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On a Class of Stochastic Damped Wave Equation

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Abstract

The present work considers a wave equation with multiplicative Gaussian white noise and weak dissipative term on a bounded domain. We first give a theorem including the local existence of mild solutions. An energy bound and a differential inequality are used to give sufficient conditions that provide the blow-up of mild local solutions of the stochastic wave equation. The paper's main contribution comes from handling a multiplicative noise and a general source term contrary to the articles that exist in the literature.

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1 Introduction

The wave equation is an essential partial differential equation that appears in several fields, such as electromagnetic, traffic flows, acoustics, fluid dynamics, general relativity, atmosphere and ocean dynamics, chemical reactions, and biological sciences. The widespread use of wave equations led many mathematicians to study different aspects of the equation, such as the existence and uniqueness, decay, and explosion of the solutions [1, 2, 3, 4].

To include the fluctuating properties of the media in the model, a noise term must be added to the wave equation. Such inclusions led to stochastic wave equations in the 1960's. They have become a very important tool in the past few decades for phenomena experiencing random changes. Inspired by these facts we devoted this paper to the following wave equation that models a variety of physical situations with a multiplicative Gaussian noise:

$$du_t + [\mu u_t + \alpha u - \Delta u]\, dt = \varphi(u)\, dt + h(u, u_t, \nabla u, x, t)\, dW(x,t), \quad (t,x) \in (0,T) \times \Omega \tag{1}$$

$$u(x,0) = u_0, \quad u_t(x,0) = v_0, \quad x \in \Omega, \tag{2}$$

$$u = 0, \quad (t,x) \in (0,T) \times \partial\Omega. \tag{3}$$

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Here α, $\mu > 0$, u_t is a weak dissipative (damping) term, $\varphi(u)$ is a nonlinear source term, and $W(x,t)$, the properties of which will be determined later, is a Wiener process. The weak dissipative term μu_t, with $\mu > 0$ stands for the dynamical friction ([5]). The $h = 0$ state in Eq. (1) corresponds to the deterministic wave equation and has been considered in a vast number of articles [6, 7, 8, 9, 10, 11]. The simplest form of the deterministic wave equation with Cauchy data, on which there are many studies, is the equation given below:

$$\begin{cases} v_{tt} - \Delta v = f(v), & x \in \mathbb{R}^n, t > 0, \\ v(x,0) = \phi(x), \quad v_t(x,0) = \psi(x), & x \in \mathbb{R}^n. \end{cases}$$

The critical exponent $p_* = p_*(n)$ given by

$$\begin{cases} p_*(n) := \frac{n+1+(n^2+10n-7)^{1/2}}{2(n-1)} \text{ for } n \geq 2, \\ p_*(1) := \infty \end{cases}$$

determines the threshold between the existence of a global weak solution with small data and the blow-up of a local weak solution with small data for a power nonlinearity $f(v) = |v|^p$ [12]. The stochastic wave equation has received extensive attention from a mathematical viewpoint [13, 14, 15, 16, 17, 18, 19, 20, 21, 22]. For the motivation of our paper, let us examine these studies in detail. Chow [13] studied the nonlinear stochastic wave equation on a bounded domain $\Omega \subset R^d$

$$v_{tt} = c^2 \Delta v - \alpha v + g(v) + h(v, Dv, x, t)\partial_t W(x,t), \tag{4}$$

where $d \leq 3$, $D = \partial_x$ is the gradient operator, $c, \alpha > 0$ are parameters, $g(s)$ and $h(s,q,x,t)$ are nonlinear functions growing polynomially, and locally Lipschitz continuous. By imposing some conditions on f and h, and an energy bound, Chow proved that solutions of Eq. (4) blow-up in the mean-square sense in finite time. The restriction $d \leq 3$ on dimension is made in order to work with polynomial nonlinear terms. Local and global existence of solutions in some Sobolev spaces with $\alpha = 0$ and nonlinear terms of polynomial degree by a truncation technique was studied in [23]. In [14], Chow investigated the semilinear stochastic wave equation

$$\partial_t^2 v = [A(x,D) - 2\alpha\partial_t]v + f(v, Dv, x, t) + h(v, Dv, x, t)\partial_t W(x,t), \tag{5}$$

with initial boundary conditions. Here A is a strongly elliptic operator. The existence of solutions for finite and infinite time intervals and the asymptotic behavior of solutions are discussed in [14]. An invariant measure's existence was proved in Brzezniak et. al. [20] for damped wave equation

$$u_{tt} = \Delta u - m^2 u - au|u|^{p-1} - \beta u_t + F + \eta g(u)\dot{W}$$

on R^d, where $m, \beta \geq 0$, $a > 0$, $F \in L^2$, $\eta \in L^\infty$ and W is a Wiener process of cylindrical type. In [21], the authors deal with explosive solutions of stochastic damped wave equation

$$u_{tt} + Au_t - \Delta u = \gamma |u|^p u + g(u, u_t, Du)\partial_t W(x,t) \tag{6}$$

on R^d. Weak and strong damped cases are discussed separately. Using a truncation technique, they approximated mild solutions of (6) by a sequence of strong solutions in infinite dimension so that they were able to apply the Ito rule. Afterward, they proved that the solutions of the Cauchy problem of (6) cease to exist in finite time in L^2 sense.

