EJAM
www.ejamjournal.com

Electronic Journal of Applied Mathematics

Research article

Mersin University Publishing
DOI: 10.61383/ejam.20231347
2023, Vol. 1, No. 3, 26–37

The traveling wave solutions to a variant of the Boussinesq equation

Muhammad Nadeem[1,†], Loredana Florentina Iambor[2]

1. School of Mathematics and Statistics, Qujing Normal University, Qujing 655011, China
2. Department of Mathematics and Computer Science, University of Oradea, 1 University Street, 410087 Oradea, Romania

[†]Corresponding author

Abstract

In this study, we study a variant of the Boussinesq equation called as $B(n+1, 1, n)$ equation, and construct some traveling wave solutions by using an effective approach called the extended trial equation method. Thus, the soliton solutions, rational function solutions, elliptic function solutions and Jacobi elliptic function solutions, which show the existence of various mathematical and physical structures and events in the fundamental equation considered, have been constructed. In order to make a more detailed examination of the physical behavior of these solutions, two- and three-dimensional graphs of some solution functions were drawn with the help of the Mathematica package program. In the section of Discussion, we suggest a more general version of the trial equation method for nonlinear differential equations.

Key words: the extended trial equation method, $B(n+1,1,n)$ equations, soliton solution, elliptic solutions
2020 Mathematics Subject Classification: 35C07, 35C08, 35Q51
Article history: Received 03 Sep 2023; Accepted 22 Dec 2023; Online 24 Dec 2023

1 Introduction

Partial differential equations (PDEs) are mathematical equations that describe how quantities such as temperature, pressure, or velocity vary in space and time. They are widely used in various fields, including physics, engineering, biology, and finance. While many PDEs can be solved numerically using computer algorithms, finding exact solutions to PDEs is often challenging and requires sophisticated mathematical techniques. Exact solutions to PDEs are useful as they provide insights into the behavior of the system being described by the equation. They can reveal symmetries, conservation laws, and other fundamental properties of the system. Moreover, exact solutions can serve as benchmarks for numerical algorithms, helping

Contact: Muhammad Nadeem ✉ nadeem@mail.qjnu.edu.cn; Loredana Florentina Iambor ✉ iambor.loredana@gmail.com

to validate their accuracy and efficiency. Finding exact solutions to PDEs involves solving the equation analytically, i.e., finding a mathematical expression that satisfies the equation. This is typically done by assuming a certain functional form for the solution and then determining the parameters or functions that make the equation hold. The choice of the functional form depends on the specific PDE and the physical problem being described. There are several techniques for constructing exact solutions to PDEs. Some of the commonly used methods include separation of variables, the method of characteristics, similarity transformations, and integral transforms. Each method has its own set of assumptions and limitations, and the choice of method depends on the particular PDE and its boundary or initial conditions. Exact solutions to PDEs can take various forms, including simple analytical expressions, series expansions, or special functions such as Bessel functions or hypergeometric functions. These solutions can be classified into different types, such as traveling wave solutions, soliton solutions, or periodic solutions, depending on their behavior. In addition to finding exact solutions to PDEs, it is also important to analyze their stability and dynamical properties. Perturbation analysis, linear stability analysis, and bifurcation theory are some of the tools used to study the behavior of solutions to PDEs. Despite the challenges involved, constructing exact solutions to PDEs is a crucial endeavor in nonlinear science. It provides theoretical understanding and insight into complex phenomena, and it has practical applications in various fields. The development of new techniques and methods for solving PDEs is an active area of research, aiming to uncover the secrets hidden within these fundamental equations. These problems can be analyzed by the numerous methods such as Sardar-subequation method [1, 2], the extended F-expansion method [3], the modifed simple equation method [4], Jacobi elliptic function expansion method [5], the generalized unified method [6, 7], the generalized exponential rational function method [8–10], the linear superposition principle [11, 12]. These methods, that give the exact solutions, are used to solve nonlinear problems where an analytical solution is not readily available. Also, these methods provide valuable tools for solving nonlinear problems and can be adapted to different types of equations. This approach has been successful in providing a systematic way to identify and classify the different types of traveling wave solutions that can exist for a given nonlinear evolution equation. The complete discrimination system for a polynomial involves the use of a set of discriminant constraints that are derived from the polynomial equation describing the nonlinear evolution equation. These discriminant constraints are used to determine the different types of solitary wave solutions, such as solitons, breathers, and kink waves, that can exist for the equation [13, 14]. Liu has applied this approach to a variety of nonlinear PDEs. By using the complete discrimination system, Liu has been able to systematically identify and classify the different types of traveling wave solutions for these equations. Using the trial equation method proposed by Liu, several authors have been able to analyze complex physical problems and derive valuable results in wave theory [15–20]. On the other hand, the extended trial equation method, is introduced by Gurefe et al. [21–23], is based on the concept of solitons and integrable systems. It involves constructing trial equations for the solutions of the nonlinear PDEs by incorporating soliton solutions and elliptic integral functions. The trial equation method also allows for the construction of solutions involving elliptic integral functions and Jacobi elliptic functions. These special functions arise in the study of mathematical physics and have a wide range of applications. Generally, the trial equation method provides a systematic approach for obtaining solutions to nonlinear PDEs with generalized evolution. It allows for the construction of a variety of solution types, including solitons, singular solitons, elliptic integral functions, and Jacobi elliptic functions. In the case of the variant of the Boussinesq equation with generalized evolution, it can be started by considering the original Boussinesq equation, which is a PDE that describes the propagation of long waves in shallow water [24]. To apply the extended trial equation method to this variant, it can be assumed a trial equation that satisfies the general form of the Boussinesq equation, including the generalized evolution terms. The $B\left(n+1,1,n\right)$ equation can be shown in the form of

