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Dynamics of a diffusive two predators-one prey system

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Abstract

The present article examines a diffusive model of predator-prey interaction, which includes a single prey species and two predator species. The model implements a modified Leslie-Gower term with a Holling type II scheme and is subject to the homogeneous Neumann boundary condition. Local stability condition is derived through the application of Routh-Hurwitz criterion. Global asymptotic stability of the singular positive steady state is shown by fitting a suitable Lyapunov function in the presence of self-diffusion, the cross-diffusion has the potential to generate stationary patterns, and therefore enables non-constant positive steady state. By taking cross-diffusion as a bifurcation parameter, it is possible to show the existence of positive non-constant solutions with the help of bifurcation theory. A brief conclusion completes the paper.

Key words: Cross-diffusion, Leslie-Gower model, global stability, priori estimates, bifurcation
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1 Introduction

In recent years, many mathematical models have been formulated to predict the behavior of different interacting species. In particular, the model dealing with predator-prey populations where the predation process follows Holling type II response has received much attention from the researchers. Considering real factors, a model that includes a modified Leslie-Gower (LG) term is important to examine. This term signifies a relation between the carrying capacity of the predator with the density of the prey. Various works on this scheme can be found in [1–3]. In [4], the authors investigated a prey-predator interaction with LG term and obtained boundedness, the viability of the steady state, and global stability of the positive steady state.

As long as the species are uniformly distributed, the study of the emergence and stability of the constant equilibrium point in a mathematical model is significant. When the environment faces non-homogeneity in species dispersion, the appearance of a non-constant positive solution addresses an important aspect of a dynamical system. The investigation of finding such a steady state is demonstrated in [5–7]. Ko and Ryu [8] studied the diffusive prey-predator interaction

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with LG term and observed that no non-constant positive solution exists when the population is homogeneously distributed but a steady state which is positive as well as non-constant may appear in case of general functional response. Hu and Li [9] found non-constant steady state in a diffusive prey-predator system with the LG term. Quo and Guo [10] analyzed a diffusive and advective LG model. Li et al. [31] discussed global asymptotic stability and stationary pattern of a diffusive prey-predator system with LG term and diffusion. Predator-prey models with more than two species create a major interest to the researchers. In the environment, populations are mixed up and are not confined to just two species in a single habitat. So models are formed to address complex dynamics for multiple species. The two predator-one prey model can be found in [11]. Recently, Mirella Cappelletti and Lisena [12] analyzed the impact of diffusion on a single prey and two predator system. They studied the asymptotic properties of the solutions of the model. Recently, some articles have shown the role of diffusion on predator-prey interactions [13–16].

Motivated by the above works, we are interested in examining a diffusive three species predator-prey system with modified LG term and Holling type II schemes as follows:

$$\begin{aligned} \frac{\partial p}{\partial t} &= \delta_1 \Delta p + p(r_1 - ap - \frac{c_1 q_1}{h_1 + p} - \frac{c_2 q_2}{h_2 + p}), x \in \Gamma, t > 0 \\ \frac{\partial q_1}{\partial t} &= \Delta(\delta_2 q_1 + \delta_3 p q_1) + q_1(r_2 - \frac{f_1 q_1}{h_1 + p}), x \in \Gamma, t > 0 \\ \frac{\partial q_2}{\partial t} &= \Delta(\delta_4 q_2 + \delta_5 p q_2) + q_2(r_3 - \frac{f_2 q_2}{h_2 + p}), x \in \Gamma, t > 0 \\ \frac{\partial p}{\partial n} &= \frac{\partial q_1}{\partial n} = \frac{\partial q_2}{\partial n} = 0, x \in \partial\Gamma \\ p(x,0) &= p_0(x) > 0, q_1(x,0) = q_{10}(x) > 0, q_2(x,0) = q_{20}(x) > 0, x \in \Gamma \end{aligned} \tag{1}$$

where $p(t), q_1(t)$ and $q_2(t)$ stand for the biomasses of prey, first predator, and the second predator population respectively. r_1, r_2 and r_3 represent the intrinsic growth rate of the prey and two predators respectively. a represents the intra-specific competition coefficient among the prey species. c_1 and c_2 denote the predation rate p. f_1 and f_2 carry the same meaning as of c_1 and c_2. h_1 and h_2 stand for the environmental protection for predator q_1 and q_2 respectively. We choose a domain Γ in $\mathbb{R}^{\mathbb{N}}$ where $\mathbb{N}$ is a positive integer. The boundary $\partial\Gamma$ is assumed to be smooth and the outward unit normal vector is designated as n. The diffusion terms $\delta_i (i = 1, 2, 3)$ is known as self diffusion coefficient, which means how the species move in a dispersive manner. The constant δ_3 and δ_4 measures the cross diffusion effect, which signifies an associated intervene between two interacting populations. In this model, q_1 diffuses with flux:

$$J_1 = -\nabla(\delta_2 q_1 + \delta_3 p q_1) = -(\delta_2 + \delta_3 p)\nabla q_1 - \delta_3 q_1 \nabla p.$$

We note that the part $-\delta_3 q_1 \nabla p$ of the flux is in the direction of the population density of p whose density decreases that means the prey population gather and make a large community to avoid predation effect. Similarly q_2 diffuses with flux:

$$J_2 = -\nabla(\delta_4 q_2 + \delta_5 p q_2) = -(\delta_4 + \delta_5 p)\nabla q_2 - \delta_5 q_2 \nabla p.$$

In recent years, there has been a major research interest in determining the spatial and temporal patterns in ecological and chemical systems. In this article, our main aim is for a system of more than two interacting species to examine the coexistence of all the species. When the interacting population is homogeneously distributed, coexistence implies that the system finally approaches an equilibrium point that is constant in nature whereas, for inhomogeneous cases, it means the existence of non-constant positive solutions, which is referred to as a stationary pattern.