Barbu and Da Prato [17] studied the stochastic wave equation with dissipative terms in the form below

$$dV_t + (AV + V_t + g(V))dt = \sqrt{Q}dW(t)$$

on a bounded domain of R^n, for $n \leq 3$, where $A = -\Delta$, W is a cylindrical Wiener process and Q is a nonnegative symmetric operator. The invariant measure's existence was proved by the Krylov–Bogolyubov procedure.

In [15], the author investigated a semilinear damped wave equation with initial conditions and random noise

$$u_{tt} + 2\gamma u_t + \beta u - \Delta u = f(t,x,u) + \sum_{j=1}^{\infty} \sigma_j(t,x,u) \frac{dB_j(t)}{dt} \tag{7}$$

where $\beta, \gamma > 0$ are constants, $x \in R^3$ and B_j s are standard Brownian motions. The author first proved the existence of a pathwise unique solution, and then give the existence of periodic and invariant measures for the Cauchy problem of (7). They also demonstrated how their method can be applied to an initial boundary value problem. In the present paper, the solution's blow-up for problem (1)-(3) is investigated. Blow-up is a type of singularity that can be explained by the fact that the solution becomes infinite in a finite time. In a general sense, the global existence for the above equation occurs in situations where the damping term (u_t) dominates the source term $(f(u))$, while blow-up happens in the opposite case and when the initial data is large enough. In all the above-mentioned papers, either the problem is studied for a particular case of source term $f(u)$ or the problem is handled from a different aspect.

The paper is designed in the following way: In the second section, we first give the properties of Wiener process, and then give a local existence theorem. The last section includes the main result and its proof.

We will complete this section by giving some notations on spaces. $L^p(\Omega)$ is the space of all measurable functions u on Ω. Its norm is denoted by $\|.\|_{L^p}$. For $p = 2$, the norm and the inner product is given by $(\cdot,\cdot)$, and $\|.\|$, respectively. $W^{m,p}(\Omega)$, is the space of Sobolev. Its norm and inner product is denoted by $\|u\|_{W^{m,p}(\Omega)}$, and $(u,v)_m$, $m = 1,2,3,...$ respectively. If $p = 2$, we denote the space $W^{m,2}(\Omega)$ by $H^m(\Omega)$. H stands for $H^0 = L^2$. We demonstrate the norm of Sobolev space H_0^1 with $\|u\|_{H_0^1}$. We also introduce the space $\mathcal{H} := H_0^1 \times H$ accompanied with the following norm

$$\|\phi\|_{\mathcal{H}} = \left\{ \|u\|_1^2 + \|v\|^2 \right\}^{1/2}$$

for any $\phi = (u,v) \in \mathcal{H}$.

2 Preliminaries

Let us define the complete probability space $(\Omega, \mathcal{F}, P)$ with a filtration $\{\mathcal{F}_t,\ t \geq 0\}$. Consider an $H-$ valued Wiener process $W(x,t)$, $x \in \Omega$, $t \geq 0$ having covariance operator Q on $(\Omega, \mathcal{F}, P)$. Q is of trace class and $TrQ < \infty$. One way of defining Q by means of eigenvalues $\{\lambda_k\}$ and eigenfunctions $\{e_k\}$ is

$$Qe_k = \lambda_k e_k. \tag{8}$$

Here eigenvalues $\{\lambda_k\}$ are bounded and nonnegative, and eigenfunctions $\{e_k\}$ form a complete orthonormal base in H. So, $W(x,t)$ has the following expansion

$$W(x,t) = \sum_{i=1}^{\infty} \sqrt{\lambda_k} B_k(t) e_k,$$

where $\{B_k(t)\}$ denotes a real valued sequence of independent Brownian motions. The above series is convergent in L^2 [24].

