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New Estimations for Quasi-Convex Functions and $(h, m)-$Convex Functions with the Help of Caputo-Fabrizio Fractional Integral Operator

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Abstract

The main motivation of the paper is to provide new integral inequalities for different kinds of convex functions by using a fractional integral operator with a non-singular kernel. The findings involve several new integral inequalities for quasi-convex functions and $(h, m)-$convex functions. We have used the algebraic properties of Caputo-Fabrizio fractional operator, definitions of convex functions, and elementary analysis methods for the proof steps.

Key words: Caputo-Fabrizio fractional integral operator, Quasi-convex functions, $(h, m)-$convex functions, Hadamard-Type Inequalities
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1 Introduction

The concept of convexity, which has an important place in inequality theory, has been used by many researchers and has been used extensively, especially in the field of inequality theory. Convex functions are fundamental concept in mathematical analysis with applications in optimization, economics, engineering, and various other fields. A real-valued function $f : \mathbb{R}^n \to \mathbb{R}$ is considered convex if, roughly speaking, the line segment between any two points on the graph of the function lie above the graph itself. The formal definition can be given as follows.

Definition 1.1 (See [1]). Let I be on interval in $\mathbb{R}$. Then $\Xi : I \to \mathbb{R}$ is said to be convex, if

$$\Xi\left(\zeta\rho + (1-\zeta)\varrho\right) \leq \zeta\Xi\left(\rho\right) + (1-\zeta)\Xi\left(\varrho\right)$$

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holds for all $\rho,\varrho\in I$ and $\zeta\in[0,1]$.

We remind that the notion of $h-$convex functions as follows.

Definition 1.2. (See [2]) Let $h: J\subseteq\mathbb{R}\to\mathbb{R}$ be a positive function. We say that $\Xi: I\subseteq\mathbb{R}\to\mathbb{R}$ is a $h-$convex function, or that Ξ belongs to the class $SX(h,I)$, if Ξ is non-negative and for all $\rho,\varrho\in I$, and $\zeta\in(0,1)$, we have

$$\Xi\left(\zeta\rho+(1-\zeta)\varrho\right)\leq h(\zeta)\,\Xi(\rho)+h(1-\zeta)\,\Xi(\varrho). \tag{1}$$

If the inequality (1) is reversed, then Ξ is said to be $h-$concave function.

We recall that the notion of $(h,m)-$convex functions is the following.

Definition 1.3. (See [3]) Let $h: J\subseteq\mathbb{R}\to\mathbb{R}$ be a non-negative function. We say that $\Xi:[0,\varsigma]\to\mathbb{R}$ is a $(h-m)-$convex function, if Ξ is non-negative and for all $\rho,\varrho\in[0,\varsigma]$, $m\in[0,1]$ and $\beta\in(0,1)$, we have

$$\Xi\left(\beta\rho+m(1-\beta)\varrho\right)\leq h(\beta)\,\Xi(\rho)+mh(1-\beta)\,\Xi(\varrho). \tag{2}$$

If the inequality (2) is reversed, then Ξ is said to be $(h-m)$-concave function on $[0,\varsigma]$

We will proceed with the definition of quasi-convex functions as follows.

Definition 1.4. (See [4]) A function $\Xi:[\sigma,\varsigma]\to\mathbb{R}$ is said to be quasi-convex on $[\sigma,\varsigma]$ if

$$\Xi(\zeta\rho+(1-\zeta)\varrho)\leq\max\left\{\Xi(\rho),\Xi(\varrho)\right\},\ \text{for all } \rho,\varrho\in[\sigma,\varsigma].$$

The relationship between the concepts of convexity and quasi-convexity can be found in the papers [5–9].

In [10], Iscan defined quasi-geometrically convex functions as the following:

Definition 1.5. A mapping $\Xi: I\subset(0,\infty)\to\mathbb{R}$ is called as quasi-geometrically convex on I, if

$$\Xi\left(\rho^{1-\zeta}\varrho^{\zeta}\right)\leq\sup\left\{\Xi(\rho),\Xi(\varrho)\right\},$$

for any $\rho,\varrho\in I$ and $\zeta\in[0,1]$.

Iscan also gave an example of quasi-geometrically convex functions in [10]:

Example 1. The function $\Xi:(0,4]\to\mathbb{R}$,

$$\Xi(\varpi)=\begin{cases}1, & \varpi\in(0,1]\\ (\varpi-2)^2, & \varpi\in[1,4]\end{cases}$$

is neither $GA-$convex nor geometrically convex on $(0,4]$, but it is a quasi-geometrically convex function.