From the literature [17], we know that diffusion and cross-diffusion have been identified as reasons for the spontaneous occurrence of ordered structure, specifically a stationary pattern, in a variety of circumstances when there is no equilibrium, for example Gierer-Meinhardt model [18, 19], the Sel'kov model [20, 21], and the biological models [22–24] etc.

In this work, we mainly investigate the viability of positive stationary solutions of (1). For this, we consider the following:

$$\begin{aligned}
-\delta_1 \Delta p &= p(r_1 - ap - \frac{c_1 q_1}{h_1 + p} - \frac{c_2 q_2}{h_2 + p}), x \in \Gamma, t > 0 \\
-\Delta(\delta_2 q_1 + \delta_3 p q_1) &= q_1(r_2 - \frac{f_1 q_1}{h_1 + p}), x \in \Gamma, t > 0 \\
-\Delta(\delta_4 q_2 + \delta_5 p q_2) &= q_2(r_3 - \frac{f_2 q_2}{h_2 + p}), x \in \Gamma, t > 0 \\
\frac{\partial p}{\partial n} &= \frac{\partial q_1}{\partial n} = \frac{\partial q_2}{\partial n} = 0, x \in \partial\Gamma \\
p(x,0) &= p_0(x) > 0, q_1(x,0) = q_{10}(x) > 0, q_2(x,0) = q_{20}(x) > 0, x \in \Gamma.
\end{aligned} \tag{2}$$

By direct computation, we can get a constant positive steady state of (2) $X_0 = (p^*, q_1^*, q_2^*)$, where

$$p^* = \frac{1}{a}(r_1 f_1 f_2 - c_1 r_2 f_2 - c_2 r_3 f_1), q_1^* = \frac{r_2(h_1 + p^*)}{f_1}, q_2^* = \frac{r_3(h_2 + p^*)}{f_2}$$

provided that

$$r_1 f_1 f_2 > c_1 r_2 f_2 + c_2 r_3 f_1. \tag{3}$$

In the entire article, $0 = \rho_0 < \rho_1 < \rho_2 < \ldots < \rho_n < \ldots$ stands for the eigenvalues of $-\Delta$ in Γ with the homogeneous Neumann boundary condition. $m(\rho_i)$ represents the multiplicity of ρ_i for any $i \geq 0$.

The paper is structured in the following manner. Local and global stability of (p^*, q_1^*, q_2^*) is presented in Section 2. A priori estimate of the solutions of (2) which are positive is derived in Section (3). We discussed the situation when the positive solutions exist and are non-constant in Section 4. Section 5 establishes the same findings in the earlier section by the application of bifurcation theory.

2 Nature of constant interior equilibrium

In this part, we do not take cross-diffusion in system (1) and present the following system:

$$\begin{aligned}
\frac{\partial p}{\partial t} &= \delta_1 \Delta p + p(r_1 - ap - \frac{c_1 q_1}{h_1 + p} - \frac{c_2 q_2}{h_2 + p}), x \in \Gamma, t > 0 \\
\frac{\partial q_1}{\partial t} &= \delta_2 \Delta q_1 + q_1(r_2 - \frac{f_1 q_1}{h_1 + p}), x \in \Gamma, t > 0 \\
\frac{\partial q_2}{\partial t} &= \delta_4 \Delta q_2 + q_2(r_3 - \frac{f_2 q_2}{h_2 + p}), x \in \Gamma, t > 0 \\
\frac{\partial p}{\partial n} &= \frac{\partial q_1}{\partial n} = \frac{\partial q_2}{\partial n} = 0, x \in \partial\Gamma \\
p(x,0) &= p_0(x) > 0, q_1(x,0) = q_{10}(x) > 0, q_2(x,0) = q_{20}(x) > 0, x \in \Gamma.
\end{aligned} \tag{4}$$

2.1 Local stability

First, we state the result on the local behavior of the equilibrium point which is positive and constant.

Theorem 1. *Assume that* (3) *holds. If*

$$a > \frac{c_1 q_1^*}{(h_1+p^*)^2} + \frac{c_2 q_2^*}{(h_2+p^*)^2}$$

then X_0 *is locally asymptotically stable*

Proof. Let $X = (p, q_1, q_2)^T, X_0 = (p^*, q_1^*, q_2^*)$ and we denote

$$M(X) = \begin{pmatrix} p(r_1 - ap - \frac{c_1 q_1}{h_1+p} - \frac{c_2 q_2}{h_2+p}) \\ q_1(r_2 - \frac{f_1 q_1}{h_1+p}) \\ q_2(r_3 - \frac{f_2 q_2}{h_2+p}) \end{pmatrix}.$$

Therefore,

$$M_X(X_0) = \begin{pmatrix} -ap^* + \frac{c_1 q_1^* p^*}{(h_1+p^*)^2} + \frac{c_2 q_2^* p^*}{(h_2+p^*)^2} & -\frac{c_1 p^*}{h_1+p^*} & -\frac{c_2 p^*}{h_2+p^*} \\ \frac{f_1 q_1^{*2}}{(h_1+p^*)^2} & -\frac{f_1 q_1^*}{h_1+p^*} & 0 \\ \frac{f_2 q_2^{*2}}{(h_2+p^*)^2} & 0 & -\frac{f_2 q_2^*}{h_2+p^*} \end{pmatrix}.$$

Linearising (4) at X_0, we have

$$X_t = (D\Delta + M_{X_0})X,$$

where

$$D = \begin{pmatrix} \delta_1 & 0 & 0 \\ 0 & \delta_2 & 0 \\ 0 & 0 & \delta_4 \end{pmatrix}.$$

