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Existence of Common Coupled Fixed Points of Generalized Contractive Mappings in Ordered Multiplicative Metric Spaces

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Abstract

In order to generalize coupled fixed point results in the setup of partially ordered multiplicative metric spaces, we employing the concept of w^*- compatible mappings and generalized contractive condition and prove some coupled coincidence point and common coupled fixed points results. We also provide illustrative examples in support of our new results. Moreover, some applications to integral equations are presented. Our established results generalize, extend and unify various results in the existing literature.

Key words: Coupled coincidence point, common coupled fixed point, w^*- compatible map, partially ordered set, multiplicative metric space
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1 Introduction and preliminaries

Fixed point theory in metric spaces has attracted considerable attention due to numerous applications in areas such as variational and linear inequalities, optimization, and approximation theory. In 2004, Ran and Reurings [1] investigated the existence of fixed points in partially ordered metric spaces and then by Nieto and Lopez [2]. Further interesting results in this direction were proved (see [3–8]). Results on weak contractive mappings in such spaces, together with applications to differential equations, were obtained by Harjani and Sadarangani in [9].

Guo and Lakshmikantham in [10] initiated the study of coupled fixed points in partially ordered metric spaces and then attracted many researchers, see for example [11–14] and references therein. Bhaskar and Lakshmikantham [15] introduced the notions of mixed monotone mapping and obtained some coupled fixed point results. As an application, they studied the existence and uniqueness of a solution for a periodic boundary value problem associated with

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a first order ordinary differential equation. Lakshmikantham and Ćirić in [16] introduced the concepts of coupled coincidence and coupled common fixed point for nonlinear mappings in partially ordered complete metric spaces and generalized the concept of the mixed monotone property. Choudhury and Kundu [17] obtained coupled coincidence point results in partially ordered metric spaces for compatible mappings. Abbas et al. [18] proved coupled coincidence and common coupled fixed point results in cone metric spaces for $w-$ compatible mappings. Recently, Hussain et al. [19] established some coupled coincidence point results for a generalized compatible pair of mappings.

Banach contraction principle has been generalized either by extending the domain of the mapping or by considering a more general contractive condition on the mappings. Ozavsar and Cevikel [20] proved an analogous of Banach contraction principle in the framework of multiplicative metric spaces. They also studied some topological properties of the relevant multiplicative metric space. Bashirov et al. [21] studied the concept of multiplicative calculus and proved a fundamental theorem of multiplicative calculus. They also illustrated the usefulness of multiplicative calculus with some interesting applications. Multiplicative calculus provides natural and straightforward way to compute the derivative of product and quotient of two functions ([22]). It was shown that the multiplicative differential equations are more suitable than the ordinary differential equations in investigating some problems in economics and finance. Due to its operational simplicity and support to Newtonian calculus, it has attracted the attention of several researchers in the recent years. Furthermore, based on the definition of multiplicative absolute value function, they defined the multiplicative distance between two nonnegative real numbers and between two positive square matrices. This provided the basis for multiplicative metric spaces. Florack and Assen [23] gave applications of multiplicative calculus in biomedical image analysis. Hxiaoju et al. [24] studied common fixed points for weak commutative mappings on a multiplicative metric space (see also, [25]). Recently, Yamaod and Sintunavarat [26] obtained some fixed point results for generalized contraction mappings with cyclic (α, β)-admissible mapping in multiplicative metric spaces.
In this paper, unique common coupled fixed point results for w^*- compatible maps, which are more general than commuting and $w-$ compatible mappings, are obtained in partially ordered multiplicative metric spaces, without exploiting the notion of continuity. The results presented in this paper carry various comparable results in the existing literature (e.g. [11, 13, 15]).

By $\mathbb{R}$, $\mathbb{R}^+$, $\mathbb{R}^+_n$ and $\mathbb{N}$, we denote the set of all real numbers, the set of all nonnegative real numbers, the set of all n-tuples of positive real numbers and the set of all natural numbers, respectively.

Consistent with [21] and [20], the following definitions and results will be needed in the sequel.

Definition 1.1. [21] Let X be a nonempty set. A multiplicative metric is a mapping $d : X \times X \to \mathbb{R}^+$ satisfying the following conditions:

(i) $d(u,v) \geq 1$ for all $u, v \in X$ and $d(u,v) = 1$ iff $u = v$;
(ii) $d(u,v) = d(v,u)$ for all $u, v \in X$;
(iii) $d(u,v) \leq d(u,w) \cdot d(w,v)$ for all $u, v, w \in X$.
The pair (X, d) is called a multiplicative metric space.

Definition 1.2. [20] The multiplicative absolute value function $|\cdot| : \mathbb{R}^+ \to \mathbb{R}^+$ is defined as

$$|x| = \begin{cases} x, & \text{if } x \geq 1; \\ \dfrac{1}{x}, & \text{if } x < 1. \end{cases}$$

Using the definition of multiplicative absolute value function, we can prove the following proposition.

Proposition 1.3. *For arbitrary $x, y \in \mathbb{R}^+$, the multiplicative absolute value function $|\cdot| : \mathbb{R}^+ \to \mathbb{R}^+$ satisfies the following:*

(1) $|x| \geq 1$.
(2) $\frac{1}{|x|} \leq x \leq |x|$.
(3) $|\frac{1}{x}| = |x|$.
(4) $|x| \leq y$ *if and only if* $\frac{1}{y} \leq x \leq y$.
(5) $|x \cdot y| \leq |x||y|$.

Example 1.1. [20] Let $X = \mathbb{R}_n^+$ be the collection of all n-tuples of positive real numbers. Then $d(x, y) = |\frac{x_1}{y_1}| \cdot |\frac{x_2}{y_2}| \cdot \ldots \cdot |\frac{x_n}{y_n}|$ defines a multiplicative metric on X.

Definition 1.4. [20] Let multiplicative metric space (X, d), x_0 be an arbitrary point in X and $\varepsilon > 1$. A multiplicative open ball $B(x_0, \varepsilon)$ of radius ε centered at x_0 is the set $\{z \in X : d(z, x_0) < \varepsilon\}$.

A sequence $\{x_n\}$ in multiplicative metric space (X, d) is said to be multiplicative convergent to a point $x \in X$ if for any given $\varepsilon > 1$, there is $N \in \mathbb{N}$ such that $x_n \in B(x, \varepsilon)$ for all $n \geq N$. If $\{x_n\}$ converges to x, we write $x_n \to x$ as $n \to \infty$.

Definition 1.5. [20] Let (X, d) be a multiplicative metric space. A sequence $\{x_n\}$ in X is multiplicative convergent to x in X if and only if $d(x_n, x) \to 1$ as $n \to \infty$.