$$u_{tt}-a\left(u^{n+1}\right)_{xx}-b(u(u^{n})_{xx})_{xx}=0, \tag{1}$$

where a and b are real valued constants. The purpose of this paper is to obtain the classification of the wave solutions to Eq. (1), we employ the extended trial equation method. This method allows us to find the general form of the solutions and categorize them based on their characteristics. One of the new solutions we discover is the singular soliton, which is characterized by a localized wave profile with strong amplitude modulation. Another set of new solutions we derive are the elliptic integral functions F, E, and pi, and Jacobi elliptic function solutions. These solutions are expressed in terms of special functions that arise in the theory of elliptic integrals and elliptic functions. The newly discovered solutions, such as the singular soliton and the elliptic integral and Jacobi elliptic function solutions, open up new possibilities in the study and utilization of wave phenomena in various scientific and technological disciplines.

2 The extended trial equation method

Step 1. The general form of a PDE in two variables, x and t, and a dependent variable, $u\left(x,t\right)$, can be written as:

$$P\left(u,u_{t},u_{x},u_{xx},u_{xxx},\cdots\right)=0, \tag{2}$$

and under the general wave transformation

$$u\left(x_{1},x_{2},\cdots,x_{N},t\right)=u\left(\eta\right),\quad \eta=\lambda\left(\sum_{j=1}^{N}x_{j}-ct\right), \tag{3}$$

where $\lambda\neq0$ and $c\neq0$. By substituting Eq. (3) into Eq. (2), a nonlinear differential equation can be found as follows:

$$N\left(u,u',u'',u''',\cdots\right)=0. \tag{4}$$

Step 2. The solution function is as follows:

$$u=\sum_{i=0}^{\delta}\tau_{i}\,\Gamma^{i}, \tag{5}$$

where the trial equation is

$$\left(\Gamma'\right)^{2}=\Lambda\left(\Gamma\right)=\frac{\Phi\left(\Gamma\right)}{\Psi\left(\Gamma\right)}=\frac{\xi_{\theta}\Gamma^{\theta}+\cdots+\xi_{1}\Gamma+\xi_{0}}{\zeta_{\varepsilon}\Gamma^{\varepsilon}+\cdots+\zeta_{1}\Gamma+\zeta_{0}}. \tag{6}$$

From the relations (5) and (6), we get

$$\left(u'\right)^{2}=\frac{\Phi\left(\Gamma\right)}{\Psi\left(\Gamma\right)}\left(\sum_{i=0}^{\delta}i\,\tau_{i}\Gamma^{i-1}\right)^{2}, \tag{7}$$

$$u''=\frac{\Phi'\left(\Gamma\right)\Psi\left(\Gamma\right)-\Phi\left(\Gamma\right)\Psi'\left(\Gamma\right)}{2\Psi^{2}\left(\Gamma\right)}\left(\sum_{i=0}^{\delta}i\,\tau_{i}\Gamma^{i-1}\right)+\frac{\Phi\left(\Gamma\right)}{\Psi\left(\Gamma\right)}\left(\sum_{i=0}^{\delta}i\,\left(i-1\right)\tau_{i}\Gamma^{i-2}\right), \tag{8}$$

where $\Phi\left(\Gamma\right)$ and $\Psi\left(\Gamma\right)$ are different polynomials of Γ. Substituting above two relations into Eq. (4) produces an equation of polynomial $\Omega\left(\Gamma\right)$ of Γ:

$$\Omega\left(\Gamma\right)=\rho_{s}\Gamma^{s}+\cdots+\rho_{1}\Gamma+\rho_{0}=0. \tag{9}$$

A very important relation between the values of θ, ε and δ is obtained by applying the balance principle. The values θ, ε and δ required for the application of the method can be chosen by this relation.

Step 3. By setting all coefficients of $\Omega(\Gamma)$ equal to zero, a system of nonlinear algebraic equations is obtained

$$\rho_i = 0, \qquad i = 0, \cdots, s. \tag{10}$$

Solving the system (10), the values of $\xi_0, \cdots, \xi_\theta; \zeta_0, \cdots, \zeta_\varepsilon$ and $\tau_0, \cdots, \tau_\delta$ can be determined.

Step 4. Since Eq. (6) is a separable differential equation, it can be reduced to a simple form as

$$\pm(\mu - \mu_0) = \int \frac{d\Gamma}{\sqrt{\Lambda(\Gamma)}} = \sqrt{\frac{G(\Gamma)}{F(\Gamma)}} d\Gamma. \tag{11}$$

Eq. (11) can be solved with the help of the Mathematica package program by using a complete discrimination system of a polynomial used to find the roots of the algebraic equation, and thus the exact solutions of Eq. (4) can be achieved. The solutions obtained here enable the solutions of Eq. (2) to be easily reached.