The characteristic polynomial of $-\mu_i D + M_X(X_0)$ is determined by

$$\psi_i(\lambda) = \lambda^3 + a_1\lambda^2 + a_2\lambda + a_3,$$

where

$$\begin{aligned}
a_1 =& \mu_i\delta_1 - a_{11} + \mu_i\delta_2 + \frac{f_1 q_1^*}{h_1+p^*} + \mu_i\delta_4 + \frac{f_2 q_2^*}{h_2+p^*}, \\
a_2 =& (\mu_i\delta_2 + \frac{f_1 q_1^*}{h_1+p^*})(\mu_i\delta_4 + \frac{f_2 w^*}{h_2+u^*}) \\
&+ (\mu_i\delta_1 - a_{11})(\mu_i\delta_2 + \frac{f_1 q_1^*}{h_1+p^*} + \mu_i\delta_4 + \frac{f_2 q_2^*}{h_2+p^*}) \\
&+ \frac{c_1 f_1 p^* q_1^{*2}}{(h_1+p^*)^3} + \frac{c_2 f_2 p^* q_2^{*2}}{(h_1+p^*)^3}, \\
a_3 =& (\mu_i\delta_1 - a_{11})(\mu_i\delta_2 + \frac{f_1 q_1^*}{h_1+p^*})(\mu_i\delta_4 + \frac{f_2 q_2^*}{h_2+p^*}) \\
&+ \frac{c_1 f_1 p^* q_1^{*2}}{(h_1+p^*)^3}(\mu_i\delta_4 + \frac{f_2 q_2^*}{h_2+p^*}) + \frac{c_2 f_2 p^* q_2^{*2}}{(h_1+p^*)^3}(\mu_i\delta_2 + \frac{f_1 q_1^*}{h_1+p^*}), \\
a_{11} =& -p^*(a - \frac{c_1 q_1^*}{(h_1+p^*)^2} - \frac{c_2 q_2^*}{(h_2+p^*)^2}).
\end{aligned}$$

By the assumption of the theorem, a_1, a_2 and a_3 are all positive. Moreover, $a_1 a_2 - a_3 > 0$. Then using the well known criterion developed by Routh-Hurwitz, one can show that for each $i \geq 1$, all the three roots $\lambda_{i,1}, \lambda_{i,2}$ and $\lambda_{i,3}$ of $\psi_i(\lambda) = 0$ have negative real parts. Now it is possible to find a $\delta > 0$ such that

$$\text{Re}\{\lambda_{i1}\}, \text{Re}\{\lambda_{i2}\}, \text{Re}\{\lambda_{i3}\} \leq -\delta, i \geq 1. \tag{5}$$

Let $\lambda = \mu_i \eta$, then we have

$$\psi_i(\lambda) = \mu_i^3 \eta^3 + a_1 \mu_i^2 \eta^2 + a_2 \mu_i \eta + a_3 \triangleq \bar{\psi}_i(\eta).$$

Note that $\mu_i \to \infty$ as $i \to \infty$. We get

$$\lim_{i\to\infty} \frac{\bar{\psi}_i(\eta)}{\mu_i^3} = \eta^3 + (\delta_1 + \delta_2 + \delta_4)\eta^2 + (\delta_2\delta_4 + \delta_1\delta_2 + \delta_1\delta_4)\eta + \delta_1\delta_2\delta_4 \triangleq \bar{\psi}(\eta)$$

It is to be noted that all the three roots η_1, η_2 and η_3 of $\bar{\psi}(\eta) = 0$ have negative real parts. So, we can find a $\delta > 0$ such that

$$\text{Re}\{\eta_1\}, \text{Re}\{\eta_2\}, \text{Re}\{\eta_3\} \leq -\delta. \tag{6}$$

By continuity, we get $i_0 \in \mathbb{N}$ such that the three roots η_{i1}, η_{i2} and η_{i3} of $\bar{\psi}_i(\eta) = 0$ satisfy

$$\text{Re}\{\eta_{i1}\}, \text{Re}\{\eta_{i2}\}, \text{Re}\{\eta_{i3}\} \leq -\frac{\bar{\delta}}{2}, i \geq i_0. \tag{7}$$

this leads that

$$\text{Re}\{\lambda_{i1}\}, \text{Re}\{\lambda_{i2}\}, \text{Re}\{\lambda_{i3}\} \leq -\frac{\mu_i \bar{\delta}}{2}, i \geq i_0. \tag{8}$$

Choose $-\bar{\delta} = \max\{\text{Re}\{\lambda_{i1}\}, \text{Re}\{\lambda_{i2}\}, \text{Re}\{\lambda_{i3}\}$, then $\bar{\delta}$ is positive and (5) is satisfied whenever $\delta = \min\{\bar{\delta}.\frac{\bar{\delta}}{2}\}$. Applying Theorem 5.1.1 in [25], we complete the proof. □

2.2 Global stability

Now, our next aim in this section is to determine the condition for global behavior of (p^*, q_1^*, q_2^*) by applying the method used in [26] when there is no cross diffusion.