Definition 1.6. Let (X, d_X) and (Y, d_Y) be two multiplicative metric spaces and $f : X \to Y$ be a map. Let x_0 be arbitrary but fixed element of X. Then the map f is said to be multiplicative continuous at x_0 if and only if $x_n \to x_0$ in (X, d_X) implies $f(x_n) \to f(x_0)$ in (Y, d_Y) for every multiplicative convergent sequence $\{x_n\}$ in X. That is, given arbitrary $\varepsilon > 1$, there exists $\delta > 1$ which depend on x_0 and ε such that $d_Y(fx, fx_0) < \varepsilon$ whenever $d_X(x, x_0) < \delta$ for all $x \in X$.

Definition 1.7. [20] Let (X, d) be a multiplicative metric space.

(i) A sequence $\{x_n\}$ in X is said to be multiplicative Cauchy sequence if for any $\varepsilon > 1$, there exists $N \in \mathbb{N}$ such that $d(x_n, x_m) < \varepsilon$ for all $m, n \geq N$.
(ii) A multiplicative metric space (X, d) is said to be complete if every Cauchy sequence $\{x_n\}$ in X is multiplicative convergent to a point $x \in X$.

Definition 1.8. [20] Let (X, d) be a multiplicative metric space. A sequence $\{x_n\}$ in X is multiplicative Cauchy if and only if $d(x_n, x_m) \to 1$ as $n, m \to \infty$.

Example 1.2. Let $X = C^*[a, b]$ be the collection of all real-valued multiplicative continuous functions over $[a, b] \subseteq \mathbb{R}^+$ with the multiplicative metric d defined by

$$d(f, g) = \sup_{x \in [a,b]} \left|\frac{f(x)}{g(x)}\right| \text{ for arbitrary } f,\ g \in X$$

and $|.| : \mathbb{R}^+ \to \mathbb{R}^+$ is a multiplicative absolute valued function defined in Definition 1.2. Then $(C^*[a, b], d)$ is complete.

Recall that if $(X, \preceq)$ is a partially ordered set and $f : X \to X$ is such that for $x, y \in X$, $x \preceq y$ implies $f(x) \preceq f(y)$, then the mapping f is said to be nondecreasing. Similarly, a nonincreasing mapping is defined.

Definition 1.9. [16] Let $(X, \preceq)$ be a partially ordered set. A mapping $F : X \times X \to X$ is said to have a mixed monotone property with respect to $g : X \to X$, if for any $x, y \in X$,

$$x_1, x_2 \in X,\ gx_1 \preceq gx_2 \text{ implies } F(x_1, y) \preceq F(x_2, y),$$

and

$$y_1, y_2 \in X,\ gy_1 \preceq gy_2 \text{ implies } F(x, y_2) \preceq F(x, y_1).$$

If we take $g = I_X$ (an identity mapping on X), then F is said to has the mixed monotone property [15].

Definition 1.10. [15] An element $(x, y) \in X \times X$ is called a coupled fixed point of mapping $F : X \times X \to X$ if $x = F(x, y)$ and $y = F(y, x)$.

Definition 1.11. [18] An element $(x, y) \in X \times X$ is called:

(c_1) a coupled coincidence point of mappings $F : X \times X \to X$ and $g : X \to X$ if $g(x) = F(x, y)$ and $g(y) = F(y, x)$, and the pair (gx, gy) is called coupled point of coincidence.
(c_2) a common coupled fixed point of mappings $F : X \times X \to X$ and $g : X \to X$ if $x = g(x) = F(x, y)$ and $y = g(y) = F(y, x)$.

We denote the set of coupled coincidence points of F and g with

$$CC(F, g) = \{(x, y) : g(x) = F(x, y) \text{ and } g(y) = F(y, x)\}.$$

Definition 1.12. [18] Mappings $F : X \times X \to X$ and $g : X \to X$ are called:

(w_1) $w-$ compatible if $g(F(x, y)) = F(gx, gy)$ whenever $g(x) = F(x, y)$ and $g(y) = F(y, x)$;
(w_2) w^*- compatible if $g(F(x, x)) = F(gx, gx)$ whenever $g(x) = F(x, x)$.

Note that, every $w-$ compatible pair of mapping F and g is also w^*- compatible. We present an example in which of mappings are w^*- compatible but not $w-$ compatible.

Example 1.3. Let $X = \mathbb{R}^+$ and $F : X \times X \to X$, $g : X \to X$ be define as

$$F(x, y) = \begin{cases} 4 & \text{if } (x, y) = (1, 2), \\ 6 & \text{if } (x, y) = (2, 1), \\ 8 & \quad \text{otherwise}, \end{cases}$$

and

$$g(x) = \begin{cases} 4 & \text{if } x = 1, \\ 6 & \text{if } x = 2, \\ 8 & \text{if } x = 8, \\ 10 & \text{otherwise}. \end{cases}$$

Then F and g are not $w-$ compatible because $g(F(1, 2)) = g(4) = 10 \neq 8 = F(4, 6) = F(g(1), g(2))$ whereas $g(1) = 4 = F(1, 2)$ and $g(2) = 6 = F(2, 1)$. However, F and g are w^*- compatible maps, since $F(x, x) = 8 = g(x)$ implies that $g(F(x, x)) = 8 = F(gx, gx)$.

Definition 1.13. Let X be a nonempty set. Then $(X, d, \preceq)$ is called partially ordered multiplicative metric space if and only if d is a multiplicative metric on a partially ordered set $(X, \preceq)$.

Let g be a self map on X. We define $\Delta_g, \Delta \subseteq X \times X$ as follows:

$$\Delta_g = \{(x, y, u, v) \in X^4 : gx \preceq gu \text{ and } gy \succeq gv\}$$

and

$$\Delta = \{(x, y, u, v) \in X^4 : x \preceq u \text{ and } y \succeq v\}.$$

Note that, if $(x, y, u, v) \in \Delta_g$ or Δ, then $(v, u, y, x) \in \Delta_g$ or Δ and vice versa.

2 Main results

Now, we start with the following result of common coupled fixed point.