3 Application to a variant of the Boussinesq equation

In Section 2, we proposed a method called the extended trial equation method to solve a specific equation. Now, in this section, we will apply that method to $B(n+1, 1, n)$ equation we are currently working with. To construct the traveling wave solutions of Eq. (1), we apply the traveling wave transformation $u(x,t) = u(\eta)$, $\eta = x - ct$ where is an arbitrary constant. Then, integrating this equation with respect to η twice and equating the integration constant to zero, we can easily write

$$c^2 u - a u^{n+1} - b\left(u(u^n)''\right) = 0. \tag{12}$$

We substitute the following transformation into Eq. (12)

$$u = v^{-\frac{1}{n}}. \tag{13}$$

Eq. (12) turns into the following equation

$$c^2 v^3 - a v^2 - 2b(v')^2 - b v v'' = 0. \tag{14}$$

According to the balance procedure, substituting Eqs. (7) and (8) into Eq. (14) yields the balance relation as $\theta = \varepsilon + \delta + 2$. By applying the solution procedures explained in detail in Section 2, the solutions as follows are obtained:

Case 1. If we get $\varepsilon = 0$, $\delta = 1$ and $\theta = 3$, then

$$(v')^2 = \frac{\tau_1^2 \left(\xi_0 + \xi_1 \Gamma + \xi_2 \Gamma^2 + \xi_3 \Gamma^3\right)}{\zeta_0},$$

$$v'' = \frac{\tau_1(\xi_1 + 2\xi_2\Gamma + 3\xi_3\Gamma^2)}{2\zeta_0}, \tag{15}$$

where $\xi_3 \neq 0$, $\zeta_0 \neq 0$. The system of the nonlinear algebraic equations, which was revealed with the help of the above equations and the coefficients of the polynomial Γ, is coded in the

Mathematica package program and the parametric solutions of this system are obtained as follows:

$$\xi_0 = -\frac{11\xi_1^2}{16\xi_2},\ \xi_1 = \xi_1,\ \xi_2 = \xi_2,\ \xi_3 = \frac{24\xi_2^2}{121\xi_1},\ \tau_0 = \tau_0,\ \tau_1 = \frac{4\xi_2\tau_0}{11\xi_1},\ \zeta_0 = \frac{21b\xi_2}{11a},\ c = \sqrt{\frac{a}{\tau_0}}. \quad (16)$$

When the coefficients determined as seen in Eq. (16) are written into Eqs. (6) and (11), respectively, an integral form as follows is obtained

$$\pm(\eta - \eta_0) = \sqrt{\frac{231b\xi_1}{24a\xi_2}} \int \frac{1}{\sqrt{\Gamma^3 + \frac{121\xi_1}{24\xi_2}\Gamma^2 + \frac{121\xi_1^2}{24\xi_2^2}\Gamma - \frac{1131\xi_1^3}{384\xi_2^3}}} d\Gamma. \quad (17)$$

The various solutions for Eq. (1) have been constructed by integrating Eq. (17):

$$\pm(\eta - \eta_0) = -\frac{2A}{\sqrt{\Gamma - \alpha_1}}, \quad (18)$$

$$\pm(\eta - \eta_0) = \frac{2A}{\sqrt{\alpha_2 - \alpha_1}} \arctan\left(\sqrt{\frac{\Gamma - \alpha_2}{\alpha_2 - \alpha_1}}\right), \quad \alpha_2 > \alpha_1, \quad (19)$$

$$\pm(\eta - \eta_0) = \frac{A}{\sqrt{\alpha_1 - \alpha_2}} \ln\left|\frac{\sqrt{\Gamma - \alpha_2} - \sqrt{\alpha_1 - \alpha_2}}{\sqrt{\Gamma - \alpha_2} + \sqrt{\alpha_1 - \alpha_2}}\right|, \quad \alpha_1 > \alpha_2, \quad (20)$$

$$\pm(\eta - \eta_0) = \frac{2A}{\sqrt{\alpha_1 - \alpha_3}} F(\varphi, l), \quad \alpha_1 > \alpha_2 > \alpha_3, \quad (21)$$

where

$$A = \sqrt{\frac{231b\xi_1}{24a\xi_2}}, \quad F(\varphi, l) = \int_0^{\varphi} \frac{1}{\sqrt{1 - l^2 \sin^2\phi}} d\phi, \quad (22)$$

and

$$\varphi = \arcsin\left(\sqrt{\frac{\Gamma - \alpha_3}{\alpha_2 - \alpha_3}}\right), \quad l^2 = \frac{\alpha_2 - \alpha_3}{\alpha_1 - \alpha_3}. \quad (23)$$

Here, the values α_1, α_2 and α_3 are the roots of an algebraic equation arising from a 3rd degree polynomial

$$\Gamma^3 + \frac{\xi_2}{\xi_3}\Gamma^2 + \frac{\xi_1}{\xi_3}\Gamma + \frac{\xi_0}{\xi_3} = 0. \quad (24)$$

When Eqs. (18)-(21) were substituted in Eqs. (5) and (13), respectively, the rational, hyperbolic and Jacobi elliptic function solutions are obtained:

$$u_{1,1}(x,t) = \left[\tau_0 + \tau_1\alpha_1 + \frac{4\tau_1 A^2}{\left(x - \sqrt{\frac{a}{\tau_0}}t - \eta_0\right)^2}\right]^{-\frac{1}{n}}, \quad (25)$$

$$u_{1,2}(x,t) = \left\{\tau_0 + \tau_1\alpha_2 + \tau_1(\alpha_1 - \alpha_2)\tanh^2\left[\frac{\sqrt{\alpha_1 - \alpha_2}}{2A}\left(x - \sqrt{\frac{a}{\tau_0}}t - \eta_0\right)\right]\right\}^{-\frac{1}{n}}, \quad (26)$$

$$u_{1,3}(x,t) = \left\{\tau_0 + \tau_1\alpha_1 + \tau_1(\alpha_1 - \alpha_2)\,\mathrm{csc\,h}^2\left[\frac{\sqrt{\alpha_1 - \alpha_2}}{2A}\left(x - \sqrt{\frac{a}{\tau_0}}t - \eta_0\right)\right]\right\}^{-\frac{1}{n}}, \quad (27)$$

