to these schemes are given in the author's paper [2], which also gives due consideration to the turbulence generated in the medium, with a comparatively simple numerical method of the highly nonlinear equations, which determine the surface elevation ζ and the flow velocity (U, V) . A heuristic argument is used in which the horizontal fluxes (Q, R) , where $Q = \zeta U$ and $R = \zeta V$ are assumed to be functions of ζ alone, as in the case of long tidal waves Lamb [1]. The present paper endeavors to find rough bounds on the flux gradients for permissible nonlinear finite amplitude wave motion over the equilibrium surface $\zeta = 1$. This treatment of the highly nonlinear system to find the restrictions on the flux of flow is apparently applicable to other models of free surface flow as well, by assuming flux to be a function of the vertical elevation of the free surface leading to easier numerical solution of the complex equations.

5 Declarations

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Competing Interests

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Ethical Approval

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Author's Contributions

All authors contributed equally. All the authors read and approved the final manuscript.

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