Theorem 2.1. *Let $(X, d, \preceq)$ be a partially ordered multiplicative metric space. Suppose that a mapping $F : X \times X \to X$ has a mixed monotone property with respect to $g : X \to X$ and*

$$d(F(x,y), F(u,v)) \leq \left(M_g(x,y,u,v)\right)^{\lambda}, \tag{1}$$

where

$$\begin{aligned} M_g(x,y,u,v) \;=\; & \max\{d(gx,gu) \cdot d(gy,gv), d(F(x,y),gx) \cdot d(F(x,y),gu), \\ & d(gy,gv) \cdot d(F(x,y),gx), d(gy,gv) \cdot d(F(x,y),gu)\} \end{aligned}$$

for all $(x,y,u,v) \in \Delta_g$, where $\lambda \in [0, \frac{1}{2})$. If $F(X \times X)$ is contained in a complete set $g(X)$ and X has the property that for any two sequences $\{x_n\}$, $\{y_n\}$ with $(x_{n+1}, y_{n+1}, x_n, y_n) \in \Delta$ such that $x_n \to x$, $y_n \to y$ as $n \to \infty$ implies that $(x_n, y, x, y_n) \in \Delta$. Then $CC(F, g)$ is nonempty provided that there exist $x_0, y_0 \in X$ such that $(g(x_0), g(y_0), F(x_0, y_0), F(y_0, x_0)) \in \Delta$. If $CC(F, g) \subseteq \Delta_g$, then F and g have a unique coupled coincidence point in X. Moreover, if $gy_0 \preceq gx_0$ with F and g are w^*- compatible, then F and g have a common coupled fixed point.

Proof. Let $x_0, y_0 \in X$ be such that $g(x_0) \preceq F(x_0, y_0)$ and $g(y_0) \succeq F(y_0, x_0)$. Set $gx_1 = F(x_0, y_0)$ and $gy_1 = F(y_0, x_0)$, this can be done as $F(X \times X) \subseteq g(X)$. Similarly, $g(x_2) = F(x_1, y_1)$ and $g(y_2) = F(y_1, x_1)$. Continuing this process we can construct sequences $\{x_n\}$ and $\{y_n\}$ in X such that

$$g(x_{n+1}) = F(x_n, y_n) \text{ and } g(y_{n+1}) = F(y_n, x_n) \text{ for all } n \geq 0. \tag{2}$$

We shall show that $g(x_n) \preceq g(x_{n+1})$ and $g(y_n) \succeq g(y_{n+1})$ for all $n \geq 0$.
By induction, let $n = 0$. Since $gx_0 \preceq F(x_0, y_0)$ and $gy_0 \succeq F(y_0, x_0)$ also $gx_1 = F(x_0, y_0)$ and $gy_1 = F(y_0, x_0)$, so that $gx_0 \preceq gx_1$ and $gy_0 \succeq gy_1$. Now, let it holds for some fixed $n \geq 0$. Since $gx_n \preceq gx_{n+1}$ and $gy_n \succeq gy_{n+1}$, and as F has the mixed monotone property with respect to g, so that $gx_{n+1} = F(x_n, y_n) \preceq F(x_{n+1}, y_n)$ and $gy_{n+1} = F(y_n, x_n) \succeq F(y_{n+1}, x_n)$. Also $gx_{n+2} = F(x_{n+1}, y_{n+1}) \succeq F(x_{n+1}, y_n)$ and $F(y_{n+1}, x_n) \succeq F(y_{n+1}, x_{n+1}) = gy_{n+2}$. Hence $gx_{n+1} \preceq gx_{n+2}$ and $gy_{n+1} \succeq gy_{n+2}$. Thus by the mathematical induction we conclude that for all $n \geq 0$,

$$\begin{aligned} gx_0 &\preceq gx_1 \preceq ... \preceq gx_n \preceq gx_{n+1} \preceq ..., \text{ and} \\ gy_0 &\succeq gy_1 \succeq ... \succeq gy_n \succeq gy_{n+1} \succeq .... \end{aligned}$$

We will suppose that $d(gx_n, gx_{n+1}) > 1$ and $d(gy_n, gy_{n+1}) > 1$ for all n, since if $d(gx_n, gx_{n+1}) = 1$ and $d(gy_n, gy_{n+1}) = 1$ for some n, then $gx_n = gx_{n+1}$ and $gy_n = gy_{n+1}$. And from (2), we have

$$gx_n = F(x_n, y_n) \text{ and } gy_n = F(y_n, x_n),$$

that is, F and g have a coupled coincidence point (x_n, y_n), and so we have finished the proof. Now from (1), we have

$$\begin{aligned} & d(gx_n, gx_{n+1}) = d(F(x_{n-1}, y_{n-1}), F(x_n, y_n)) \\ \leq\; & (\max\{d(gx_{n-1}, gx_n) \cdot d(gy_{n-1}, gy_n), \\ & d(F(x_{n-1}, y_{n-1}), gx_{n-1}) \cdot d(F(x_{n-1}, y_{n-1}), gx_n), \\ & d(gy_{n-1}, gy_n) \cdot d(F(x_{n-1}, y_{n-1}), gx_{n-1}), d(gy_{n-1}, gy_n) \cdot d(F(x_{n-1}, y_{n-1}), gx_n)\})^{\lambda} \\ =\; & (\max\{d(gx_{n-1}, gx_n) \cdot d(gy_{n-1}, gy_n), d(gx_n, gx_{n-1}) \cdot d(gx_n, gx_n), \\ & d(gy_{n-1}, gy_n) \cdot d(gx_n, gx_{n-1}), d(gy_{n-1}, gy_n) \cdot d(gx_n, gx_n)\})^{\lambda} \\ =\; & (\max\{d(gx_{n-1}, gx_n) \cdot d(gy_{n-1}, gy_n), d(gx_n, gx_{n-1}), d(gy_{n-1}, gy_n)\})^{\lambda}, \end{aligned}$$

and hence

$$d(gx_n, gx_{n+1}) \leq (d(gx_{n-1}, gx_n) \cdot d(gy_{n-1}, gy_n))^{\lambda}. \quad (3)$$

Similarly,

$$d(gy_n, gy_{n+1}) \leq (d(gy_{n-1}, gy_n) \cdot d(gx_{n-1}, gx_n))^{\lambda}. \quad (4)$$

From (3) and (4), we obtain

$$\begin{aligned} d(gx_n, gx_{n+1}) \cdot d(gy_n, gy_{n+1}) &\leq (d(gx_{n-1}, gx_n) \cdot d(gy_{n-1}, gy_n))^{2\lambda} \\ &= (d(gx_{n-1}, gx_n) \cdot d(gy_{n-1}, gy_n))^{\kappa}, \end{aligned} \quad (5)$$

where $\kappa = 2\lambda$. Obviously, $\kappa < 1$. Now

$$\begin{aligned} & d(gx_n, gx_{n+1}) \cdot d(gy_n, gy_{n+1}) \\ \leq\ & (d(gx_{n-1}, gx_n) \cdot d(gy_{n-1}, gy_n))^{\kappa} \\ \leq\ & (d(gx_{n-2}, gx_{n-1}) \cdot d(gy_{n-2}, gy_{n-1}))^{\kappa^2} \\ \leq\ & \ldots \\ \leq\ & (d(gx_0, gx_1) \cdot d(gy_0, gy_1))^{\kappa^n}. \end{aligned}$$