Inequality theory, broadly speaking, encompasses the study of mathematical inequalities and their properties. Inequalities are expressions that describe a relationship between two mathematical objects, typically numbers, vectors, functions, or operators. Inequality theory is a fundamental and pervasive part of mathematics with applications in various branches. Inequality theory is pervasive in various branches of mathematics, including analysis, algebra, number theory, optimization, probability, and functional analysis. It is fundamental in establishing bounds, proving the convergence of series, and optimizing functions subject to constraints.

In summary, inequality theory is a rich and versatile area of mathematics, providing tools and techniques for reasoning about relationships between numbers, functions, and mathematical objects across different mathematical disciplines.

Fractional integral operators are mathematical operators that generalize the concept of integration to non-integer orders. They are fundamental in the field of fractional calculus and have applications in various scientific and engineering disciplines. The two most common types of fractional integral operators are the Riemann-Liouville fractional integral and the Caputo fractional integral. Fractional integrals capture non-local effects, considering the entire history of the function up to the order of integration. The fractional integral operators exhibit a semigroup property, meaning that the composition of two fractional integrals of different orders is equivalent to a fractional integral of another order and also they serve as an inverse operation to fractional derivatives. Applying a fractional derivative followed by a fractional integral (or vice versa) may yield the original function, depending on the order of integration and differentiation. In conclusion, fractional integral operators are powerful tools for modeling and analyzing systems with memory effects and non-local behaviors. Their applications span various scientific and engineering domains where traditional integer-order calculus may fall short in capturing the complexity of the underlying processes.

Definition 1.6. (See [11–13]) Let $\Xi \in H^1(\sigma,\varsigma)$, $\sigma<\varsigma$, $\beta\in[0,1]$, then the definition of the left fractional derivative in the sense of Caputo and Fabrizio becomes

$$\left({}_{\sigma}^{CFC}D^{\beta}\Xi\right)(\zeta)=\frac{B(\beta)}{1-\beta}\int_{\sigma}^{t}\Xi'(\rho)\,e^{\frac{-\beta(t-\rho)^{\beta}}{1-\beta}}d\rho$$

and the associated fractional integral is

$$\left({}_{\sigma}^{CF}I^{\beta}\Xi\right)(\zeta)=\frac{1-\beta}{B(\beta)}\Xi(\zeta)+\frac{\beta}{B(\beta)}\int_{\sigma}^{t}\Xi(\rho)\,d\rho$$

where $B(\beta)>0$ is a normalization function satisfying $B(0)=B(1)=1$. For the right fractional derivative we have

$$\left({}^{CFC}D_{\varsigma}^{\beta}\Xi\right)(\zeta)=\frac{-B(\beta)}{1-\beta}\int_{t}^{\varsigma}\Xi'(\rho)\,e^{\frac{-\beta(\rho-t)^{\beta}}{1-\beta}}d\rho$$

and the associated fractional integral is

$$\left({}^{CF}I_{\varsigma}^{\beta}\Xi\right)(\zeta)=\frac{1-\beta}{B(\beta)}\Xi(\zeta)+\frac{\beta}{B(\beta)}\int_{t}^{\varsigma}\Xi(\rho)\,d\rho.$$

For more information related to different kinds of fractional operators, we recommend to the readers the following papers (see [14**?** –26]).

The study of new inequalities for various types of convex functions is motivated by the overarching goal of advancing mathematical tools for optimization, analysis, and understanding of the properties of complex mathematical systems. By exploring and establishing new inequalities tailored to different classes of convex functions, researchers aim to enhance the theoretical foundation of convex analysis, offering more refined tools for problem-solving and optimization. These inequalities contribute not only to the theoretical aspects of mathematics but also find practical applications in fields such as control theory, machine learning, and signal processing, where precise characterizations of convex functions and their relationships are vital for designing efficient algorithms and understanding the dynamics of real-world systems. Thus, the study of new inequalities for various convex functions align with the broader objective of advancing mathematical methodologies with widespread implications across scientific and applied disciplines. In this viewpoint, the paper includes several new integral inequalities for different kinds of convex functions via non-singular fractional integral operators.