Problem (1)- (3) can be converted to a system as

$$\begin{cases} du = vdt, \\ dv = ((-\alpha I + \Delta)\, u - \mu v + \varphi(u))dt + h(u, v, \nabla u)dW\,(x,t)\,, \\ u(x,0) = u_0\,(x)\,, u_t(x,0) = u_1\,(x)\,, \\ u(x,t) = 0. \end{cases}$$

Without loss of generality, we may take the positive constants μ and α as $\mu = \alpha = 1$. Then problem (1)- (3) can be reduced to

$$\begin{cases} d\mathcal{Y}(t) = \Lambda\mathcal{Y}(t)dt + \Psi(\mathcal{Y}(t))dt + \Sigma(\mathcal{Y}(t))dW(t), \\ \mathcal{Y}(0) = \mathcal{Y}_0 = (u_0, u_1)^T, \end{cases} \tag{9}$$

where

$$\mathcal{Y}\,(t) = \begin{pmatrix} u\,(t) \\ v\,(t) \end{pmatrix}, \quad \Lambda = \begin{pmatrix} 0 & I \\ -I+\Delta & 0 \end{pmatrix},$$
$$\Psi\,(\mathcal{Y}\,(t)) = \begin{pmatrix} 0 \\ -v + \varphi\,(u\,(t)) \end{pmatrix}, \quad \Sigma\,(\mathcal{Y}\,(t)) = \begin{pmatrix} 0 \\ h(u, v, \nabla u) \end{pmatrix}$$

Remark 2.1. A predictable, $\mathcal{F}_t$ adapted $\mathcal{H}-$ valued process $\mathcal{Y}\,(t)\,,\ t \in [0,T]$ satisfying the following equation

$$\mathcal{Y}(t) = e^{t\Lambda}\mathcal{Y}(0) + \int_0^t e^{(t-s)\Lambda}\Psi(\mathcal{Y}(s))ds + \int_0^t e^{(t-s)\Lambda}\Sigma(\mathcal{Y}(s)dW(s)).$$

is said to be a mild solution to (9).

We indicate that Λ is an analytic semigroup's ($\{e^{t\Lambda},\ t \geq 0\}$) infinitesimal generator on $\mathcal{H}$. Throughout the paper, the following hypotheses are imposed on φ and h:

L1. For $u^1, u^2 \in H_0^1$ and there exist $C_1, C_2 > 0$ such that the nonlinear term satisfies

$$\varphi\,(u) \leq C_1\,(1 + |u|^p\,u)$$

$$\left|\varphi\left(u^1\right) - \varphi\left(u^2\right)\right| \leq C_2\left(1 + \left|u^1\right|^p + |u^2|^p\right)\left|u^1 - u^2\right|$$

.

L2. Assume that $h(\cdot) : [0,T] \to L(H)$ isa continuous map for any $h : R^{d+2} \to R$. For any $u^1, v^1, u^2, v^2 \in \mathcal{H}$ and $\nabla u^1, \nabla u^2 \in H_0^1$, there exist $C_3, C_4 > 0$ such that

$$|h\,(u,v,\nabla u)|^2 \leq C_3\left(1 + |u|^{2(p+1)} + |v|^2 + |\nabla u|^2\right),$$

and

$$\left|h\left(u^1, v^1, \nabla u^1\right) - h\left(u^2, v^2, \nabla u^2\right)\right|^2 \leq C_4\Big[(1+\left|u^1\right|^{2p} + |u^2|^{2p})\left|u^1 - u^2\right|^2 + \left|v^1 - v^2\right|^2 + \left|\nabla u^1 - \nabla u^2\right|^2\Big].$$

L3. The Wiener process W with covariance operator R satisfying $TrQ < \infty$ takes values from H.

Here p satisfy

$$p \in (0,\infty) \text{ if } d = 1,2, \text{ and } 0 < p+1 \le \frac{d+2}{d-2} \text{ otherwise.} \tag{10}$$

The following consequence of the continuously compact embedding of $H_0^1(\Omega)$ into $L^p(\Omega)$ is necessary for local existence.

Lemma 2.2. *[25] Let $u, v \in \mathcal{H}$ and p satisfy (10). Then there is a positive constant C_0 such that*

$$\|u\|_{(p+1)} \le C_0 \|\nabla u\|_{H_0^1}, \quad \forall u \in H_0^1$$
$$\|u^p v\| \le C_0^{p+1} \|\nabla u\|^p \|\nabla v\|, \quad \forall u, v \in H_0^1.$$

The following theorem is related to the local existence of solutions of problem (9) or equivalently problem (1)-(3).

Theorem 2.3. *Assume that conditions L1-L3 hold, $u_0 \in H_0^1$ and $u_1 \in H$. Then problem (9) possesses a unique local mild solution (u,v) satisfying $u \in C([0,\varsigma], H_0^1)$, $v \in C([0,\varsigma], H)$ and*

$$\limsup_{t\to\varsigma} \|\nabla u\| = +\infty.$$

and

$$\mathcal{Y}(t\wedge\tau_N) = e^{(t\wedge\tau_N)\Lambda}\mathcal{Y}(0) + \int_0^{t\wedge\tau_N} e^{(t\wedge\tau_N - s)\Lambda}\Psi(\mathcal{Y}(s))ds + \int_0^{t\wedge\tau_N} e^{(t\wedge\tau_N - s)\Lambda}\Sigma(\mathcal{Y}(s)dW(s)), \tag{11}$$

where the stopping time ς is described by

$$\varsigma = \lim_{N\to\infty} \tau_N, \quad \tau_N = \inf\{t \ge 0; \|\nabla u\| \ge N\}$$

for any natural integer N.