Since $\lim_{n\to\infty} (d(gx_0, gx_1) \cdot d(gy_0, gy_1))^{\kappa^n} = 1$. Hence

$$\lim_{n\to\infty} d(gx_n, gx_{n+1}) \cdot d(gy_n, gy_{n+1}) = 1,$$

which implies that

$$\lim_{n\to\infty} d(gx_n, gx_{n+1}) = 1 \text{ and } \lim_{n\to\infty} d(gy_n, gy_{n+1}) = 1. \quad (6)$$

Now, for any $m, n \in N$ with $m > n \geq n_0$, we claim that

$$\lim_{n\to\infty} d(gx_n, gx_m) = 1 \quad (7)$$

$$\text{and } \lim_{n\to\infty} d(gy_n, gy_m) = 1. \quad (8)$$

We prove the inequality (7) by induction on m. The inequality (7) holds for $m = n + 1$ by using (6). Assume that (7) holds for $m = k$. Since $gx_n \preceq gx_k$ and $gy_n \succeq gy_k$, so that for $m = k + 1$, we have

$$\begin{aligned} & d(gx_n, gx_m) = d(gx_n, gx_{k+1}) \\ \leq\ & d(gx_n, gx_{n+1}) \cdot d(gx_{n+1}, gx_{k+1}) \\ =\ & d(gx_n, gx_{n+1}) \cdot d(F(x_n, y_n), F(x_k, y_k)) \\ \leq\ & d(gx_n, gx_{n+1}) \cdot (\max\{d(gx_n, gx_k) \cdot d(gy_n, gy_k), d(F(x_n, y_n), gx_n) \cdot d(F(x_n, y_n), gx_k), \\ & d(gy_n, gy_k) \cdot d(F(x_n, y_n), gx_n), d(gy_n, gy_k) \cdot d(F(x_n, y_n), gx_k)\})^{\lambda} \\ =\ & d(gx_n, gx_{n+1}) \cdot (\max\{d(gx_n, gx_k) \cdot d(gy_n, gy_k), d(gx_{n+1}, gx_n) \cdot d(gx_{n+1}, gx_k), \\ & d(gy_n, gy_k) \cdot d(gx_{n+1}, gx_n), d(gy_n, gy_k) \cdot d(gx_{n+1}, gx_k)\})^{\lambda}. \end{aligned}$$

On taking limit as $n \to \infty$ implies that

$$\lim_{n\to\infty} d(gx_n, gx_m) = 1.$$

Similarly, we obtain

$$\lim_{n\to\infty} d(gy_n, gy_m) = 1.$$

By induction on m, we conclude that (7) and (8) hold for $m > n \geq n_0$. Hence $\{gx_n\}$ and $\{gy_n\}$ are multiplicative Cauchy sequences in $g(X)$, so there exists x and y in X such that $\{gx_n\}$ and $\{gy_n\}$ converges to gx and gy respectively, that is,

$$\lim_{n\to\infty} d(gx_n, gx) = 1$$

and

$$\lim_{n\to\infty} d(gy_n, gy) = 1.$$

Now, we prove that $F(x,y) = gx$ and $F(y,x) = gy$.
Since $gx_n \preceq gx$ and $gy_n \succeq gy$ for all $n \geq 0$, so that we have

$$\begin{aligned}
& d(F(x,y), gx) \leq d(F(x,y), gx_{n+1}) \cdot d(gx_{n+1}, gx) \\
= \; & d(F(x_n,y_n), F(x,y)) \cdot d(gx_{n+1}, gx) \\
\leq \; & (\max\{d(gx_n, gx) \cdot d(gy_n, gy), d(F(x_n,y_n), gx_n) \cdot d(F(x_n,y_n), gx), \\
& d(gy_n, gy) \cdot d(F(x_n,y_n), gx_n), d(gy_n, gy) \cdot d(F(x_n,y_n), gx)\})^{\lambda} \cdot d(gx_{n+1}, gx) \\
= \; & (\max\{d(gx_n, gx) \cdot d(gy_n, gy), d(gx_{n+1}, gx_n) \cdot d(gx_{n+1}, gx), \\
& d(gy_n, gy) \cdot d(gx_{n+1}, gx_n), d(gy_n, gy) \cdot d(gx_{n+1}, gx)\})^{\lambda} \cdot d(gx_{n+1}, gx).
\end{aligned}$$

On taking limit as $n \to \infty$, we obtain that

$$d(F(x,y), gx) = 1,$$

and hence $F(x,y) = gx$. Similarly, it can be shown that $F(y,x) = gy$. Hence (x,y) is a coupled coincidence point and (gx, gy) is coupled point of coincidence of mappings F and g.

Now we show that F and g have a unique coupled coincidence point. As $CC(F,g) \subseteq \Delta_g$. Let $(g\widetilde{x}, g\widetilde{y})$ be another coupled point of coincidence of F and g such that $(gx, gy) \neq (g\widetilde{x}, g\widetilde{y})$, then $gx \preceq g\widetilde{x}$ and $gy \succeq g\widetilde{y}$ and from (1), we have

$$\begin{aligned}
& d(gx, g\widetilde{x}) = d(F(x,y), F(\widetilde{x},\widetilde{y})) \\
\leq \; & (\max\{d(gx, g\widetilde{x}) \cdot d(gy, g\widetilde{y}), d(F(x,y), gx) \cdot d(F(x,y), g\widetilde{x}), \\
& d(gy, g\widetilde{y}) \cdot d(F(x,y), gx), d(gy, g\widetilde{y}) \cdot d(F(x,y), g\widetilde{x})\})^{\lambda} \\
= \; & (d(gx, g\widetilde{x}) \cdot d(gy, g\widetilde{y}))^{\lambda}.
\end{aligned} \tag{9}$$

Similarly, by using $g\widetilde{y} \preceq gy$ and $g\widetilde{x} \succeq gx$, we have

$$d(gy, g\widetilde{y}) \leq (d(gx, g\widetilde{x}) \cdot d(gy, g\widetilde{y}))^{\lambda}. \tag{10}$$