2 New Inequalities for Quasi-Convex Functions

Theorem 2.1. *Let $I \subseteq R$. Suppose that $\Xi : [\sigma, \varsigma] \subseteq I \to [0, \infty)$ is a quasi-convex function on $[\sigma, \varsigma]$ such that $\Xi \in L_1[\sigma, \varsigma]$. Then, we have*

$$\left({}_{\sigma}^{CF}I^{\beta}\Xi\right)(k) + \left({}^{CF}I_{\varsigma}^{\beta}\Xi\right)(k) \leq \frac{2(1-\beta)\Xi(k) + \beta(\varsigma-\sigma)\max\{\Xi(\sigma), \Xi(\varsigma)\}}{B(\beta)}$$

where $B(\beta) > 0$ is the normalization function $\beta \in [0,1]$.

Proof. By using the definition of quasi-convex function, we can write

$$\Xi(\zeta\sigma + (1-\zeta)\varsigma) \leq \max\{\Xi(\sigma), \Xi(\varsigma)\}.$$

By applying integration for the variable ζ over $[0,1]$, we obtain

$$\int_0^1 \Xi(\zeta\sigma + (1-\zeta)\varsigma)\,d\zeta \leq \int_0^1 \max\{\Xi(\sigma), \Xi(\varsigma)\}\,d\zeta.$$

By changing of the variable as $\rho = \zeta\sigma + (1-\zeta)\varsigma$ and calculating the right hand side, we get

$$\frac{1}{\varsigma-\sigma}\int_{\sigma}^{\varsigma} \Xi(\rho)\,d\rho \leq \max\{\Xi(\sigma), \Xi(\varsigma)\}.$$

By multiplying both sides of the above inequality with $\frac{\beta(\varsigma-\sigma)}{B(\beta)}$ and adding $\frac{2(1-\beta)}{B(\beta)}\Xi(k)$, we have

$$\frac{2(1-\beta)}{B(\beta)}\Xi(k) + \frac{\beta}{B(\beta)}\int_{\sigma}^{\varsigma} \Xi(\rho)\,d\rho \leq \frac{2(1-\beta)}{B(\beta)}\Xi(k) + \frac{\beta(\varsigma-\sigma)\max\{\Xi(\sigma), \Xi(\varsigma)\}}{B(\beta)}.$$

By simplifying the inequality, we get the following result

$$\begin{aligned} &\left(\frac{(1-\beta)}{B(\beta)}\Xi(k) + \frac{\beta}{B(\beta)}\int_{\sigma}^{k} \Xi(\rho)\,d\rho\right) + \left(\frac{(1-\beta)}{B(\beta)}\Xi(k) + \frac{\beta}{B(\beta)}\int_{k}^{\varsigma} \Xi(\rho)\,d\rho\right) \\ \leq\ & \frac{2(1-\beta)}{B(\beta)}\Xi(k) + \frac{\beta(\varsigma-\sigma)\max\{\Xi(\sigma), \Xi(\varsigma)\}}{B(\beta)}. \end{aligned}$$

Namely,

$$\left({}_{\sigma}^{CF}I^{\beta}\Xi\right)(k) + \left({}^{CF}I_{\varsigma}^{\beta}\Xi\right)(k) \leq \frac{2(1-\beta)\Xi(k) + \beta(\varsigma-\sigma)\max\{\Xi(\sigma), \Xi(\varsigma)\}}{B(\beta)}.$$

This completes the proof. □

Theorem 2.2. *Let $I \subseteq R$. Suppose that $\Xi : [\sigma, \varsigma] \subseteq I \to [0, \infty)$ is an integrable function and $|\Xi|$ is a quasi-convex function on $[\sigma, \varsigma]$ such that $\Xi \in L_1[\sigma, \varsigma]$. Then, we have*

$$\left({}_{\sigma}^{CF}I^{\beta}\Xi\right)(k) + \left({}^{CF}I_{\varsigma}^{\beta}\Xi\right)(k) \leq \frac{2(1-\beta)\Xi(k)pq + \beta(\varsigma-\sigma)\left(q\left(\max\{|\Xi(\sigma)|, |\Xi(\varsigma)|\}\right)^p + p\right)}{B(\beta)pq}$$

where $B(\beta) > 0$ is the normalization function $q > 1$, $\frac{1}{p} + \frac{1}{q} = 1$ and $\beta \in [0,1]$.