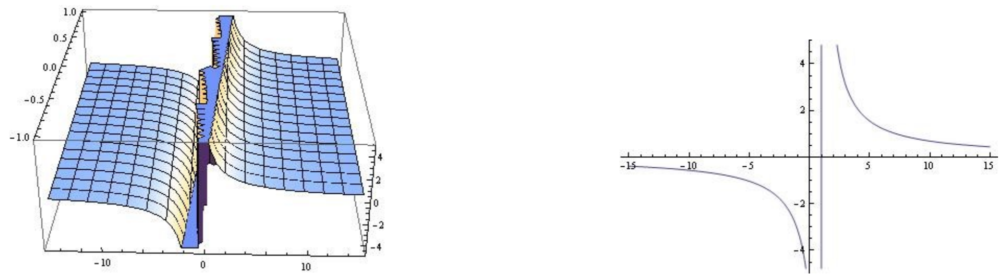

Figure 1: The solution (29) is shown at the above two or three dimensional graphs for the values $\tau_0 = a = b = -1, \quad \tau_1 = \alpha_1 = \xi_1 = \xi_2 = 1, \quad n = -2, \quad t = 1.$

$$u_{1,4}(x,t) = \left\{ \tau_0 + \tau_1\alpha_3 + \tau_1(\alpha_2 - \alpha_3)\,\mathrm{sn}^2\left[\pm\frac{\sqrt{\alpha_1 - \alpha_3}}{2A}\left(x - \sqrt{\frac{a}{\tau_0}}t - \eta_0 \right), \frac{\alpha_2 - \alpha_3}{\alpha_1 - \alpha_3} \right] \right\}^{-\frac{1}{n}}. \quad (28)$$

If we specially choose $\tau_0 = -\tau_1\alpha_1$ and $\eta_0 = 0$, then the solutions given in Eqs. (25)-(28) can be easily transformed to the rational, 1-soliton, singular soliton solutions respectively,

$$u_{1,1}(x,t) = \left(\frac{\widetilde{A}}{x - \kappa t} \right)^{-\frac{2}{n}}, \quad (29)$$

$$u_{1,2}(x,t) = \frac{A_1}{\cosh^{-\frac{2}{n}}\left[B(x - \kappa t) \right]}, \quad (30)$$

$$u_{1,3}(x,t) = \frac{A_2}{\sinh^{-\frac{2}{n}}\left[B(x - \kappa t) \right]}, \quad (31)$$

where

$$\widetilde{A} = 2A\sqrt{\tau_1},\ A_1 = (\tau_1(\alpha_2 - \alpha_1))^{-\frac{1}{n}},\ A_2 = (\tau_1(\alpha_1 - \alpha_2))^{-\frac{1}{n}},\ B = \frac{\sqrt{\alpha_1 - \alpha_2}}{2A},\ \kappa = \sqrt{-\frac{a}{\tau_1\alpha_1}}.$$

While κ and B define the velocity and inverse amplitude of the above solitons, respectively, A_1 and A_2 indicate the amplitudes of the same solitons. From here, it can be easily seen that the solitons exist for $\tau_1 > 0$. Also, for $\tau_0 = -\tau_1\alpha_3$ and $\eta_0 = 0$, the solution in Eq. (28) can be reduced to the following Jacobi elliptic function form

$$u_{1,4}(x,t) = A_3\,\mathrm{sn}^{-\frac{2}{n}}\left[\pm\widetilde{B}(x - \kappa t), \frac{\alpha_2 - \alpha_3}{\alpha_1 - \alpha_3} \right], \quad (32)$$

where $A_3 = (\tau_1(\alpha_2 - \alpha_3))^{-\frac{1}{n}}$, $\widetilde{B} = \frac{\sqrt{\alpha_1 - \alpha_3}}{2A}$.

Remark 3.1 This means that if we assign the same values to certain parameters in solutions (30) and (31), they will match exactly with solutions (26) and (27) mentioned in Ref. [24].

Remark 3.2 In this paper, we have used the extended trial equation method to obtain solutions (29)-(32) for Eq. (1). We have verified these solutions using Mathematica. To the best of our knowledge, the rational function solution, the singular soliton solution, and the Jacobi elliptic function solutions that we have found in this paper have not been previously shown in the literature. These solutions represent new traveling wave solutions for Eq. (1).

Case 2. If we take $\varepsilon = 0$, $\delta = 2$ and $\theta = 4$, then

$$(v')^2 = \frac{(\tau_1 + 2\tau_2\Gamma)^2\left(\zeta_0 + \zeta_1\Gamma + \zeta_2\Gamma^2 + \zeta_3\Gamma^3 + \zeta_4\Gamma^4 \right)}{\zeta_0},$$

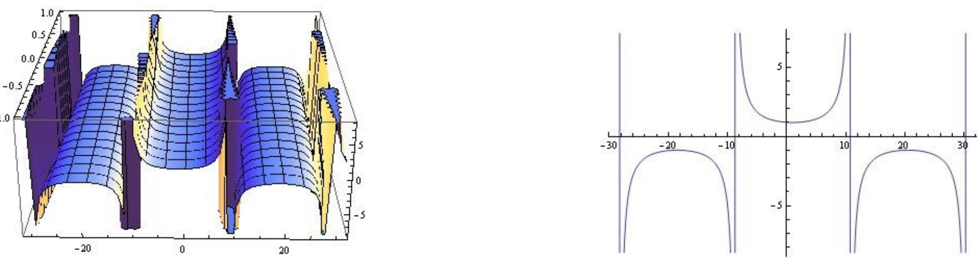

Figure 2: The solution (30) is shown at the above two or three dimensional graphs for the values $\tau_0 = a = b = -1, \quad \tau_1 = \alpha_1 = \xi_1 = \xi_2 = 1, \quad \alpha_2 = 2, \quad n = -2, \quad t = 1.$