Thus by multiplying above two inequalities,

$$d(gx, g\widetilde{x}) \cdot d(gy, g\widetilde{y}) \leq (d(gx, g\widetilde{x}) \cdot d(gy, g\widetilde{y}))^{2\lambda},$$

a contradiction as $\lambda < \frac{1}{2}$. Thus $(gx, gy) = (g\widetilde{x}, g\widetilde{y})$.
Now, we shall show that $gx = gy$. If not, then since $gx_0 \succeq gy_0$, implies $gy \preceq gy_n \preceq gy_0 \preceq gx_0 \preceq gx_n \preceq gx$ for all $n \geq 0$. Using (1), we have

$$\begin{aligned}
& d(gy, gx) = d(F(y,x), F(x,y)) \\
\leq \; & (\max\{d(gy, gx) \cdot d(gx, gy), d(F(y,x), gy) \cdot d(F(y,x), gx), \\
& d(gx, gy) \cdot d(F(y,x), gy), d(gx, gy) \cdot d(F(y,x), gx)\})^{\lambda} \\
= \; & (\max\{d(gy, gx) \cdot d(gx, gy), d(gy, gy) \cdot d(gy, gx), \\
& d(gx, gy) \cdot d(gy, gy), d(gx, gy) \cdot d(gy, gx)\})^{\lambda} \\
= \; & (d(gx, gy))^{2\lambda},
\end{aligned}$$

a contradiction as $2\lambda < 1$. Hence $gx = gy$.
Now we show that F and g have common coupled fixed point.
For this, let $g(x) = u$. Then we have $u = gx = F(x,x)$. By w^*- compatibility of F and g, we have

$$g(u) = g(gx) = g(F(x,x)) = F(gx,gx) = F(u,u).$$

Then (gu, gu) is a coupled point of coincidence of F and g. By the uniqueness of coupled point of coincidence, we have $gu = gx$. Therefore $u = gu = F(u,u)$, and (u,u) is the common coupled fixed point of F and g. □

Following example illustrates the fact that, condition $CC(F,g) \subseteq \Delta_g$ is essential to obtain unique coupled coincidence point in X.

Example 2.1. Let $X = [0,1]$ be an ordered set with the natural ordering of real numbers and $d : X \times X \to \mathbb{R}^+$ be the multiplicative metric defined by $d(x,y) = e^{|x-y|}$. Let $F : X \times X \to X$ and $g : X \to X$ be defined by

$$F(x,y) = \begin{cases} \frac{x+y+1}{6} & x \geq y, \\ 0 & x < y, \end{cases}$$

and

$$g(x) = \frac{5x}{6} \text{ for all } x \in X.$$

Note that $F(X \times X) \subseteq g(X)$.
Now for $(x,y,u,v) \in \Delta_g$ and $\frac{1}{5} \leq \lambda < \frac{1}{2}$, we have

$$\begin{aligned} d(F(x,y),F(u,v)) &= e^{\frac{1}{6}|x+y-u-v|} \\ &\leq e^{\frac{1}{6}[u-x+y-v]} \leq e^{\frac{5\lambda}{6}[u-x+y-v]} \\ &= (e^{\frac{5}{6}[(u-x)+(y-v)]})^{\lambda} = (d(gx,gu)\cdot d(gy,gv))^{\lambda} \\ &\leq (\max\{d(gx,gu)\cdot d(gy,gv), d(F(x,y),gx)\cdot d(F(x,y),gu), \\ &\quad d(gy,gv)\cdot d(F(x,y),gx), d(gy,gv)\cdot d(F(x,y),gu)\})^{\lambda}. \end{aligned}$$

Thus (1) is satisfied and F and g have coupled coincidence points. As $(\frac{1}{4},0)$, $(\frac{1}{3},\frac{1}{3}) \in CC(F,g)$ but not in Δ_g, so $CC(F,g) \nsubseteq \Delta_g$. Moreover, the set $CC(F,g)$ is not singleton. □

Example 2.2. Let $X = \mathbb{R}^+$ be a partially ordered set with the natural ordering of real numbers and d be a multiplicative metric space on X defined by $d(x,y) = a^{|x-y|}$, where $a > 1$ is a real number. Consider the mappings $F : X \times X \to X$ and $g : X \to X$ defined by

$$F(x,y) = \begin{cases} \dfrac{x^2-y^2}{8}, \text{ if } x \geq y, \\ 0, \quad \text{if } x < y, \end{cases}$$

$$g(x) = \frac{7}{8}x^2 \text{ for all } x \in X.$$

Note that $F(X \times X)$ is contained in a complete set $g(X)$.
Now, for $(x,y,u,v) \in \Delta_g$ with $\lambda = \frac{1}{7}$, we obtain

$$\begin{aligned} d(F(x,y),F(u,v)) &= a^{\frac{1}{8}\left|x^2-y^2-(u^2-v^2)\right|} \\ &= a^{\frac{1}{8}\left(u^2-x^2+y^2-v^2\right)} = a^{\frac{7\lambda}{8}\left(u^2-x^2+y^2-v^2\right)} \\ &= (a^{\frac{7}{8}\left|x^2-u^2\right|}\cdot a^{\frac{7}{8}\left|y^2-v^2\right|})^{\lambda} = (d(gx,gu)\cdot d(gy,gv))^{\lambda} \\ &\leq (\max\{d(gx,gu)\cdot d(gy,gv), d(F(x,y),gx)\cdot d(F(x,y),gu) \\ &\quad d(gy,gv)\cdot d(F(x,y),gx), d(gy,gv)\cdot d(F(x,y),gu)\})^{\lambda}. \end{aligned}$$

Thus mappings F and g satisfy all the conditions of Theorem 2.1. Moreover $(0,0)$ is the unique common coupled fixed point of F and g. $\square$

Corollary 2.2. *Let $(X, d, \preceq)$ be a partially ordered multiplicative metric space. Suppose that a mapping $F : X \times X \to X$ has a mixed monotone property with respect to $g : X \to X$ and*

$$d(F(x,y), F(u,v)) \leq (d(gx, gu) \cdot d(gy, gv))^k \tag{11}$$

for all $(x, y, u, v) \in \Delta_g$, where $k \in [0, \frac{1}{2})$. If $F(X \times X)$ is contained in a complete set $g(X)$ and X has the following property:

i. if a non-decreasing sequence $\{x_n\} \to x$, then $x_n \preceq x$ for all n,
ii. if a non-increasing sequence $\{y_n\} \to y$, then $y \preceq y_n$ for all n.

Then $CC(F, g)$ is nonempty provided that there exist $x_0, y_0 \in X$ such that

$$g(x_0) \preceq F(x_0, y_0) \text{ and } g(y_0) \succeq F(y_0, x_0).$$

If the set $CC(F, g)$ is contained in Δ_g, then F and g have a unique coupled coincidence point in X. Moreover, if $gy_0 \preceq gx_0$ with F and g are w^*- compatible, then F and g have a common coupled fixed point.