Proof. From the definition of quasi-convex function, we have

$$\left|\Xi\left(\zeta\sigma+\left(1-\zeta\right)\varsigma\right)\right| \leq \max\left\{\left|\Xi\left(\sigma\right)\right|,\left|\Xi\left(\varsigma\right)\right|\right\}.$$

By applying integration for the variable ζ on $[0,1]$, we obtain

$$\int_0^1 \left|\Xi\left(\zeta\sigma+\left(1-\zeta\right)\varsigma\right)\right| d\zeta \leq \int_0^1 \max\left\{\left|\Xi\left(\sigma\right)\right|,\left|\Xi\left(\varsigma\right)\right|\right\} d\zeta.$$

By using the celebrated Young's inequality for the above inequality, we get

$$\int_0^1 \left|\Xi\left(\zeta\sigma+\left(1-\zeta\right)\varsigma\right)\right| d\zeta \leq \left(\frac{1}{p}\int_0^1 \left(\max\left\{\left|\Xi\left(\sigma\right)\right|,\left|\Xi\left(\varsigma\right)\right|\right\}\right)^p d\zeta + \frac{1}{q}\int_0^1 1^q d\zeta\right).$$

By changing of the variable as $\rho = \zeta\sigma + (1-\zeta)\varsigma$ and calculating the right hand side, it yields that

$$\frac{1}{\varsigma-\sigma}\int_\sigma^\varsigma \left|\Xi\left(\rho\right)\right| d\rho \leq \frac{\left(\max\left\{\left|\Xi\left(\sigma\right)\right|,\left|\Xi\left(\varsigma\right)\right|\right\}\right)^p}{p} + \frac{1}{q}$$

If we product both sides of the above inequality with $\frac{\beta(\varsigma-\sigma)}{B(\beta)}$ and adding $\frac{2(1-\beta)}{B(\beta)}\Xi\left(k\right)$, we can write

$$\begin{aligned}&\frac{2\left(1-\beta\right)}{B\left(\beta\right)}\Xi\left(k\right)+\frac{\beta}{B\left(\beta\right)}\int_\sigma^\varsigma \left|\Xi\left(\rho\right)\right| d\rho \\ &\qquad\leq \frac{2\left(1-\beta\right)}{B\left(\beta\right)}\Xi\left(k\right)+\frac{\beta\left(\varsigma-\sigma\right)}{B\left(\beta\right)}\left(\frac{\left(q\left(\max\left\{\left|\Xi\left(\sigma\right)\right|,\left|\Xi\left(\varsigma\right)\right|\right\}\right)^p+p\right)}{pq}\right).\end{aligned}$$

By a simple arrangement, we get the result. □

Theorem 2.3. *Let $I \subseteq R$. Suppose that $\Xi : [\sigma,\varsigma] \subseteq I \to [0,\infty)$ is a quasi geometrically-convex function on $[\sigma,\varsigma]$ such that $\Xi \in L_1[\sigma,\varsigma]$. Then, we have*

$$\left({}_{\sigma}^{CF}I^{\beta}\hbar\right)(k)+\left({}^{CF}I_{\varsigma}^{\beta}\hbar\right)(k) \leq \frac{2\left(1-\beta\right)\hbar\left(k\right)+\beta\left(\ln\varsigma-\ln\sigma\right)\sup\left\{\Xi\left(\sigma\right),\Xi\left(\varsigma\right)\right\}}{B\left(\beta\right)}$$

where $B(\beta) > 0$ is the normalization function $\hbar(\rho) = \frac{\Xi(\rho)}{\rho}$ and $\beta \in [0,1]$.

Proof. By using the definition of quasi geometrically-convex function and integrating both sides of the inequality over $[0,1]$ with respect to ζ, we can write , we get

$$\int_0^1 \Xi\left(\sigma^\zeta\varsigma^{1-\zeta}\right) d\zeta \leq \int_0^1 \sup\left\{\Xi\left(\sigma\right),\Xi\left(\varsigma\right)\right\} d\zeta.$$

By changing of the variable as $\rho = \sigma^\zeta\varsigma^{1-\zeta}$ and calculating the right hand side of the above inequality, we obtain

$$\frac{1}{\ln\varsigma-\ln\sigma}\int_\sigma^\varsigma \frac{\Xi\left(x\right)}{x}dx \leq \sup\left\{\Xi\left(\sigma\right),\Xi\left(\varsigma\right)\right\}$$

If we take $\hbar(\rho) = \frac{\Xi(\rho)}{\rho}$, we obtain

$$\frac{1}{\ln\varsigma-\ln\sigma}\int_\sigma^\varsigma \hbar\left(\rho\right)d\rho \leq \sup\left\{\Xi\left(a\right),\Xi\left(b\right)\right\}$$