Proof. The proof of the theorem is carried out via a truncation method as in [21], where the damped wave equation is studied with the source term $\varphi(u) = \mu |u|^p u$. We will sketch the outlines here for the motivation of the paper. For $N > 0$, let $\eta_N(\cdot) : \mathbb{R}^+ = [0,\infty) \to \mathbb{R}^+$ be a function in C^1 such that $\eta_N(s) = 1$ for $|s| \le N$, $\eta_N(s) \in (0,1)$ for $N < |s| < N+1$, and $\eta_N(s) = 0$ for $|s| \ge N+1$. Furthermore, $\|\eta_N'\|_\infty \le 2$. Let us define $\varphi_N(u(t)) = \eta_N(\|u\|_{H_0^1})\varphi(u)$ and $h_N(u, \nabla u, x, t) = \eta_N(\|u\|_{H_0^1})h(u, \nabla u, x, t)$. Then system (9) turns into the following truncated system

$$\begin{aligned} &d\mathcal{Y}_N(t) = \Lambda\mathcal{Y}_N(t)dt + \Psi_N(\mathcal{Y}_N(t))dt + \Sigma_N(\mathcal{Y}_N(t))dW(t), \\ &\mathcal{Y}(0) = \mathcal{Y}_0, \end{aligned} \tag{12}$$

where $\mathcal{Y}_N = (u_N(t), v_N(t))$, $\Psi_N(\mathcal{Y}) = \begin{pmatrix} 0 \\ -v + \varphi_N(u(t)) \end{pmatrix}$, $[\Sigma_N(\mathcal{Y})](x) = h_N(u, \nabla u, x, t)(x)$. Then from Lemma 2.2 and Hölder inequality, we get

$$\left\|\varphi_N(u^1) - \varphi_N(u^2)\right\|^2 \le C_5(N,p)\left\|\nabla u^1 - \nabla u^2\right\|^2$$

which yields

$$\begin{aligned}\left\|\Psi_N(\mathcal{Y})-\Psi_N(\mathcal{Y}')\right\|^2 &\leq \left\|\varphi_N(u^1)-\varphi_N(u^2)+v^1-v^2\right\|^2\\ &\leq \left\|\varphi_N(u^1)-\varphi_N(u^2)\right\|^2+\left\|v^1-v^2\right\|^2\\ &\leq C_6(N,p)\left\|\nabla\mathcal{Y}-\nabla\mathcal{Y}'\right\|^2,\end{aligned}$$

where $\mathcal{Y}=\left(u^1,v^1\right)^T, \mathcal{Y}'=\left(u^2,v^2\right)^T$. Moreover,

$$\begin{aligned}\left\|\Psi_N(u)\right\|^2 &= \left\|\varphi_N(u)-v\right\|^2\\ &\leq C_7(N,p)\left(1+|\nabla\mathcal{Y}|^p\right)|\nabla\mathcal{Y}|.\end{aligned}$$

Similarly,

$$Tr[h_N(u)Qh_N^*(u)]\leq c_8(N,p)(1+\|\nabla\mathcal{Y}\|^2),$$

and

$$Tr\left[\left(h_N(u^1)-h_N(u^2)\right)Q\left(h_N(u^1)-h_N(u^2)\right)^*\right]\leq C_9(N,p)\left\|\nabla\mathcal{Y}-\nabla\mathcal{Y}'\right\|^2$$

The above computations show that for any $N>0$, Lipschitz continuity and linear growth conditions are also fulfilled by φ_N and h_N on bounded sets in $\mathcal{H}$. By applying the existence theorem given in [24] (see Theorem 7.2) we deduce that there exists a unique mild solution $\mathcal{Y}_N=(u_N(t),v_N(t))\in\mathcal{H}$ to the truncated system (12). For each $N>0$, let us define a stopping time

$$\tau_N=\inf\{t\geq 0;\|\nabla u\|\geq N\},$$

then by the uniqueness of the solution of (12), for $M>N$, $\mathcal{Y}_M(t)=\mathcal{Y}_N(t)$ on $[0,\tau_N]$. Thus we can describe a local solution by $\mathcal{Y}(t)=\mathcal{Y}_N(t)$ to (12) on $t\in[0,\tau_N\wedge T]$. Let $\varsigma=\lim_{N\to\infty}\tau_N$, then by the continuity of $t\to\mathcal{Y}(t)$, $\mathcal{Y}(t)$ is the unique continuous solution with a finite lifespan ς and fulfilling (11). □

In the next section, we provide conditions for blow-up of solutions that rely on energy identity $\vartheta(t):\mathcal{H}\to R$ related to (1)

$$\vartheta(t)=\frac{1}{2}\|v\|^2+\frac{1}{2}\|u\|^2_{H_1^0}-\int_\Omega\varphi(u)dx, \tag{13}$$

where $t\geq 0$. We utilize the following lemmas including differential inequalities to show the blow-up of solutions.