Theorem 2. *If*

$$a > \frac{c_1 r_2 h_2 f_2 + c_2 r_3 h_1 f_1}{h_1 h_2 f_1 f_2}$$

then (p^*, q_1^*, q_2^*) is globally asymptotically stable

Proof. Consider the Lyapunov function

$$V(t) = \int_\Gamma [\int_{p^*}^{p} \frac{\tau - p^*}{d\tau} d\tau + \lambda_1 \int_{q_1^*}^{q_1} \frac{\zeta - q_1^*}{d\zeta} d\zeta + \lambda_2 \int_{q_2^*}^{q_2} \frac{\eta - q_2^*}{d\eta} d\eta] dx$$

where λ_1 and λ_2 are positive constants to be chosen later. We have

$$
\begin{aligned}
\frac{dV}{dt} &= \int_\Gamma \Big(\frac{p-p^*}{p}\frac{\partial p}{\partial t} + \frac{q_1-q_1^*}{p}\frac{\partial q_1}{\partial t} + \frac{q_2-q_2^*}{q_2}\frac{\partial q_2}{\partial t}\Big)dx \\
&= \int_\Gamma (p-p^*)\Big(r_1 - ap - \frac{c_1 q_1}{h_1+p} - \frac{c_2 q_2}{h_2+p} - r_1 + ap^* + \frac{c_1 q_1^*}{h_1+p^*} + \frac{c_2 q_2^*}{h_2+p^*}\Big)dx \\
&\quad + \lambda_1 \int_\Gamma (q_1-q_1^*)\Big(r_2 - \frac{f_1 q_1}{h_1+p} - r_2 + \frac{f_1 q_1^*}{h_1+p^*}\Big)dx \\
&\quad + \lambda_2 \int_\Gamma (q_2-q_2^*)\Big(r_3 - \frac{f_2 q_2}{h_2+p} - r_3 + \frac{f_2 q_2^*}{h_2+p^*}\Big)dx \\
&\quad - \delta_1 p^* \int_\Gamma \frac{|\nabla p|^2}{p^2}dx - \lambda_1\delta_2 q_1^* \int_\Gamma \frac{|\nabla q_1|^2}{q_1^2}dx - \lambda_2\delta_4 q_2^* \int_\Gamma \frac{|\nabla q_2|^2}{q_2^2}dx
\end{aligned}
$$

then

$$
\begin{aligned}
\frac{dV}{dt} =& -a\int_\Gamma (p-p^*)^2 dx + \frac{c_1-\lambda_1 f_1 q_1^*}{h_1+p^*}\int_\Gamma \frac{(p-p^*)(q_1-q_1^*)}{h_1+p}dx \\
&+ \frac{c_1 q_1^*}{h_1+p^*}\int_\Gamma \frac{(p-p^*)^2}{h_1+p}dx + \frac{c_2-\lambda_2 f_2 q_2^*}{h_2+p^*}\int_\Gamma \frac{(p-p^*)(q_2-q_2^*)}{h_2+p}dx \\
&+ \frac{c_2 q_2^*}{h_2+p^*}\int_\Gamma \frac{(p-p^*)^2}{h_2+p}dx - \lambda_1 f_1 \int_\Gamma \frac{(q_1-q_1^*)^2}{h_1+p}dx \\
&- \lambda_2 f_2 \int_\Gamma \frac{(q_2-q_2^*)^2}{h_2+p}dx - \delta_1 p^* \int_\Gamma \frac{|\nabla p|^2}{p^2}dx - \lambda_1\delta_2 q_1^* \int_\Gamma \frac{|\nabla q_1|^2}{q_1^2}dx \\
&- \lambda_2\delta_4 q_2^* \int_\Gamma \frac{|\nabla q_2|^2}{q_2^2}dx.
\end{aligned}
$$

Choose $\lambda_1 = \frac{c_1}{f_1 q_1^*}$, and $\lambda_2 = \frac{c_2}{f_2 q_2^*}$ then

$$
\begin{aligned}
\frac{dV}{dt} \le& -\Big[a - \frac{c_1 q_1^*}{h_1(h_1+p^*)} - \frac{c_2 q_2^*}{h_2(h_2+p^*)}\Big]\int_\Gamma (p-p^*)^2 dx \\
&- \lambda_1 f_1 \int_\Gamma \frac{(q_1-q_1^*)^2}{h_1+p}dx - \lambda_2 f_2 \int_\Gamma \frac{(q_2-q_2^*)^2}{h_2+p}dx - \delta_1 p^* \int_\Gamma \frac{|\nabla p|^2}{p^2}dx \\
&- \lambda_1\delta_2 q_1^* \int_\Gamma \frac{|\nabla q_1|^2}{q_1^2}dx - \lambda_2\delta_4 q_2^* \int_\Gamma \frac{|\nabla q_2|^2}{q_2^2}dx \\
=& -\Big[a - \frac{c_1 r_2}{h_1 f_1} - \frac{c_2 r_3}{h_2 f_2}\Big]\int_\Gamma (p-p^*)^2 dx - \lambda_1 f_1 \int_\Gamma \frac{(q_1-q_1^*)^2}{h_1+p}dx \\
&- \lambda_2 f_2 \int_\Gamma \frac{(q_2-q_2^*)^2}{h_2+p}dx - \delta_1 p^* \int_\Gamma \frac{|\nabla p|^2}{p^2}dx \\
&- \lambda_1\delta_2 q_1^* \int_\Gamma \frac{|\nabla q_1|^2}{q_1^2}dx - \lambda_2\delta_4 q_2^* \int_\Gamma \frac{|\nabla q_2|^2}{q_2^2}dx \le 0
\end{aligned}
$$

which implies the desired assertion. □

3 Estimation of positive solution of (2)

In this part, we now estimate of those solutions of (2) which are positive. We need two lemmas of which first one is developed in [27] and the second one in [28] respectively. We state the lemma on maximum principle.

Lemma 3. *Assume that* $f \in C(\bar{\Gamma} \times \mathbb{R})$.

i. *Let* $r \in C^2(\Gamma) \cap C^1(\bar{\Gamma})$ *and fulfills*

$$\begin{cases} \Delta r(x) + f(x, r(x)) \geq 0 \; in \; \Gamma, \\ \frac{\partial r}{\partial n} \leq 0 \; in \; \partial\Gamma \end{cases}$$

If $r(x_0) = max_{\bar{\Gamma}} r(x)$, *then* $f(x_0, r(x_0)) \geq 0$.

ii. *Suppose that* $r \in C^2(\Gamma) \cap C^1(\bar{\Gamma})$ *and satisfies*

$$\begin{cases} \Delta r(x) + f(x, r(x)) \leq 0 \; in \; \Gamma, \\ \frac{\partial r}{\partial n} \geq 0 \; in \; \partial\Gamma. \end{cases}$$

If $r(x_0) = min_{\bar{\Gamma}} r(x)$, *then* $f(x_0, r(x_0)) \leq 0$.