Figure 3: The solution (32) is shown at the above two or three dimensional graphs for the values $\tau_0 = a = b = -1, \quad \tau_1 = \alpha_3 = \xi_1 = \xi_2 = 1, \quad \alpha_1 = 3, \quad \alpha_2 = 2, \quad n = -2, \quad t = 1.$

$$v'' = \frac{(\tau_1 + 2\tau_2\Gamma)\left(\xi_1 + 2\xi_2\Gamma + 3\xi_3\Gamma^2 + 4\xi_4\Gamma^3\right) + 4\tau_2\left(\xi_0 + \xi_1\Gamma + \xi_2\Gamma^2 + \xi_3\Gamma^3 + \xi_4\Gamma^4\right)}{2\zeta_0}, \tag{33}$$

where $\xi_4 \neq 0, \zeta_0 \neq 0$. The system of the nonlinear algebraic equations, which is determined with the help of Eq. (33) and the coefficients of the polynomial Γ, is created the calculation codes in the Mathematica package program and the parametric solutions of this system are found as follows:

$$\tau_0 = \tau_0,\ \tau_1 = \tau_1,\ \tau_2 = \tau_2,\ \xi_0 = \frac{a\zeta_0\tau_0^2}{3b\left(4\tau_0\tau_2 - \tau_1^2\right)},\ \xi_1 = \frac{2a\zeta_0\tau_0\tau_1}{3b\left(4\tau_0\tau_2 - \tau_1^2\right)},\ \xi_2 = \frac{a\zeta_0\left(\tau_1^2 + 2\tau_0\tau_2\right)}{3b\left(4\tau_0\tau_2 - \tau_1^2\right)},$$

$$\xi_3 = -\frac{2a\zeta_0\tau_1\tau_2}{3b\left(\tau_1^2 - 4\tau_0\tau_2\right)}\ \xi_4 = -\frac{a\zeta_0\tau_2^2}{3b\left(\tau_1^2 - 4\tau_0\tau_2\right)},\ \zeta_0 = \zeta_0,\ c = \sqrt{\frac{14a\tau_2}{3\left(4\tau_0\tau_2 - \tau_1^2\right)}}. \tag{34}$$

When the coefficients written in Eq. (34) are substituted into Eqs. (6) and (11), respectively, the following integral form is obtained

$$\pm(\eta - \eta_0) = \sqrt{\frac{3b\left(4\tau_0\tau_2 - \tau_1^2\right)}{a\tau_2^2}} \int \frac{1}{\sqrt{\Gamma^4 + \frac{\xi_3}{\xi_4}\Gamma^3 + \frac{\xi_2}{\xi_4}\Gamma^2 + \frac{\xi_1}{\xi_4}\Gamma + \frac{\xi_0}{\xi_4}}} d\Gamma. \tag{35}$$

The following wave solutions to Eq. (1) are computed by integrating Eq. (35):

$$\pm(\eta - \eta_0) = -\frac{A}{\Gamma - \alpha_1}, \tag{36}$$

$$\pm(\eta - \eta_0) = \frac{2A}{\alpha_1 - \alpha_2}\sqrt{\frac{\Gamma - \alpha_2}{\Gamma - \alpha_1}}, \tag{37}$$

$$\pm(\eta - \eta_0) = \frac{A}{\alpha_1 - \alpha_2}\ln\left|\frac{\Gamma - \alpha_1}{\Gamma - \alpha_2}\right|, \tag{38}$$

$$\pm(\eta-\eta_0)=\frac{2A}{\sqrt{(\alpha_1-\alpha_2)(\alpha_1-\alpha_3)}}\ln\left|\frac{\sqrt{(\Gamma-\alpha_2)(\alpha_1-\alpha_3)}-\sqrt{(\Gamma-\alpha_3)(\alpha_1-\alpha_2)}}{\sqrt{(\Gamma-\alpha_2)(\alpha_1-\alpha_3)}+\sqrt{(\Gamma-\alpha_3)(\alpha_1-\alpha_2)}}\right|, \tag{39}$$

$$\pm(\eta-\eta_0)=\frac{2A}{\sqrt{(\alpha_1-\alpha_3)(\alpha_2-\alpha_4)}}F(\varphi,l), \tag{40}$$

where

$$A=\sqrt{\frac{3b\left(4\tau_0\tau_2-\tau_1^2\right)}{a\tau_2^2}},\ \varphi=\arcsin\left(\sqrt{\frac{(\Gamma-\alpha_1)(\alpha_2-\alpha_4)}{(\Gamma-\alpha_2)(\alpha_1-\alpha_4)}}\right),\ l^2=\frac{(\alpha_2-\alpha_3)(\alpha_1-\alpha_4)}{(\alpha_1-\alpha_3)(\alpha_2-\alpha_4)}. \tag{41}$$

Here, the values α_1, α_2, α_3 and α_4 are called as the roots of an algebraic equation arising from the following 4rd degree polynomial

$$\Gamma^4+\frac{\xi_3}{\xi_4}\Gamma^3+\frac{\xi_2}{\xi_4}\Gamma^2+\frac{\xi_1}{\xi_4}\Gamma+\frac{\xi_0}{\xi_4}=0. \tag{42}$$