Lemma 2.4. *[26] Suppose that $\eta>0$ and $K(t)>0$ is a C^2 function satisfying the following differential inequality*

$$K''-4(\eta+1)K'+4(\eta+1)K\geq 0. \tag{14}$$

If the following condition is satisfied

$$K'(0)>p_1K(0)+M_0,$$

then for $t>0$, $K'(t)>0$, where M_0 is a constant and $p_1=2(\eta+1)^{1/2}[(\eta+1)^{1/2}-(\eta)^{1/2}]$ denotes the smallest root of

$$p^2-4(\eta+1)p+4(\eta+1)=0.$$

Lemma 2.5. *[26] If $G(t)$ is a function that is nonincreasing for $0 \leq t_1 < \infty$ and fulfills the following inequality*

$$G'(t)^2 \geq A_0 + A_1\, G(t)^{2+\frac{1}{\gamma}}, \;\; t \geq t_1,$$

then for a finite time T_0, the following equality is satisfied

$$\lim_{t \to T_0^-} G(t) = 0,$$

where $A_0 > 0$, $A_1 \in R$. Furthermore, the upper bound of T_0 is predicted as

- *(i) If $A_1 > 0$, then*

$$T_0 \leq t_1 + 2^{(3\alpha+1)/2\alpha} \frac{\alpha A_3}{A_0^{1/2}} \left\{ 1 - \left(1 + A_3 G\left(t_1\right)\right)^{-1/2\alpha} \right\},$$

or

$$T_0 \leq t_1 + \frac{A_1\left(t_1\right)}{\sqrt{A_0}},$$

where $A_3 = \left(\frac{A_1}{A_2}\right)^{2+1/\alpha}$.

- *(ii) If $A_1 = 0$, then $T_0 \leq t_1 + \frac{G(t_1)}{\sqrt{A_0}}$,*
- *(iii) If $A_1 < 0$ and $G\left(t_1\right) < \min\{1, \sqrt{A_0 / -A_1}\}$, then*

$$T_0 \leq t_1 + \frac{1}{\sqrt{-A_1}} \ln \frac{\sqrt{A_0 / -A_1}}{\sqrt{A_0 / -A_1} - G\left(t_1\right)}.$$

3 Blow-up of solutions

In this section, we are interested in a type of singularity for Problem (1)-(3) or (12), the blow-up of solutions.

For this purpose, we utilize the following auxiliary functional for $t \geq 0$

$$Y(t) = E \int_0^t \|u\|^2 d\tau + E\|u\|^2. \tag{15}$$

Direct computations yield

$$Y'(t) = E(u,u) + 2E(u,v),$$

and

$$Y''\left(t\right) = 2E\|v\|^2 - 2E\|\nabla u\|^2 - 2E\|u\|^2 + 2E(u, \varphi(u)), \tag{16}$$

where (16) is obtained by multiplying (1) by u and then taking expectations in both sides of the arising equation.

Remark 3.1. Hereafter, we assume that h satisfies the condition

$$|h(s,r,\nabla s,x,t)| \leq \frac{2\gamma}{(2\gamma+1)c_*^2 TrQ} |r|^2, \tag{17}$$

instead of L2, where $\gamma > 0$ is a constant. The nonlinear term φ satisfies an Ambrosetti-Rabinowitz type condition

$$s\varphi(s) \geq (2 + 4\gamma)\Theta(s) \tag{18}$$

where $\Theta\left(u\right) = \int_0^u \varphi\left(s\right) ds$.

Now, we provide an energy inequality that has a decisive role in the behavior of the solution to our problem.

Lemma 3.2. *Suppose that (u,v) is a mild solution in $\mathcal{H}$ under the conditions of Theorem 2.3. Then the inequality*

$$E\downarrow(t) \leq E\downarrow(0) - E\int_0^t \|v\|^2 ds + \frac{1}{2}c_*^2 TrQ \int_0^t \int_\Omega h^2 dxds \tag{19}$$

and equality

$$\begin{aligned} E(u,v) = & E(u_0,v_0) + E\int_0^t (v,v)ds - E\int_0^t \|u\|_{H_0^1}^2 ds - \frac{1}{2}E\|u\|_{H_0^1}^2 \\ & + \frac{1}{2}E\|u_0\|_{H_0^1}^2 + E\int_0^t (u,\varphi)\, ds \end{aligned} \tag{20}$$

are satisfied.