Our next lemma is designated as Harnack inequality.

Lemma 4. *Let* $r \in C^2(\Gamma) \cap C^1(\bar{\Gamma})$ *and satisfy* $\Delta r(x) + c(x) r(x) = 0$ *in* Γ, *suchthat* $c \in C(\bar{\Gamma})$, *follows the homogeneous Neumann boundary condition. So one can find* $C > 0$, *depending only on* B *satisfying* $\|c\|_\infty \leq B$ *for which*

$$max_{\bar{\Gamma}} r(x) \leq C min_{\bar{\Gamma}} r(x)$$

Now we state the result on the upper bound.

Theorem 5. *Suppose that solution* $X(x) = (p(x), q_1(x), q_2(x))^T$ *of* (2) *is positive. Then it fulfills*

$$max_{\bar{\Gamma}} p(x) \leq M_1, max_{\bar{\Gamma}} q_1(x) \leq M_2, max_{\bar{\Gamma}} q_2(x) \leq M_3 \tag{9}$$

where

$$M_1 = \frac{r_1}{a}, M_2 = \frac{r_2(a\delta_2 + \delta_3 r_1)(ah_1 + r_1)}{a^2 f_1 \delta_2}, M_3 = \frac{r_3(a\delta_4 + \delta_5 r_1)(ah_2 + r_1)}{a^2 f_2 \delta_4}.$$

Proof. Choose $x_0 \in \bar{\Gamma}$ such that $p(x_0) = \max_{\bar{\Gamma}} p(x)$. From Lemma 3, we find

$$r_1 - ap(x_0) - \frac{c_1 p(x_0)}{h_1 + p(x_0)} - \frac{c_2 q_2(x_0)}{h_2 + p(x_0)} \geq 0$$

and hence

$$p(x_0) = \max_{\bar{\Gamma}} p(x) \leq \frac{r_1}{a} \triangleq M_1.$$

Let $\psi(x) = \delta_2 q_1 + \delta_3 p q_1$ and $\psi(x_1) = \max_{\bar{\Gamma}} \psi$. Thus

$$q_1(x_1)(r_2 - \frac{f_1 q_1(x_1)}{h_1 + p(x_1)} \geq 0.$$

It follows from above that

$$q_1(x_1) \leq \frac{r_2(ah_1 + r_1}{af_1}.$$

Then we have

$$\begin{aligned} \delta_2 \max_{\bar{\Gamma}} q_1 &= \psi(x_1) = \delta_2 q_1(x_1) + \delta_3 q_1(x_1) p(x_1) \\ &\leq (\delta_2 + \delta_3 \max_{\bar{\Gamma}} p) q_1(x_1) \leq (\delta_2 + \frac{\delta_3 r_1}{a}) \frac{r_2(ah_1 + r_1)}{af_1} \end{aligned}$$

which implies that

$$\max_{\bar{\Gamma}} q_1 \leq \frac{r_2(a\delta_2+\delta_3 r_1)(ah_1+r_1)}{a^2 f_1 \delta_2} = M_1(\text{say}).$$

From the first equation of (2), we find by using Lemma 4 that

$$\max_{\bar{\Gamma}} p \leq k_1 \min_{\bar{\Gamma}} p$$

where $k_1 > 0$. The second equation of (2) can be rewritten as:

$$-\Delta\psi = \psi\left(r_2 - \frac{f_1 q_1}{h_1+p}\right)(\delta_2+\delta_3 p)^{-1} = q\psi \tag{10}$$

Now

$$\|q\|_\infty = \left\|\left(r_2 - \frac{f_1 q_1}{h_1+p}\right)(\delta_2+\delta_3 p)^{-1}\right\|_\infty \leq \frac{1}{\delta_2}\left(r_2+\frac{M_1}{r_1}\right).$$

Applying the Harnack inequality to (10), we find a positive constant k_2 such that

$$\max_{\bar{\Gamma}} \psi \leq k_2 \min_{\bar{\Gamma}} \psi.$$

From the formula of ψ, we have $q_1 = \frac{\psi}{\delta_2+\delta_3 p}$ and

$$\frac{\max_{\bar{\Gamma}} q_1}{\min_{\bar{\Gamma}} q_1} \leq \frac{\max_{\bar{\Gamma}} \psi}{\delta_2+\delta_3 \min_{\bar{\Gamma}} p} \Big/ \frac{\min_{\bar{\Gamma}} \psi}{\delta_2+\delta_3 \max_{\bar{\Gamma}} p} \leq \frac{\max_{\bar{\Gamma}} \psi}{\min_{\bar{\Gamma}} \psi}\frac{\max_{\bar{\Gamma}} p}{\min_{\bar{\Gamma}} p} \leq k_1 k_2.$$

It is deduced from the above

$$\max_{\bar{\Gamma}} q_1 \leq k_3 \min_{\bar{\Gamma}} q_1.$$

Similarly, we can show that

$$\max_{\bar{\Gamma}} q_2 \leq \frac{r_3(a\delta_4+\delta_5 r_1)(ah_2+r_1)}{a^2 f_2 \delta_4} = M_3(\text{say})$$

and

$$\max_{\bar{\Gamma}} q_2 \leq k_4 \min_{\bar{\Gamma}} q_2$$

where k_4 is a positive constant. □

We state results on lower bounds.