Corollary 2.3. *Let $(X, d, \preceq)$ be a partially ordered multiplicative metric space. Suppose that a mapping $F : X \times X \to X$ has the mixed monotone property and*

$$\begin{aligned} & d(F(x,y), F(u,v)) \\ \leq \; & (\max\{d(x,u) \cdot d(y,v), d(F(x,y),x) \cdot d(F(x,y),u), \\ & d(y,v) \cdot d(F(x,y),x), d(y,v) \cdot d(F(x,y),u)\})^{\lambda} \end{aligned} \tag{12}$$

for all $(x, y, u, v) \in \Delta_g$ with $\lambda \in [0, \frac{1}{2})$. If X is a complete with following property:

i. if a non-decreasing sequence $\{x_n\} \to x$, then $x_n \preceq x$ for all n,
ii. if a non-increasing sequence $\{y_n\} \to y$, then $y \preceq y_n$ for all n.

Then F has a coupled fixed point in X provided that there exist $x_0, y_0 \in X$ such that

$$x_0 \preceq F(x_0, y_0) \text{ and } y_0 \succeq F(y_0, x_0).$$

If the set of coupled fixed point of F is contained in Δ, then F has a unique coupled fixed point in X.

Proof. The results follows by taking $g = I$ (identity mapping) in Theorem 2.1. $\square$

Corollary 2.4. *Let $(X, d, \preceq)$ be a partially ordered multiplicative metric space. Suppose that a mapping $F : X \times X \to X$ has the mixed monotone property and*

$$d(F(x,y), F(u,v)) \leq (d(x,u) \cdot d(y,v))^{\lambda} \tag{13}$$

for all $(x, y, u, v) \in \Delta$ with $\lambda \in [0, \frac{1}{2})$. If X is complete and has a property that for any two sequences $\{x_n\}$, $\{y_n\}$ with $(x_{n+1}, y_{n+1}, x_n, y_n) \in \Delta$ such that $x_n \to x$, $y_n \to y$ as $n \to \infty$ implies that $(x_n, y, x, y_n) \in \Delta$. Then F has a fixed point in X provided that there exist $x_0, y_0 \in X$ such that $(x_0, y_0, F(x_0, y_0), F(y_0, x_0)) \in \Delta$.

Theorem 2.5. *Let $(X, d, \preceq)$ be a partially ordered multiplicative metric space. Suppose that a mapping $F : X \times X \to X$ has a mixed monotone property with respect to $g : X \to X$ and*

$$\begin{aligned} & d(F(x,y),F(u,v)) \\ \leq\ & d(gx,gu)^{k_1} \cdot d(gy,gv)^{k_2} \cdot d(F(x,y),gx)^{k_3} \\ & \cdot d(F(u,v),gu)^{k_4} \cdot d(F(x,y),gu)^{k_5} \end{aligned} \tag{14}$$

for all $(x,y,u,v) \in \Delta_g$, with nonnegative real numbers k_i, $i = 1,2,...,5$ and $\sum_{i=1}^{5} k_i < 1$. If $F(X \times X)$ is contained in a complete set $g(X)$ and X has the property that for any two sequences $\{x_n\}$, $\{y_n\}$ with $(x_{n+1}, y_{n+1}, x_n, y_n) \in \Delta$ such that $x_n \to x$, $y_n \to y$ as $n \to \infty$ implies that $(x_n, y, x, y_n) \in \Delta$. Then $CC(F,g)$ is nonempty provided that there exist $x_0, y_0 \in X$ such that $(g(x_0), g(y_0), F(x_0,y_0), F(y_0,x_0)) \in \Delta$. If $CC(F,g) \subseteq \Delta_g$, then F and g have a unique coupled coincidence point in X. Moreover, if $gy_0 \preceq gx_0$ with F and g are w^*- compatible, then F and g have a common coupled fixed point.

Proof. Let $x_0, y_0 \in X$ be such that $g(x_0) \preceq F(x_0,y_0)$ and $g(y_0) \succeq F(y_0,x_0)$. Using the similar arguments to those given in Theorem 2.1, we construct sequences $\{x_n\}$ and $\{y_n\}$ in X such that

$$g(x_{n+1}) = F(x_n,y_n) \text{ and } g(y_{n+1}) = F(y_n,x_n) \text{ for all } n \geq 0,$$

and for all $n \geq 0$,

$$\begin{aligned} gx_0 &\preceq gx_1 \preceq ... \preceq gx_n \preceq gx_{n+1} \preceq ..., \text{ and} \\ gy_0 &\succeq gy_1 \succeq ... \succeq gy_n \succeq gy_{n+1} \succeq .... \end{aligned}$$

Now we will suppose that $d(gx_n, gx_{n+1}) > 1$ and $d(gy_n, gy_{n+1}) > 1$ for all n, otherwise, F and g have a coupled coincidence point at (x_n, y_n), and so we have finished the proof. From (14), we have

$$\begin{aligned} & d(gx_n, gx_{n+1}) \\ =\ & d(F(x_{n-1},y_{n-1}),F(x_n,y_n)) \\ \leq\ & d(gx_{n-1},gx_n)^{k_1} \cdot d(gy_{n-1},gy_n)^{k_2} \cdot d(F(x_{n-1},y_{n-1}),gx_{n-1})^{k_3} \\ & \cdot d(F(x_n,y_n),gx_n)^{k_4} \cdot d(F(x_{n-1},y_{n-1}),gx_n)^{k_5} \\ =\ & d(gx_{n-1},gx_n)^{k_1} \cdot d(gy_{n-1},gy_n)^{k_2} \cdot d(gx_n,gx_{n-1})^{k_3} \\ & \cdot d(gx_{n+1},gx_n)^{k_4} \cdot d(gx_n,gx_n)^{k_5} \\ =\ & d(gx_{n-1},gx_n)^{k_1+k_3} \cdot d(gy_{n-1},gy_n)^{k_2} \cdot d(gx_{n+1},gx_n)^{k_4}, \end{aligned}$$

from which it follows

$$d(gx_n,gx_{n+1})^{1-k_4} \leq d(gx_{n-1},gx_n)^{k_1+k_3} \cdot d(gy_{n-1},gy_n)^{k_2}. \tag{15}$$

Similarly,

$$d(gy_n,gy_{n+1})^{1-k_4} \leq d(gy_{n-1},gy_n)^{k_1+k_3} \cdot d(gx_{n-1},gx_n)^{k_2}. \tag{16}$$