When Eqs. (36)-(40) were substituted in Eqs. (5) and (13), respectively, the rational, exponential, hyperbolic and Jacobi elliptic function solutions are determined as follows, and for simplicity, if we choose , then we can get the following solutions:

$$u_{2,1}(x,t)=\left[\sum_{i=0}^{2}\tau_i\left(\alpha_1\mp\frac{A}{x-ct}\right)^i\right]^{-\frac{1}{n}}, \tag{43}$$

$$u_{2,2}(x,t)=\left[\sum_{i=0}^{2}\tau_i\left(\alpha_1-\frac{4A^2(\alpha_1-\alpha_2)}{4A^2-[(\alpha_1-\alpha_2)(x-ct)]^2}\right)^i\right]^{-\frac{1}{n}}, \tag{44}$$

$$u_{2,3}(x,t)=\left[\sum_{i=0}^{2}\tau_i\left(\alpha_2+\frac{\alpha_2-\alpha_1}{-1+\exp[B_3(x-ct)]}\right)^i\right]^{-\frac{1}{n}}, \tag{45}$$

$$u_{2,4}(x,t)=\left[\sum_{i=0}^{2}\tau_i\left(\alpha_1+\frac{\alpha_1-\alpha_2}{-1+\exp[B_3(x-ct)]}\right)^i\right]^{-\frac{1}{n}}, \tag{46}$$

$$u_{2,5}(x,t)=\left[\sum_{i=0}^{2}\tau_i\left(\alpha_1-\frac{2(\alpha_1-\alpha_2)(\alpha_1-\alpha_3)}{2\alpha_1-\alpha_2-\alpha_3+(\alpha_3-\alpha_2)\cosh[C(x-ct)]}\right)^i\right]^{-\frac{1}{n}}, \tag{47}$$

$$u_{2,6}(x,t)=\left[\sum_{i=0}^{2}\tau_i\left(\alpha_2+\frac{(\alpha_1-\alpha_2)(\alpha_4-\alpha_2)}{\alpha_4-\alpha_2+(\alpha_1-\alpha_4)\operatorname{sn}^2(\chi,l)}\right)^i\right]^{-\frac{1}{n}}, \tag{48}$$

where

$$B_3=\frac{\alpha_1-\alpha_2}{A},\ C=\frac{\sqrt{(\alpha_1-\alpha_2)(\alpha_1-\alpha_3)}}{2A},$$
$$\chi=\frac{\sqrt{(\alpha_1-\alpha_3)(\alpha_2-\alpha_4)}}{2A}(x-ct),\ l^2=\frac{(\alpha_2-\alpha_3)(\alpha_1-\alpha_4)}{(\alpha_1-\alpha_3)(\alpha_2-\alpha_4)}. \tag{49}$$

Remark 3.3 In this paper, we have used the extended trial equation method to obtain solutions (43)-(48) for Eq. (1). We have verified these solutions using Mathematica. To the best of our knowledge, the rational function solution, the singular soliton solution, and the Jacobi elliptic function solutions that we have found in this paper have not been previously shown in the literature. These solutions represent new traveling wave solutions for Eq. (1).

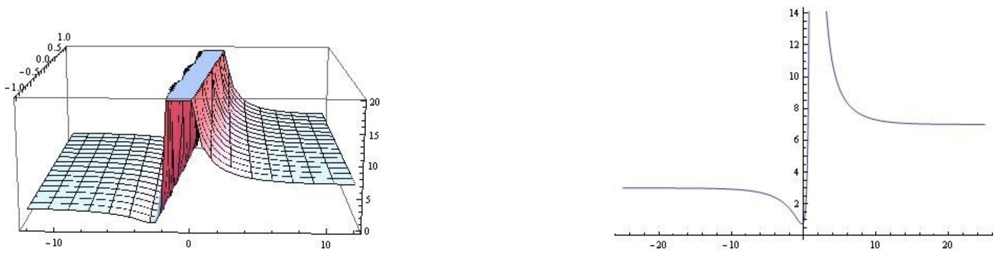

Figure 4: The solution (46) is shown at the above two or three dimensional graphs for the values $\tau_0 = \tau_1 = \tau_2 = \alpha_1 = a = b = 1, \quad \alpha_2 = 2, \quad n = -1, \quad t = 1.$

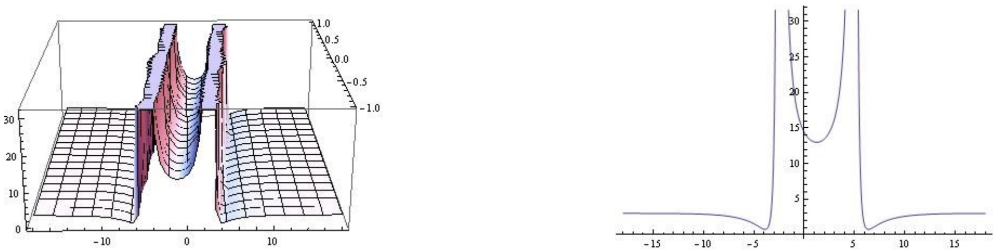

Figure 5: The solution (47) is shown at the above two or three dimensional graphs for the values $\tau_0 = \tau_1 = \tau_2 = \alpha_1 = a = b = 1, \quad \alpha_2 = 2, \quad \alpha_3 = 3, \quad n = -1, \quad t = 1.$