Theorem 6. *Let*

$$r_1 > \frac{c_1 M_1 h_2 + c_2 M_2 h_1}{h_1 h_2}.$$

For each positive solution $(p(x), q_1(x), q_2(x))$ of (2), one has

$$\min_{\bar{\Gamma}} p(x) \geq l, \min_{\bar{\Gamma}} q_1(x) \geq m, \max_{\bar{\Gamma}} q_2(x) \geq n, \tag{11}$$

where

$$l = \frac{1}{a}\left(r_1 - \frac{c_1 M_1}{h_1} - \frac{c_2 M_2}{h_2}\right), m = \frac{r_2(h_1+l)}{f_1}, n = \frac{r_3(h_2+l)}{f_2}$$

Proof. Let $x_0 \in \bar{\Gamma}$ such that $p(x_0) = \min_{\bar{\Gamma}}$. Clearly from Lemma 3, we get

$$p(x_0)\{r_1 - ap(x_0) - \frac{c_1 q_1(x_0)}{h_1 + p(x_0)} - \frac{c_2 q_2(x_0)}{h_2 + p(x_0)}\} \leq 0.$$

It follows from above that

$$ap(x_0) \geq r_1 - \frac{c_1 q_1(x_0)}{h_1 + p(x_0)} - \frac{c_2 q_2(x_0)}{h_2 + p(x_0)} \geq r_1 - \frac{c_1 M_1}{h_1} - \frac{c_2 M_2}{h_2}$$

which implies that

$$p(x_0) \geq \frac{1}{a}(r_1 - \frac{c_1 M_1}{h_1} - \frac{c_2 M_2}{h_2}) = l.$$

Let $x_1 \in \bar{\Gamma}$ such that $q_1(x_1) = \min_{\bar{\Gamma}} q_1$. Then by Lemma 3, it is evident that

$$q_1(x_1)(r_2 - \frac{f_1 q_1(x_1)}{h_1 + p(x_1)}) \leq 0.$$

It follows from above that

$$q_1(x_1) \geq \frac{r_2(h_1 + p(x_1))}{f_1} \geq \frac{r_2(h_1 + l)}{f_1} = m.$$

Let $x_2 \in \bar{\Gamma}$ such that $q_2(x_2) = \min_{\bar{\Gamma}} q_2$. Then by Lemma 3, we get

$$q_2(x_2)(r_3 - \frac{f_2 q_2(x_2)}{h_2 + p(x_2)}) \leq 0.$$

It follows from above that

$$q_2(x_2) \geq \frac{r_3(h_2 + p(x_2))}{f_1} \geq \frac{r_2(h_2 + l)}{f_2} = n.$$

□

4 Emergence of non-constant positive solution

In this part, we will show that the cross-diffusion can ensure the existence of non-constant positive solution to (2). Let

$$X = (p, q_1, q_2)^T,$$
$$X_0 = (p^*, q_1^*, q_2^*),$$
$$\Phi(X) = (\delta_1 p, \delta_2 q_1 + \delta_3 p q_1, \delta_4 q_2 + \delta_5 p q_2)^T,$$

then system (2) becomes $-\Delta\Phi(X) = M(X)$, where $M(X)$ is already defined in Theorem 1. Clearly, the solution X of (2) is positive whenever

$$P(X) = X - (I - \Delta)^{-1}\{\Phi_X^{-1}(X)[M(X) + \nabla X \Phi_{XX}(X) \nabla X^T] + X\} = 0. \tag{12}$$

Using a result in [29], and the similar reasoning as in [30], we realize that to simplify our calculation of index $(I - P, X_0)$, we require to find out the sign of $N(\theta)$, where $N(\theta)$ is defined by

$$N(\theta) = \det\{\Phi_X^{-1}(X_0)\}\det\{\theta\Phi_X(X_0) - M_X(X_0)\}. \tag{13}$$

After some calculation, we can show that $\det\{\Phi_X^{-1}(X_0)\} > 0$ and

$$\det\{\theta\Phi_X(X_0) - M_X(X_0)\} = Q_3(\delta_3)\theta^3 + Q_2(\delta_3)\theta^2 + Q_1(\delta_3)\theta + Q_0(\delta_3) = Q(\delta_3;\theta)$$

where

$$\begin{aligned}
Q_3(\delta_3) =& \delta_1(\delta_2+\delta_3 p^*)(\delta_4+\delta_5 p^*),\\
Q_2(\delta_3) =& -a_{11}(\delta_2+\delta_3 p^*)(\delta_4+\delta_5 p^*) + [\delta_1\{(\delta_2+\delta_3 p^*)\frac{f_2 q_2^*}{h_2+p^*}\\
&+(\delta_4+\delta_5 p^*)\frac{f_1 q_1^*}{h_1+p^*}\} - \frac{c_1\delta_3 p^* q_1^*(\delta_4+\delta_5 p^*)}{h_1+p^*} - \frac{c_2\delta_5 p^* q_2^*(\delta_2+\delta_3 p^*)}{h_2+p^*}],\\
Q_1(\delta_3) =& \frac{\delta_1 f_1 f_2 q_1^* q_2^*}{(h_1+p^*)(h_2+p^*)} - a_{11}\{(\delta_2+\delta_3 p^*)\frac{f_2 q_2^*}{h_2+p^*} + (\delta_4+\delta_5 p^*)\frac{f_1 q_1^*}{h_1+p^*}\}\\
&- \frac{c_1 p^*}{h_1+p^*}\{\frac{\delta_3 f_2 q_1^* q_2^*}{h_2+p^*} - \frac{f_1 q_1^* 2(\delta_4+\delta_5 p^*)}{(h_1+p^*)^2}\}\\
&- \frac{c_2 p^*}{h_2+p^*}\{\frac{\delta_5 f_1 q_1^* q_2^*}{h_1+p^*} - \frac{f_2 q_2^* 2(\delta_2+\delta_3 p^*)}{(h_2+p^*)^2}\},\\
Q_0(\delta_3) =& -\frac{f_1 f_2 q_1^* q_2^*}{(h_1+p^*)(h_2+p^*)}\{a_{11} - \frac{c_1 p^* q_1^*}{(h_1+p^*)^2} - \frac{c_2 p^* q_2^*}{(h_2+p^*)^2}\}.
\end{aligned}$$