From (15) and (16), we obtain

$$\begin{aligned} & (d(gx_n,gx_{n+1}) \cdot d(gy_n,gy_{n+1}))^{1-k_4} \\ \leq\ & (d(gy_{n-1},gy_n) \cdot d(gx_{n-1},gx_n))^{k_1+k_2+k_3}, \end{aligned}$$

that is

$$d(gx_n,gx_{n+1}) \cdot d(gy_n,gy_{n+1}) \leq [d(gx_{n-1},gx_n) \cdot d(gy_{n-1},gy_n)]^{\lambda}, \tag{17}$$

where $\lambda = \dfrac{k_1 + k_2 + k_3}{1 - k_4}$. Obviously, $0 \leq \lambda < 1$. Now

$$\begin{aligned}
d(gx_n, gx_{n+1}) \cdot d(gy_n, gy_{n+1}) &\leq [d(gx_{n-1}, gx_n) \cdot d(gy_{n-1}, gy_n)]^{\lambda} \\
&\leq [d(gx_{n-2}, gx_{n-1}) \cdot d(gy_{n-2}, gy_{n-1})]^{\lambda^2} \\
&\leq \ldots \\
&\leq [d(gx_0, gx_1) \cdot d(gy_0, gy_1)]^{\lambda^n}.
\end{aligned}$$

Then, for all $n, m \in N$, $m > n$, we have

$$\begin{aligned}
d(gx_n, gx_m) \cdot d(gy_n, gy_m) &\leq d(gx_n, gx_{n+1}) \cdot d(gx_{n+1}, gx_{x+2}) \cdot \ldots \cdot d(gx_{m-1}, gx_m) \\
&\quad \cdot d(gy_n, gy_{n+1}) \cdot d(gy_{n+1}, gy_{x+2}) \cdot \ldots \cdot d(gy_{m-1}, gy_m) \\
&= d(gx_n, gx_{n+1}) \cdot d(gy_n, gy_{n+1}) \cdot d(gx_{n+1}, gx_{x+2}) \\
&\quad \cdot d(gy_{n+1}, gy_{x+2}) \cdot \ldots \cdot d(gx_{m-1}, gx_m) \cdot d(gy_{m-1}, gy_m) \\
&\leq [d(gx_0, gx_1) \cdot d(gy_0, gy_1)]^{\frac{\lambda^n}{1-\lambda}},
\end{aligned}$$

which implies that $\lim\limits_{n,m\to\infty} [d(gx_n, gx_m) \cdot d(gy_n, gy_m)] = 1$. Hence $\lim\limits_{n,m\to\infty} d(gx_n, gx_m) = 1$ and $\lim\limits_{n,m\to\infty} d(gy_n, gy_m) = 1$. Hence $\{gx_n\}$ and $\{gy_n\}$ are multiplicative Cauchy sequences in $g(X)$, so there exists x and y in X such that $\{gx_n\}$ and $\{gy_n\}$ converges to gx and gy respectively. Now, we prove that $F(x, y) = gx$ and $F(y, x) = gy$.
Now since $gx_n \preceq gx$ and $gy_n \succeq gy$ for all $n \geq 0$, so that we have

$$\begin{aligned}
& d(F(x, y), gx) \\
\leq\ & d(F(x, y), gx_{n+1}) \cdot d(gx_{n+1}, gx) \\
=\ & d(F(x_n, y_n), F(x, y)) \cdot d(gx_{n+1}, gx) \\
\leq\ & d(gx_n, gx_n)^{k_1} \cdot d(gy_n, gy_n)^{k_2} \cdot d(F(x_n, y_n), gx_n)^{k_3} \cdot d(F(x, y), gx)^{k_4} \\
& \cdot d(F(x_n, y_n), gx)^{k_5} \cdot d(gx_{n+1}, gx) \\
=\ & d(gx_{n+1}, gx_n)^{k_3} \cdot d(F(x, y), gx)^{k_4} \cdot d(gx_{n+1}, gx)^{k_5} \cdot d(gx_{n+1}, gx).
\end{aligned}$$

On taking the limit as $n \to \infty$, we obtain that

$$d(F(x, y), gx) \leq d(F(x, y), gx)^{k_4}.$$

Since $k_4 < 1$, so that $F(x, y) = gx$. Similarly, it can be shown that $F(y, x) = gy$. Hence (x, y) is a coupled coincidence point and (gx, gy) is coupled point of coincidence of mappings F and g.

Now we show that F and g have a unique coupled coincidence point. As $CC(F, g) \subseteq \Delta_g$, so that, if $(g\widetilde{x}, g\widetilde{y})$ be another coupled point of coincidence of F and g such that $(gx, gy) \neq (g\widetilde{x}, g\widetilde{y})$, then $gx \preceq g\widetilde{x}$ and $gy \succeq g\widetilde{y}$ and from (14), we have

$$\begin{aligned}
& d(gx, g\widetilde{x}) \\
=\ & d(F(x, y), F(\widetilde{x}, \widetilde{y})) \\
\leq\ & d(gx, g\widetilde{x})^{k_1} \cdot d(gy, g\widetilde{y})^{k_2} \cdot d(F(x, y), gx)^{k_3} \\
& \cdot d(F(\widetilde{x}, \widetilde{y}), g\widetilde{x})^{k_4} \cdot d(F(x, y), g\widetilde{x})^{k_5} \\
=\ & d(gx, g\widetilde{x})^{k_1+k_5} \cdot d(gy, g\widetilde{y})^{k_2}.
\end{aligned} \tag{18}$$

Similarly, by using $g\widetilde{y} \preceq gy$ and $g\widetilde{x} \succeq gx$, we have

$$d(gy, g\widetilde{y}) \leq d(gy, g\widetilde{y})^{k_1+k_5} \cdot d(gx, g\widetilde{x})^{k_2}. \tag{19}$$

Thus by multiplying above two inequalities,

$$d(gx, g\widetilde{x}) \cdot d(gy, g\widetilde{y}) \leq (d(gx, g\widetilde{x}) \cdot d(gy, g\widetilde{y}))^{k_1+k_2+k_5},$$

a contradiction as $k_1 + k_2 + k_5 < 1$. Thus $(gx, gy) = (g\widetilde{x}, g\widetilde{y})$, that is, (gx, gy) is the unique couple coincidence point of F and g.
Now, we shall show that $gx = gy$. Since $gx_0 \succeq gy_0$, implies $gy \preceq gy_n \preceq gy_0 \preceq gx_0 \preceq gx_n \preceq gx$ for all $n \geq 0$. Using (14), we have

$$\begin{aligned} & d(gy, gx) \\ = \; & d(F(y,x), F(x,y)) \\ \leq \; & d(gy, gx)^{k_1} \cdot d(gx, gy)^{k_2} \cdot d(F(y,x), gy)^{k_3} \\ & \cdot d(F(x,y), gx)^{k_4} \cdot d(F(y,x), gx)^{k_5} \\ = \; & d(gx, gy)^{k_1+k_2+k_5}, \end{aligned}$$

implies $gx = gy$.
Now we show that F and g have common coupled fixed point.
For this, let $g(x) = u$. Then we have $u = gx = F(x, x)$. By w^*- compatibility of F and g, we have

$$g(u) = g(gx) = g(F(x, x)) = F(gx, gx) = F(u, u).$$

Then (gu, gu) is a coupled point of coincidence of F and g. By the uniqueness of coupled point of coincidence, we have $gu = gx$. Therefore $u = gu = F(u, u)$, and (u, u) is the common coupled fixed point of F and g. □