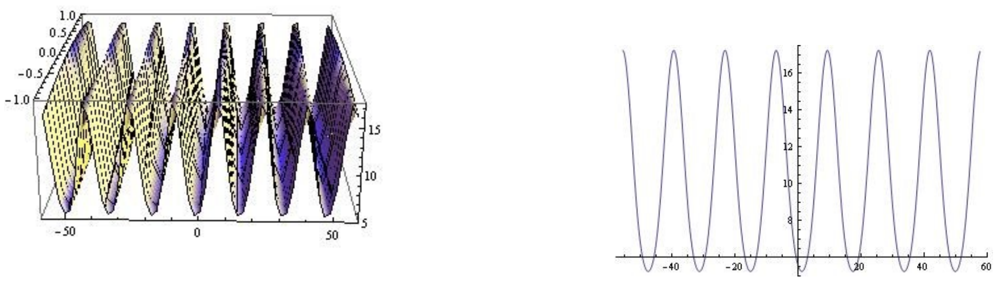

Figure 6: The solution (48) is shown at the above two or three dimensional graphs for the values $\tau_0 = \tau_2 = \alpha_1 = a = b = 1, \quad \alpha_2 = \frac{1}{2}, \quad \alpha_3 = 3, \quad \tau_1 = \alpha_4 = 2, \quad n = -1, \quad t = 1.$

4 Discussion

The solution function in the extended trial equation method was taken in rational form and a similar method was tried to be created again, and thus, a new approach was proposed, which is thought to be effective in researching the exact solutions of NPDEs and is expected to provide new solution functions that are not included in the literature, and is defined in the following steps.

Step 1. Assume that the trial function defined in Eq. (5) has the following more general form:

$$u = \frac{A(\Gamma)}{B(\Gamma)} = \frac{\sum\limits_{i=0}^{\delta} \tau_i \Gamma^i}{\sum\limits_{j=0}^{\mu} \omega_j \Gamma^j}, \tag{50}$$

where

$$(\Gamma')^2 = \Lambda(\Gamma) = \frac{\Phi(\Gamma)}{\Psi(\Gamma)} = \frac{\xi_\theta \Gamma^\theta + \cdots + \xi_1 \Gamma + \xi_0}{\zeta_\varepsilon \Gamma^\varepsilon + \cdots + \zeta_1 \Gamma + \zeta_0}. \tag{51}$$

Here, $\tau_i \ (i = 0, \cdots, \delta)$, $\omega_j \ (j = 0, \cdots, \mu)$, $\xi_\varrho \ (\varrho = 0, \cdots, \theta)$ and $\zeta_\sigma \ (\sigma = 0, \cdots, \varepsilon)$ are the constants to be determined.

Step 2. Taking trial equations (49) and (50), we derive the following equations:

$$(u')^2 = \frac{\Phi(\Gamma)\left(A'(\Gamma) B(\Gamma) - A(\Gamma) B'(\Gamma)\right)^2}{\Psi(\Gamma) B^4(\Gamma)}, \tag{52}$$

and the terms containing higher order derivatives such as u''', and so on.

Step 3. Substituting u', u'' and the higher order derivative terms into Eq. (4) we can get

$$\Omega(\Gamma) = \rho_s \Gamma^s + \cdots + \rho_1 \Gamma + \rho_0 = 0. \tag{53}$$

An important relation between the values of δ, μ, θ and ε is obtained by applying the balance principle to this method. The values δ, μ, θ and ε required for the application of the method can be determined by this balance relation.

Step 4. Letting the coefficients of $\Omega(\Gamma)$ all be zero yield a system of algebraic equations $\rho_i = 0 \ (i = 0, \cdots, s)$. After we solve this system with the help of Mathematica, we can determine the values $\tau_i \ (i = 0, \cdots, \delta)$, $\omega_j \ (j = 0, \cdots, \mu)$, $\xi_\varrho \ (\varrho = 0, \cdots, \theta)$ and $\zeta_\sigma \ (\sigma = 0, \cdots, \varepsilon)$.

Step 5. If we substitute the values computed in Step 4 into Eq. (50), and then integrate Eq. (50), we can construct the new wave solutions of Eq. (2).

5 Conclusions and Remarks

Our study focused on a variant of the Boussinesq equation, and our goal was to construct the traveling wave solutions. To achieve this, we applied the extended trial equation method. The Boussinesq equation, which is a well-known nonlinear partial differential equation, is widely used to describe various wave phenomena in different fields of science. However, the variant we consider in this paper introduces modifications to the original Boussinesq equation, rendering traditional methods insufficient for finding exact solutions. To overcome this challenge, we utilized the extended trial equation method, which had been proven to be effective in finding exact solutions for various nonlinear partial differential equations. This method involves proposing an extended trial equation, which is a solution ansatz that encompasses the original equation. Using this extended trial equation, we derived the corresponding traveling wave solutions by substituting it into the variant of the Boussinesq equation. Some of them are respectively hyperbolic, rational, elliptic and Jacobi elliptic functions. This process also led to a system of algebraic equations that can be solved to obtain the explicit forms of the traveling wave solutions. In order to solve this system, Mathematica software helped us. Through this approach, we were able to obtain a family of solutions that describe the wave behavior in the variant Boussinesq equation. Our results provide valuable insights into the wave dynamics of the variant Boussinesq equation and contribute to the understanding of nonlinear wave phenomena. The extended trial equation method proves to be a powerful tool for tackling complex nonlinear partial differential equations and discovering exact solutions. Further studies can be conducted to investigate the properties and applications of these solutions in different scientific disciplines.

6 Declarations

Funding

This research received no external funding.

Competing Interests

The authors declare that they have no competing interests.

Ethical Approval

Not applicable.

Authors's Contributions

All authors contributed equally. All the authors read and approved the final manuscript.

Availability Data and Materials

Not applicable.

Acknowledgements

The authors would like to thank the anonymous reviewer for his/her comments.