Let the three roots of $Q(\delta_3;\theta) = 0$ be $\bar\theta_1, \bar\theta_2$ and $\bar\theta_3$ respectively such that

$$\mathrm{Re}(\bar\theta_1) \le \mathrm{Re}(\bar\theta_2) \le \mathrm{Re}(\bar\theta_3).$$

Note that $Q_0 > 0$ when

$$a_{11} < \frac{c_1 p^* q_1^*}{(h_1+p^*)^2} + \frac{c_2 p^* q_2^*}{(h_2+p^*)^2}$$

and $Q_3 > 0$. Then $\bar\theta_1\bar\theta_2\bar\theta_3 = -\frac{Q_0}{Q_3} < 0$. Evidently, one of the three roots $\bar\theta_1, \bar\theta_2$ and $\bar\theta_3$ is real and negative while the product of the other two is positive. Note the limits given below:

$$\begin{aligned}
\lim_{\delta_3\to\infty}\frac{Q_0(\delta_3)}{\delta_3} &= 0,\\
\lim_{\delta_3\to\infty}\frac{Q_1(\delta_3)}{\delta_3} &= -\frac{f_2 p^* q_2^*}{h_2+p^*}\{a_{11} + \frac{c_1 q_1^*}{h_1+p^*} - \frac{c_2 q_2^* p^*}{(h_2+p^*)^2}\} \triangleq b_1,\\
\lim_{\delta_3\to\infty}\frac{Q_2(\delta_3)}{\delta_3} &= -p^*(\delta_4+\delta_5 p^*)(a_{11} + \frac{c_1 q_1^*}{h_1+p^*}) + \frac{p^* q_2^*}{h_2+p^*}(\delta_1 f_2 - c_2\delta_5 p^*) \triangleq b_2,\\
\lim_{\delta_3\to\infty}\frac{Q_3(\delta_3)}{\delta_3} &= \delta_1 p^*(\delta_4+\delta_5 p^*) \triangleq b_3.
\end{aligned}$$

It is easy to see that $b_3 > 0$. Note that

$$\lim_{\delta_3\to\infty}\frac{Q(\delta_3;\theta)}{\delta_3} = b_3\theta^3 + b_2\theta^2 + b_1\theta = \theta(b_3\theta^2 + b_2\theta + b_1).$$

The equation $b_3\theta^2 + b_2\theta + b_1 = 0$ may possess two strictly roots whenever:

$$b_1 > 0, b_2 < 0, b_2^2 - 4b_1 b_3 > 0. \tag{14}$$

For sufficiently large value of δ_3 , $\bar{\theta}_1 < 0$ and $\bar{\theta}_2$ and $\bar{\theta}_3 > 0$ as $\bar{\mu}_2\bar{\mu}_3 > 0$. Again, we have

$$\lim_{\delta_3\to\infty} \bar{\theta}_1 = 0,$$
$$\lim_{\delta_3\to\infty} \bar{\theta}_2 = \frac{-b_2 - \sqrt{b_2{}^2 - 4b_1b_3}}{2b_3} \triangleq \theta_2^* > 0,$$
$$\lim_{\delta_3\to\infty} \bar{\theta}_3 = \frac{-b_2 + \sqrt{b_2{}^2 - 4b_1b_3}}{2b_3} \triangleq \theta_3^* > 0. \quad (15)$$

We can find a $\delta_3^* > 0$ as long as $\delta_3 > \delta_3^*$, such that: $-\infty < \bar{\theta}_1 < 0 < \bar{\theta}_2 < \bar{\theta}_3$

$$\begin{cases} Q(\delta_3;\theta) < 0 \text{ if } \theta \in (-\infty, \bar{\theta}_1) \cup (\bar{\theta}_2, \bar{\theta}_3) \\ Q(\delta_3;\theta) > 0 \text{ if } \theta \in (\bar{\theta}_1, \bar{\theta}_2) \cup (\bar{\theta}_3, \infty). \end{cases}$$

Since $\bar{\theta}_2^* \in (\theta_i, \theta_{i+1})$ and $\bar{\theta}_3^* \in (\theta_j, \theta_{j+1})$ for some $j > i > 0$, $Q(\delta_3;\theta_k)$ is negative for $i+1 < k < j+1$. Thus from the previous analysis, we state the theorem below.

Theorem 7. *Suppose $\delta_1, \delta_2, \delta_4$ and δ_5 be fixed and satisfy the assumption of Theorem 6 and (14) and let θ_2^* and θ_3^* be defined in (15). If $\bar{\theta}_2^* \in (\theta_i, \theta_{i+1})$ and $\bar{\theta}_3^* \in (\theta_j, \theta_{j+1})$ for some $j > i \geq 1$, and the sum $\Sigma_{n=i+1}^{j} m(\theta_n)$ is odd, then one can find a $\delta_3^* > 0$ for which if $\delta_3 \geq \delta_3^*$, (2) possesses at least one non-constant positive solution*

Proof. We establish the result by the method of contradiction. Suppose that for some $\delta_3 = \bar{\delta}_3 \geq \delta_3^*$, system (2) does not possess non-constant solutions. Next, take $\delta_3 = \bar{\delta}_3 \geq \delta_3^*$, for $t \in [0,1]$, define

$$\Phi(t;X) = (\delta_1 p, \delta_2 q_1 + t\delta_3 p q_1, \delta_4 q_2 + t\delta_5 p q_2)^T.$$