Proof. An application of Ito formula to $\|v\|^2$ yields

$$\begin{aligned} \|v\|^2 &= \|v_0\|^2 + 2\int_0^t (v,dv) + \int_0^t (dv,dv) \\ &= \|v_0\|^2 - 2\int_0^t (\nabla u, \nabla v)\, ds - 2\int_0^t \|v\|^2 ds - 2\int_0^t (u,v)\, ds \\ &\quad + 2\int_0^t (v,\varphi)\, ds + 2\int_0^t (v,hdW) + \sum_{k=1}^{\infty} \int_0^t (hQe_k, he_k)\, ds. \end{aligned} \tag{21}$$

A direct computation yields

$$2\int_0^t (u,v)\, ds = \|u\|^2 - \|u_0\|^2, \tag{22}$$

$$2\int_0^t (\nabla u, \nabla v)\, ds = \|\nabla u\|^2 - \|\nabla u_0\|^2 \tag{23}$$

and

$$2\int_0^t (v,\varphi)\, ds = 2\int_\Omega (\Theta(u) - \Theta(u_0))\, dx. \tag{24}$$

By substituting (22)-(24) and (8) in (21), we obtain

$$\begin{aligned} \|v\|^2 = & 2\vartheta(0) - \|\nabla u\|^2 - 2\int_0^t \|v\|^2 ds - \|u\|^2 + 2\int_\Omega \Theta(u)\, dx \\ & + 2\int_0^t (v,hdW) + \sum_{k=1}^{\infty} \lambda_k \int_0^t \int_\Omega h^2 e_k^2(x)\, dxds. \end{aligned} \tag{25}$$

Defining $c_* := \sup_{k\geq 1} \|e_k\|_\infty < \infty$, taking into account $TrQ = \sum_{k=1}^{\infty} \lambda_k$ and taking expectation in (25), we get (19). Now we will prove (20). Considering global mild solution (u,v) for problem (9), for each $k \geq 1$, and an orthonormal base $\{\tilde{e}_k\}_{k\geq 1}$ of L^2, $\{(u(t), \tilde{e}_k); t \geq 0\}$ and $\{(v(t), \tilde{e}_k); t \geq 0\}$ are a continuous process with finite variation and a continuous semi-martingale, respectively and both are $\{\mathcal{F}_t\}_{t\geq 0}$ adapted. Via the Ito formula, we get

$$(u(t), \tilde{e}_k)(v(t), \tilde{e}_k) = (u_0, \tilde{e}_k)(v_0, \tilde{e}_k) + \int_0^t (u(t), \tilde{e}_k)\, d(v(t), \tilde{e}_k) + \int_0^t (v(t), \tilde{e}_k)\, d(u(t), \tilde{e}_k).$$

From the above equation, we have

$$\begin{aligned}(u,v) &= (u_0,v_0) + \int_0^t (u,dv) + \int_0^t (v,du) \\ &= (u_0,v_0) - \frac{1}{2}\|u\|_{H_0^1}^2 + \frac{1}{2}\|u_0\|_{H_0^1}^2 - \int_0^t \|u\|^2\, ds - \int_0^t \|\nabla u\|^2\, ds \\ &+ \int_0^t \|v\|^2\, ds + \int_0^t (u,\varphi)\, ds + \int_0^t (u, hdW).\end{aligned} \tag{26}$$

Taking the expectation in (26) gives the desired equality. □

Before proving the blow-up theorem we introduce the following functional

$$\mathcal{L} = \left(Y(t) + (T_* - t)\, E\,\|u_0\|^2\right)^{-\gamma}. \tag{27}$$

Simple computations yield

$$\mathcal{L}' = -\gamma \mathcal{L}^{-\gamma-1}(t)\left[Y'(t) - E\,\|u_0\|^2\right] = -\gamma \mathcal{L}^{1+1/\gamma}(t)\left[Y'(t) - E\,\|u_0\|^2\right] \tag{28}$$

and

$$\mathcal{L}'' = \gamma \mathcal{L}^{1+2/\gamma}(t)\left\{(\gamma+1)\left(Y'(t) - E\,\|u_0\|^2\right)^2 - \left[Y(t) + (T_* - t)\, E\,\|u_0\|^2\right] Y''(t)\right\}. \tag{29}$$

Using Hölder inequality and equality (22), we obtain

$$\begin{aligned}Y'(t) - E\,\|u_0\|^2 &= E\,(u,v) + E\int_0^t (u,v)\, ds \\ &\le \left(E\|u\|^2 E\|v\|^2\right)^{1/2} + \left(E\int_0^t \|u\|^2 d\tau E\int_0^t \|v\|^2 d\tau\right)^{1/2}.\end{aligned} \tag{30}$$