References

[1] Khalida Faisal, Souleymanou Abbagari, Arash Pashrashid, Alphonse Houwe, Shao-Wen Yao, and Hijaz Ahmad, *Pure-cubic optical solitons to the Schrödinger equation with three forms of nonlinearities by Sardar subequation method*, Results Phys. **48** (2023), 106412.

[2] Kang-jia Wang, and Jing-Hua Liu, *On abundant wave structures of the unsteady korteweg-de vries equation arising in shallow water*, J. Ocean Eng. Sci. **8** (2023), 595-601.

[3] Mohammed Borg, Niveen M. Badra, Hamdy M. Ahmed, and Wafaa B. Rabie, *Derivation of optical solitons in magneto-optical waveguides to coupled BMEs having Kudryashov's nonlinearity*, Opt. Quantum Electron. **55** (2023), 672.

[4] Jesmin Akter, and M. Ali Akbar, *Exact solutions to the Benney-Luke equation and the Phi-4 equations by using modified simple equation method*, Results Phys. **5** (2015), 125-130.

[5] V. Senthil Kumar, Hadi Rezazadeh, Mostafa Eslami, Franoosh Izadi, and M. S Osman, *Jacobi elliptic function expansion method for solving KdV equation with conformable derivative and dual-power law nonlinearity*, Int. J. Appl. Comput. Math. **5** (2019), 127.

[6] M.S. Osman, *Multiwave solutions of time-fractional (2+1)-dimensional Nizhnik-Novikov-Veselov equations*, Pramana-J. Phys. **88** (2017). 67.

[7] Tugba Aydemir, *Application of the generalized unified method to solve (2+1)-dimensional Kundu-Mukherjee-Naskar equation*, Opt. Quantum Electron. **55** (2023), 534.

[8] Behzad Ghanbari, and J.F. Gomez-Aguilar, *The generalized exponential rational function method for Radhakrishnan-Kundu-Lakshmanan equation with β-conformable time derivative*, Rev. Mex. Fis. **65** (2019), no. 5, 503-518.

[9] B. Gunay, Chun-Ku Kuo, and Wen-Xiu Ma, *An application of the exponential rational function method to exact solutions to the Drinfeld-Sokolov system*, Results Phys. **29** (2021), 104733.

[10] Melike Kaplan, Rubayyi T. Alqahtani, and Nadiyah Hussain Alharthi, *Wave propagation and stability analysis for Ostrovsky and symmetric regularized long-wave equations*, Mathematics **11** (2023), no. 19, 4030.

[11] Dipankar Kumar, Chun-Ku Kuo, Gour Chandra Paul, Jui Saha, and Israt Jahan, *Wave propagation of resonance multi-stripes, complexitons, and lump and its variety interaction solutions to the (2+1)-dimensional pKP equation*, Commun. Nonlinear Sci. Numer. Simul. **100** (2021), 105853.

[12] Chun-Ku Kuo, *Novel resonant multi-soliton solutions and inelastic interactions to the (3+1)-and (4+1)-dimensional Boiti-Leon-Manna-Pempinelli equations via the simplified linear superposition principle*, Eur. Phys. J. Plus. **136** (2021), no. 1, 1-11.

[13] Cheng-shi Liu, *A new trial equation method and its applications*, Commun. Theor. Phys. **45** (2006), 395-397.

[14] Cheng-shi Liu, *Applications of complete discrimination system for polynomial for classifications of traveling wave solutions to nonlinear differential equations*, Comput. Phys. Commun. **181** (2010), 317-324.

[15] Cheng Yan-jun, *Classification of traveling wave solutions to the Vakhnenko equations*, Comput. Math. Appl. **62** (2011), 3987-3996.

[16] Cheng Yan-jun, *Classification of traveling wave solutions to the modified form of the Degasperis-Procesi equation*, Math. Comput. Model. **56** (2012), 43-48.

[17] Usman Afzal, Nauman Raza, and Isma Ghulam Murtaza, *On soliton solutions of time fractional form of Sawada-Kotera equation*, Nonlinear Dyn. **95** (2019), 391-405.

[18] Damin Cao, and Lijuan Du, *The classification of the single traveling wave solutions to (1+1)-dimensional Gardner equation with variable coefficients*, Adv. Differ. Equ. **2019** (2019), 121.

[19] Yang Liu, and Xin Wang, *The construction of solutions to Zakharov-Kuznetsov equation with fractional power nonlinear terms*, Adv. Differ. Equ. **2019** (2019), 134.

[20] Tao Liu, *Exact solutions to time-fractional fifth order KdV equation by trial equation method based on symmetry*, Symmetry **11** (2019), 742.

[21] Yusuf Pandir, Yusuf Gurefe, Ugur Kadak, Emine Misirli, *Classifications of exact solutions for some nonlinear partial differential equations with generalized evolution*, Abstr. Appl. Anal. **2012** (2012), 478531.

[22] Yusuf Gurefe, Emine Misirli, Abdullah Sonmezoglu, Mehmet Ekici, *Extended trial equation method to generalized nonlinear partial differential equations*, Appl. Math. Comput. **219** (2013), 5253-5260.

[23] Yusuf Pandir, Yusuf Gurefe, Emine Misirli, *Classification of exact solutions to the generalized Kadomtsev-Petviashvili equation*, Phys. Scr. **87** (2013), 1-12.

[24] Abdul-Majid Wazwaz, *The sine-cosine and the tanh methods: Reliable tools for analytic treatment of nonlinear dispersive equations*, Appl. Math. Comput. **173** (2006), 150-164.