Then (2) becomes

$$-\Delta\Phi(t;X) = M(X), x \in \Gamma$$
$$\frac{\partial X}{\partial n} = 0,\ x \in \partial\Gamma. \quad (16)$$

Clearly, for $t \in [0,1]$, there exists only one positive solution X_0 (16) and X is a positive solution of system (2) whenever X is a positive solution of system (16) for $t = 1.X$ is a positive solution of system (16) whenever

$$P(X) = U - (I-\Delta)^{-1}\{\Phi_X^{-1}(X)[M(X) + \nabla X \Phi_{XX}(X)\nabla X^T] + X\} = 0.$$

By a priori estimates, it can be shown that the region

$$E = \{(p, q_1, q_2) : C < p, q_1, q_2 < D\}$$

contains all the positive solutions of system (16), where

$$C = \max\{l, m, n\}, D = \max\{M_1, M_2, M_3\}$$

and $P(t;X)$ is non-zero on ∂E. So $\deg(P(t;X), E, 0)$ is clear. Using topology degree theory, one can get

$$\deg(P(1,\cdot), E, 0) = \deg(P(0,\cdot), E, 0). \quad (17)$$

Note that

$$N(t;\theta) = \det\{\Phi_X^{-1}(t;X_0)\}\det\{\theta\Phi_X(t;X_0) - M_X(X_0)\} \quad (18)$$

and when $t = 0, N(0;\theta) > 0$ by the assumption (14). By means of the result
$\text{index}(I - P(t,\cdot), X_0) = (-1)^{\sigma}$ where $\sigma = \Sigma_{n\geq 1, N(\theta_n)<0} m(\theta_n)$. We get

$$\text{index}(I - P(0,\cdot), X_0) = (-1)^0 = 1. \tag{19}$$

By the assumption, that system (2) does not possess any non-constant positive solution and $\Sigma_{n=i+1}^{j} m(\theta_n)$, is odd, then we obtain $\text{index}(I - P(1,\cdot), X_0) = (-1)^{\Sigma_{n=i+1}^{j} m(\theta_n)} = 1$. Moreover, $P(1;0) = 0$ and $P(0;X) = 0$ possess a unique positive solution X_0 on E. Thus

$$\deg(P(0,\cdot), E, 0) = \text{index}(I - P(0,\cdot), X_0) = 1, \tag{20}$$
$$\deg(P(1,\cdot), E, 0) = \text{index}(I - P(1,\cdot), X_0) = -1. \tag{21}$$

Thus (20), (21) contradict with (17). This completes the proof. □

5 Non-constant solution through bifurcation

Now, with the help of bifurcation theory it is possible to show the existence of non-constant positive solutions of (2). Keeping the parameters $r_1, r_2, r_3, c_1, c_2, h_1, h_2, f_1, f_2, \delta_1, \delta_2, \delta_4$ fixed and taking δ_3 as a key parameter, we develop the result.

Definition 8. (δ_3^*, X_0) is regarded as a bifurcation point of (2) given any $\delta \in (0, \delta_3^*)$, one can find a $\delta_3 \in [\delta_3^* - \delta, \delta_3^* + \delta]$ such that system (2) has a non-constant positive solution. Otherwise, (δ_3^*, X_0) is a regular point.

Define $Q = \{\theta > 0 \,|h(\theta) = 0\}$ and $S_s = \{\theta_2, \theta_3, \theta_4, \cdots\}$, where $N(\theta)$ is defined by (17). To highlight the dependence of $N(\theta), P(\theta)$ and Q on δ_3, we designate these by $N(\delta_3;\mu), P(\delta_3;\theta)$ and $Q(\delta_3;X)$ respectively, where $P(X)$ is defined by (12). The proof of the result below is similar to a result in [30] and the proof is not given here.

Theorem 9. *Let* $\delta_3^* > 0$.

- **i.** *If* $S_s \cup Q(\delta_3^*) = \phi$, *then* (δ_3^*, X_0) *is a regular point of system* (2).
- **ii.** *Suppose* $S_s \cup Q(\delta_3^*) \neq \phi$, *and the positive roots of* $N(\delta_3^*;\theta) = 0$ *are all simple. If* $\Sigma_{\theta_j \in Q(\delta_3^*)} m(\theta_j)$ *is odd, then* (δ_3^*, X_0) *is a bifurcation point of system* (2).

6 Discussion

The current article has investigated a diffusive modified LG model of predator-prey interaction consisting of three species with Holling type II schemes. We consider the spatial inhomogeneity in the environment. The diffusive system follows homogeneous Neumann boundary states.

From Theorem 1, we note that if the intra-specific competition coefficient of the prey species exceeds a certain threshold value then the constant positive steady state is locally asymptotically stable. This result indicates that, if there is no cross-diffusion then the diffusion driven instability cannot occur. If the condition of the Theorem 1 is reversed, local stability can be achieved under the influence of self-diffusion of prey species. We have also derived the condition for global stability of the constant positive steady state by forming an appropriate Lyapunov function. By applying the maximum principle and Harnack inequality, it is possible to find the priori estimate to the positive solutions which are reflected in Theorems 5 and 6. We have shown that under certain constraints, cross-diffusion can induce stationary patterns. This fact can be considered as a modification of Turing pattern. Lastly, we have identified the parameter δ_3 to obtain bifurcation of the system. We can find a similar phenomenon by considering the parameter δ_5 also. In [31], the authors studied the same type of model with a single predator, while multiple predators are not investigated yet for obtaining coexistence results.

7 Declarations

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