Then by (16) and (18), we have

$$Y''(t) \ge 2(2+4\gamma)E\int_\Omega \Theta(u)dx + 2E(v,v) - 2E\|u\|_{H_0^1}^2 \tag{31}$$

Inserting (17) in (19) gives

$$E\vartheta(t) \le E\vartheta(0) - \frac{\gamma+1}{2\gamma+1}\int_0^t \|v\|^2 d\tau. \tag{32}$$

Employing (19) and (32) in (31) yields

$$Y''(t) \ge -(8\gamma+4)E\vartheta(0) + 4(\gamma+1)E\int_0^t \|v\|^2\, ds + 4(\gamma+1)E(v,v). \tag{33}$$

Depending on the initial energy's ($E\vartheta(0)$) sign, three different cases arise.

(i) If $E\vartheta(0) < 0$, inequality (33) yields

$$Y''(t) \ge -(8\gamma+4)E\vartheta(0). \tag{34}$$

Integration of the above inequality from 0 to t gives

$$Y'(t) \ge Y'(0) - (8\gamma+4)E\vartheta(0)t\ , \quad t \ge 0.$$

Hence, for $t \ge t_1$ we get $Y'(t) > E\|u_0\|^2$, where

$$t_1 = \max\left\{0, \frac{Y'(0) - E\|u_0\|^2}{(8\gamma+4)E\vartheta(0)}\right\}. \tag{35}$$

(ii) If $E\vartheta(0) = 0$, then $Y''(t) \geq 0$ for $t \geq 0$. Thus, if $E\|u_0\|^2 < Y'(0)$, then for $t \geq 0$, we get $Y'(t) > E\|u_0\|^2$.

(iii) If $E\vartheta(0) > 0$, then with the aid of (22) and the Young inequality, we obtain

$$E\|u_0\|^2 + E\int_0^t \|u\|^2 d\tau + E\int_0^t \|v\|^2 d\tau \geq E\|u\|^2$$

which yields

$$Y(t) + E\int_0^t \|v\|^2 d\tau + E\|u_0\|^2 + E\|v\|^2 \geq E\|u\|^2. \tag{36}$$

If this inequality is combined with (33), then one can see that the functional $Y(t)$ satisfies (14), i.e.

$$Y''(t) - 4(\gamma+1)Y'(t) + 4(\gamma+1)Y(t) + M \geq 0, \tag{37}$$

where

$$M = (8\gamma+4)E\vartheta(0) + 4(\gamma+1)E\|u_0\|^2. \tag{38}$$

Let us define

$$K(t) = Y(t) + \frac{1}{4(\gamma+1)}M$$

for $t > 0$. Then $K(t)$ satisfies the differential inequality (14). Moreover, Lemma 2.4 implies that under the condition $K'(0) > r_2K(0) + M_0$, we have $Y' > 0$ for $t > 0$.
As a consequence of the above computations, the following lemma can be written.

Lemma 3.3. *Assume that φ is local Lipschitz, and φ, h satisfies conditions (18) and (17), respectively. Moreover, suppose either of the following conditions holds*

(1) $E\vartheta(0) < 0$
(2) $E\vartheta(0) = 0$ *and* $Y'(0) > E\|u_0\|^2$
(3) $E\vartheta(0) > 0$ *and*

$$Y'(0) > r_2\left(Y(0) + \frac{M}{4(\gamma+1)}\right) + E\|u_0\|^2. \tag{39}$$

Then $Y'(t) > E\|u_0\|^2$ for $t \geq t_0$, where $t_0 = t_1$ is given explicitly in the proof for case (1), and for cases (2) and (3) $t_0 = 0$.

Theorem 3.4. *Suppose that conditions L1, L3, (17) and one of the following conditions are met*

(i) $E\vartheta(0) < 0$
(ii) $E\vartheta(0) = 0$ *and* $Y'(0) > E\|u_0\|^2$
(iii) $\frac{(E(u_0,v_0))^2}{2(T_*+1)E\|u_0\|^2} > E\vartheta(0) > 0$ *and*

$$Y'(0) > r_2\left(Y(0) + \frac{M}{4(\gamma+1)}\right) + E\|u_0\|^2. \tag{40}$$

Then

$$lim_{t\to T_0^-}\left(E\int_0^t \|u\|^2 d\tau + E\|u\|^2\,(t)\right) = +\infty. \tag{41}$$