Corollary 2.6. *Let $(X, d, \preceq)$ be a partially ordered set and d a multiplicative metric on X. Suppose that a mapping $F : X \times X \to X$ has a mixed monotone property with respect to $g : X \to X$ and*

$$d(F(x,y), F(u,v)) \leq d(F(x,y), gx)^k \cdot d(F(u,v), gu)^l \tag{20}$$

for all $(x, y, u, v) \in \Delta_g$ and $k, l \geq 0$ with $k + l < 1$. If $F(X \times X)$ is contained in a complete set $g(X)$ and X has the property that for any two sequences $\{x_n\}$, $\{y_n\}$ with $(x_{n+1}, y_{n+1}, x_n, y_n) \in \Delta$ such that $x_n \to x$, $y_n \to y$ as $n \to \infty$ implies that $(x_n, y, x, y_n) \in \Delta$. Then $CC(F, g)$ is nonempty provided that there exist $x_0, y_0 \in X$ such that $(g(x_0), g(y_0), F(x_0, y_0), F(y_0, x_0)) \in \Delta$. If $CC(F, g) \subseteq \Delta_g$, then F and g have a unique coupled coincidence point in X. Moreover, if $gy_0 \preceq gx_0$ with F and g are w^*- compatible, then F and g have a common coupled fixed point.

3 Application.

Let $\Omega = [0, T]$ be a bounded set in $\mathbb{R}$, where $T > 0$ and $X = C(\Omega, \mathbb{R})$ denote the space of real valued continuous functions on Ω. Consider the integral equations

$$\begin{aligned} x(t) &= \int_{\Omega} q(t, x(s), y(s))ds + k(t); \\ y(t) &= \int_{\Omega} q(t, y(s), x(s))ds + k(t), \end{aligned} \tag{21}$$

where $q : \Omega \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ and $k : \Omega \to \mathbb{R}$ be given continuous mappings.

We shall study sufficient condition for existence of solution of integral equations (21) in framework of multiplicative metric spaces. Define $d : X \times X \to [1, \infty)$ by

$$d(x, y) = e^{\sup_{t \in \Omega} |x(t) - y(t)|}.$$

Then (X,d) is a complete multiplicative metric space. We endow X with the partial ordered $\preceq$ given by: $x,y \in X$, $x \preceq y \Leftrightarrow x(t) \leq y(t)$, for all $t \in \Omega$. Suppose that for all $x,y,u,v \in \mathbb{R}$ with $x \leq u$ and $y \geq v$, we have

$$0 \leq q(t,x,y) - q(t,u,v) \leq \frac{\lambda}{T}(x - y - u + v),$$

for each $t \in \Omega$, where $\lambda \in [0, \frac{1}{2})$.
Then the integral equations (21) have a solution in $L^2(\Omega)$.

Proof. Define $F(x,y)(t) = \int\limits_{\Omega} q(t,x(s),y(s))ds + k(t)$ for $x,y \in X$ and $t \in \Omega$. For all $x,y \in C(\Omega,\mathbb{R})$,

$$\begin{aligned}
d(F(x,y),F(u,v)) &= e^{\sup\limits_{t\in\Omega}|F(x,y)(t)-F(u,v)(t)|} \\
&= e^{\sup\limits_{t\in\Omega}|\int\limits_{\Omega} q(t,x(s),y(s))ds-\int\limits_{\Omega} q(t,u(s),v(s))ds|} \\
&\leq e^{\sup\limits_{t\in\Omega}\int\limits_{\Omega}|q(t,x(s),y(s))-q(t,u(s),v(s))|ds} \\
&\leq e^{\frac{\lambda}{T}\int\limits_{\Omega}[\sup\limits_{t\in\Omega}|x(t)-u(t)|+\sup\limits_{t\in\Omega}|y(t)-v(t)|]ds} \\
&= e^{\lambda[\sup\limits_{t\in\Omega}|x(t)-u(t)|+\sup\limits_{t\in\Omega}|y(t)-v(t)|]} \\
&= (d(x,u)\cdot d(y,v))^{\lambda} \\
&\leq (M(x,y,u,v))^{\lambda},
\end{aligned}$$

where

$$\begin{aligned}
M(x,y,u,v) &= \max\{d(x,u)\cdot d(y,v), d(F(x,y),x)\cdot d(F(x,y),u), \\
&\quad d(y,v)\cdot d(F(x,y),x), d(y,v)\cdot d(F(x,y),u)\}
\end{aligned}$$

for $\lambda \in [0,\frac{1}{2})$. Thus (1) is satisfied. Moreover, it is easy to see that there exists $(x_0,y_0) \in C(\Omega,\mathbb{R}) \times C(\Omega,\mathbb{R})$ such that $(x_0,y_0,F(x_0,y_0),(y_0,x_0)) \in \Delta$. Thus all the condition of Theoerm 2.1 are satisfied with $g = I$. Therefore we apply Theorem 2.1 and get the the solution $(\widetilde{x},\widetilde{y}) \in C(\Omega,\mathbb{R}) \times C(\Omega,\mathbb{R})$ of integral equations (21). □

4 Conclusions and Remarks

In this paper, we used the concept of w^*- compatible mappings and obtained several common coupled fixed point results of mappings in the setup of partially ordered multiplicative metric spaces. It is worth mentioning that these results are based on without exploiting the notion of continuity of mappings. We prsented some examples that shown the validitity of obtained results. As an application of these results, we presented the existence of solution of integral equations in framework of multiplicative metric spaces. These established results generalize, extend and unify various results in the existing literature.

5 Declarations

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Competing Interests

The authors declare no conflict of interest.

Ethical Approval

Not applicable.

Author's Contributions

Conceptualization, Z.Z. and T.N.; methodology, T.N.; formal analysis, investigation, writing—original draft preparation, writing—review and editing, visualization, supervision, project administration, Z.Z. and T.N. All authors have read and agreed to the published version of the manuscript.

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No data is used in this study.

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