If we product both sides of the above inequality with $\frac{\beta(\ln\varsigma-\ln\sigma)}{B(\beta)}$ and adding $\frac{2(1-\beta)}{B(\beta)}\hbar(k)$, we can write

$$\frac{2(1-\beta)}{B(\beta)}\hbar(k)+\frac{\beta}{B(\beta)}\int_{\sigma}^{\varsigma}\hbar(\rho)\,d\rho\leq\frac{2(1-\beta)}{B(\beta)}\hbar(k)+\frac{\beta(\ln\varsigma-\ln\sigma)\sup\{\Xi(\sigma),\Xi(\varsigma)\}}{B(\beta)}.$$

By making use of the necessary arrangements, we have the following inequality

$$\begin{aligned}&\left(\frac{(1-\beta)}{B(\beta)}\hbar(k)+\frac{\beta}{B(\beta)}\int_{\sigma}^{k}\hbar(\rho)\,d\rho\right)+\left(\frac{(1-\beta)}{B(\beta)}\hbar(k)+\frac{\beta}{B(\beta)}\int_{k}^{\varsigma}\hbar(\rho)\,d\rho\right)\\ \leq\;&\frac{2(1-\beta)}{B(\beta)}\hbar(k)+\frac{\beta(\ln\varsigma-\ln\sigma)\sup\{\Xi(a),\Xi(b)\}}{B(\beta)}.\end{aligned}$$

Which implies the result. □

Theorem 2.4. *Let $I\subseteq R$. Suppose that $\Xi:[\sigma,\varsigma]\subseteq I\to[0,\infty)$ is an integrable function and $|\Xi|$ is quasi geometrically-convex function on $[\sigma,\varsigma]$ such that $\Xi\in L_1[\sigma,\varsigma]$. Then, the following inequality is valid:*

$$\left({}_{\sigma}^{CF}I^{\beta}\hbar\right)(k)+\left({}^{CF}I_{\varsigma}^{\beta}\hbar\right)(k)\leq\frac{2(1-\beta)\hbar(k)\,pq+\beta(\varsigma-\sigma)\left(q\left(\max\{|\Xi(\sigma)|,|\Xi(\varsigma)|\}\right)^{p}+p\right)}{B(\beta)\,pq}$$

where $B(\beta)>0$ is the normalization function $q>1$, $\frac{1}{p}+\frac{1}{q}=1$, $\hbar(\rho)=\frac{\Xi(\rho)}{\rho}$ and $\beta\in[0,1]$.

Proof. By a similar argument to the proof of the previous theorem but now by applying Young inequality, we get the proof. □

3 New Inequalities for $(h,m)-$Convex Functions

Theorem 3.1. *Let $I\subseteq R$. Suppose that $\Xi:[\sigma,\varsigma]\subseteq I\to R$ is a (h,m)-convex function on $[\sigma,\varsigma]$ such that $\Xi\in L_1[\sigma,\varsigma]$. Then, we get the following inequality:*

$$\begin{aligned}&\left({}_{\sigma}^{CF}I^{\beta}\Xi\right)(k)+\left({}^{CF}I_{\varsigma}^{\beta}\Xi\right)(k)\\ \leq\;&\frac{2(1-\beta)}{B(\beta)}\Xi(k)+\frac{\beta(\varsigma-\sigma)\Xi(\sigma)}{B(\beta)}\int_0^1 h(\zeta)\,d\zeta+\frac{\beta(\varsigma-\sigma)m\Xi(\varsigma)}{B(\beta)}\int_0^1 h(1-\zeta)\,d\zeta\end{aligned}$$

where $B(\beta)>0$ is the normalization function and $m,\beta\in[0,1]$.

Proof. By using the definition of (h,m)-convex function, we can write

$$\Xi(\zeta\sigma+(1-\zeta)\varsigma)\leq h(\zeta)\Xi(\sigma)+mh(1-\zeta)\Xi(\varsigma).$$

By applying integration for the variable ζ on $[0,1]$ with respect to ζ, we get

$$\int_0^1\Xi(\zeta\sigma+(1-\zeta)\varsigma)\,d\zeta\leq\Xi(\sigma)\int_0^1 h(\zeta)\,d\zeta+m\Xi(\varsigma)\int_0^1 h(1-\zeta)\,d\zeta.$$

By changing of the variable as $\rho=\zeta\sigma+(1-\zeta)\varsigma$, we obtain

$$\frac{1}{\varsigma-\sigma}\int_{\sigma}^{\varsigma}\Xi(\rho)\,d\rho\leq\Xi(a)\int_0^1 h(\zeta)\,d\zeta+m\Xi(b)\int_0^1 h(1-\zeta)\,d\zeta.$$