Proof. We try to obtain an estimate for $\mathcal{L}$. Rearranging (29) yield

$$\mathcal{L}'' = -\gamma\mathcal{L}^{1+2/\gamma}(t)\Xi(t), \tag{42}$$

where

$$\Xi(t) = \left[Y(t) + (T_* - t)\,E\,\|u_0\|^2\right]Y''(t) - (\gamma+1)\left(Y'(t) - E\,\|u_0\|^2\right)^2. \tag{43}$$

Making use of the estimates (30) and (33) in the above equation, we obtain

$$\begin{aligned}\Xi(t) &\geq \left(-(4+8\gamma)E\vartheta(0)+4(\gamma+1)\left(E\|v\|^2+E\int_0^t\|v\|^2d\tau\right)\right)(Y(t)+(T_*-t)E\|u_0\|^2)\\&-4(1+\gamma)Y(t)\left(E\|v\|^2+E\int_0^t\|v\|^2d\tau\right)\\&\geq -(4+8\gamma)E\vartheta(0)(Y(t)+(T_*-t)E\|u_0\|^2)\end{aligned} \tag{44}$$

Using (44) in (42), the following inequality is obtained

$$\mathcal{L}''(t)\leq\gamma(4+8\gamma)E\vartheta(0)\mathcal{L}^{1+\frac{1}{\gamma}}(t)\ ,\quad t\geq t_0. \tag{45}$$

Since $\mathcal{L}'(t)<0$ for $t>t_0$ due to Lemma 3.3, the following inequality is obtained by multiplying (45) by $\mathcal{L}'(t)$ and integrating from t_0 to t :

$$\mathcal{L}'(t)^2\geq A_0+A_1\mathcal{L}^{2+\frac{1}{\gamma}}(t),\quad t\geq t_0,$$

where

$$A_0=\gamma^2\mathcal{L}^{2+\frac{2}{\gamma}}(t_0)((Y'(t_0)-E\|u_0\|^2)^2-8E\vartheta(0)\mathcal{L}^{\frac{-1}{\gamma}}(t_0))\ ,\ A_1=8\gamma^2E\vartheta(0).$$

$A_0>0$ iff

$$E\vartheta(0)<\frac{(Y'(t_0)-E\|u_0\|^2)^2}{8(Y(t_0)+(T_*-t_0)E\|u_0\|^2)}.$$

Lemma 2.5 implies that $\lim_{t\to T_0}\mathcal{L}(t)=0$ in a finite time T_0, and hence, the upper bound of blow-up time T_0 is estimated with respect to the sign of $E\vartheta(0)$. In case (i),

$$T_0\leq t_0-\frac{\mathcal{L}(t_0)}{\mathcal{L}'(t_0)}. \tag{46}$$

Moreover, if $\mathcal{L}(t_0)<\left\{1,\sqrt{A_0/-A_1}\right\}$ then

$$T_0\leq t_0+\frac{1}{\sqrt{-A_1}}\ln\frac{\sqrt{A_0/-A_1}}{\sqrt{A_0/-A_1}-\mathcal{L}(t_0)},$$

where $t_0=t_1$ was given in (35). In case (ii),

$$T_0\leq-\frac{\mathcal{L}(0)}{\mathcal{L}'(0)}$$

or

$$T_0\leq\frac{\mathcal{L}(0)}{\sqrt{A_0}}.$$

For the third case,

$$T_0\leq\frac{\mathcal{L}(0)}{\sqrt{A_0}}$$

or

$$T_0\leq 2^{\frac{3\gamma+1}{2\gamma}}\frac{\gamma P}{\sqrt{A_0}}(1-(1+P\mathcal{L}(0))^{\frac{-1}{2\gamma}}).$$

Here $P=(A_0/A_1)^{2+\frac{1}{\gamma}}$. For T_*, we may select $T_*\geq T_0$ as any positive constant for the three cases. We conclude from the above that

$$\lim_{t\to T_0}Y(t)=+\infty.$$

□

Remark 3.5. Damping terms play an important role in evolution equations. If there is not a damping term in Eq. (1), the condition that continuity of the function h in x and t, and locally Lipschitz continuity in u and ∇u is sufficient. But, in the presence of damping terms, the interaction between the source term and the damping term determines the behavior of the problem. For the blow-up of the solutions of the problem investigated here, the conditions (17) and (18) were imposed on h, which means that the source term dominates the damping term. Conversely, the dominance of the damping term over the source term ensures global in time solutions. For stochastic wave equations, a large noise may prevent the blow-up of solutions in finite time, however, a small noise is not enough to prevent blow-up phenomena.

The initial boundary value problem of Eq. (1) in the absence of a stochastic term was previously studied by Xu [7].

4 Declarations

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Competing Interests

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Author's Contributions

Conceptualization, H. Taskesen; methodology, H. Taskesen and B. Yağız; investigation, H. Taskesen and B. Yağız; writing—original draft preparation, B. Yağız; writing—review and editing, H. Taskesen.

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