If we product both sides of the above inequality with $\frac{\beta(\varsigma-\sigma)}{B(\beta)}$ and adding $\frac{2(1-\beta)}{B(\beta)}\Xi(k)$, we have

$$\begin{aligned}&\frac{2(1-\beta)}{B(\beta)}\Xi(k)+\frac{\beta}{B(\beta)}\int_{\sigma}^{\varsigma}\Xi(\rho)\,d\rho\\ \leq\;&\frac{2(1-\beta)}{B(\beta)}\Xi(k)+\frac{\beta(\varsigma-\sigma)\,\Xi(\sigma)}{B(\beta)}\int_0^1 h(\zeta)\,d\zeta+\frac{\beta(\varsigma-\sigma)\,m\Xi(\varsigma)}{B(\beta)}\int_0^1 h(1-\zeta)\,d\zeta.\end{aligned}$$

By simplifying the inequality, we get the following result

$$\begin{aligned}&\left(\frac{(1-\beta)}{B(\beta)}\Xi(k)+\frac{\beta}{B(\beta)}\int_{\sigma}^{k}\Xi(\rho)\,d\rho\right)+\left(\frac{(1-\beta)}{B(\beta)}\Xi(k)+\frac{\beta}{B(\beta)}\int_{k}^{\varsigma}\Xi(\rho)\,d\rho\right)\\ \leq\;&\frac{2(1-\beta)}{B(\beta)}\Xi(k)+\frac{\beta(\varsigma-\sigma)\,\Xi(\sigma)}{B(\beta)}\int_0^1 h(\zeta)\,d\zeta+\frac{\beta(\varsigma-\sigma)\,m\Xi(\varsigma)}{B(\beta)}\int_0^1 h(1-\zeta)\,d\zeta.\end{aligned}$$

It yields that

$$\begin{aligned}&\left({}_{\sigma}^{CF}I^{\beta}\Xi\right)(k)+\left({}^{CF}I_{\varsigma}^{\beta}\Xi\right)(k)\\ \leq\;&\frac{2(1-\beta)}{B(\beta)}\Xi(k)+\frac{\beta(\varsigma-\sigma)\,\Xi(\sigma)}{B(\beta)}\int_0^1 h(\zeta)\,d\zeta+\frac{\beta(\varsigma-\sigma)\,m\Xi(\varsigma)}{B(\beta)}\int_0^1 h(1-\zeta)\,d\zeta.\end{aligned}$$

The proof is completed. □

Remark 3.2. If one chooses some special cases for the parameters β, m and the function h, some earlier findings can be provided.

4 Conclusion

Fractional calculus is a branch of mathematical analysis that generalizes the concept of differentiation and integration to non-integer orders. Unlike classical calculus, which deals with integer-order derivatives and integrals (e.g., first, second, and third derivatives), fractional calculus extends these operations to non-integer orders. Fractional calculus is a branch of mathematical analysis that generalizes the concept of differentiation and integration to non-integer orders. Unlike classical calculus, which deals with integer-order derivatives and integrals (e.g., first, second, third derivatives), fractional calculus extends these operations to non-integer orders. In essence, fractional calculus provides a richer framework for understanding and describing complex systems that exhibit non-local and memory-dependent behaviors. Its importance lies in its ability to offer more accurate models, improved control strategies, and better insights into the dynamics of diverse natural and engineered systems. The relationship between fractional calculus and inequalities is manifested in the study of fractional differential and integral inequalities. These inequalities involve fractional derivatives or integrals of functions and play a crucial role in analyzing the behavior of solutions to fractional differential equations. Fractional inequalities have applications in diverse fields such as analysis, mathematical modeling, and the study of complex systems. In this sense, we have proved some novel integral inequalities for different kinds of convex functions via Caputo-Fabrizio fractional integral operator.

Fractional calculus involves a diverse set of inequalities. Interested researchers can explore both integral and differential inequalities and understand how they interconnect. Besides, they can consider applications of fractional inequalities in different scientific and engineering domains, such as physics, biology, finance, and control systems. Showcasing the versatility of fractional inequalities can enhance their significance in the literature.

The authors will focus to explore and develop numerical methods for solving fractional inequalities in the next studies. Because it is important to provide efficient algorithms for solving fractional inequalities for practical applications and simulations.

5 Declarations

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Competing Interests

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Ethical Approval

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Authors's Contributions

All authors contributed equally. All the authors read and approved the final manuscript.

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