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Geometry of solutions of the geometric curve flows in space

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Abstract

In this study, we aim to investigate the geometry of surfaces corresponding to the geometry of solutions of the geometric curve flows in Euclidean 3-space $\mathbb{R}^3$ considering the Frenet frame. In particular, we express some geometric properties and some characterizations of u-parameter curves and t-parameter curves of some trajectory surfaces including the Hasimoto surface, the shortening trajectory surface, the minimal trajectory surface, the $\sqrt{\tau}$-normal trajectory surface in $\mathbb{R}^3$.

Key words: the Hasimoto surface, the shortening trajectory surface, the minimal trajectory surface, the $\sqrt{\tau}$-normal trajectory surface
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1 Introduction

This article investigates the geometry of surfaces defined as trajectories of evolving geometric curve flows in Euclidean 3-space $\mathbb{R}^3$. In recent years, the theory of surfaces with the connection of the geometric curve flows in different spaces and integrable non-linear equations is a subject of research attention (see e.g. [1–16]) and finding geometric curve flows which generate various types of surfaces may expand our understanding of their geometric and topological properties, in order to a deep understanding of the physical world (see e.g. [17–20]).

Hasimoto [10] in 1972 showed the connection between the binormal flow and the nonlinear Schrödinger equation. Surfaces generated by the binormal flow, referred to as the Hasimoto surfaces, have been previously considered in [1, 12]. For the Hasimoto surfaces, a lot of researches are done using the different frames (such as the Frenet frame [1], the Bishop frame [13], the Darboux frame [7, 9], the quasi-frame [11], the modified orthogonal frame [8], the hybrid frame [16] and so on) in Euclidean 3-space [1], Minkowski 3-space [5, 14, 15], Galilean 3-space [6] and pseudo-Galilean 3-space [2]. Trajectory surfaces have been studied for the special

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case of inextensible flows in [21], curves flow of elastic rods in [22], and the curve shortening flow in [23–25]. Recently, J. Minarčík and M. Beneš [26] introduced minimal surface generating flow for space curves of non-vanishing torsion in Euclidean 3-space $\mathbb{R}^3$, and analyzed some properties of space curves evolved by the minimal surface generating flow.

The aim of this paper is to study the geometry of trajectory surfaces corresponding to the geometry of solutions of the geometric curve flows in Euclidean 3-space $\mathbb{R}^3$. The paper is organized as follows. Section 2 contains an introduction to the geometric curve flow and recalls the notion of trajectory surfaces, including the Hasimoto surface, the shortening trajectory surface, the minimal trajectory surface, the $\sqrt{\tau}$-normal trajectory surface in $\mathbb{R}^3$. In Section 3, we investigate the geometric properties of some trajectory surfaces via the Frenet frame. Also, the characterization of the u-parameter curves and t-parameter curves of the trajectory surfaces is examined. In section 4, we draw the main conclusions, as well as some questions that we postpone for the future.

2 Trajectory surfaces for the geometric curve flows in space

In this section, we define the necessary notation for the parametric space curves in motion, recall the governing equations for a general geometric flow of curves in $\mathbb{R}^3$ and introduce the notion of trajectory surfaces generated by this general motion law, and finally, we obtain the Gaussian curvature, the mean curvature, the principal curvatures, the geodesic torsion, the geodesic curvature, and the normal curvature of the trajectory surfaces.

Assume the family of evolving curves is parametrized as follows:

$$\{\Gamma_t\}_{t\in[0,t_{max})} = \{\mathbf{X}(u,t), u \in \mathbb{S}^1, t \in [0,t_{max})\},$$

where $\mathbf{X} = \mathbf{X}(u,t) : \mathbb{S}^1 \times [0,t_{max}) \to \mathbb{R}^3$ is a smooth mapping, $t_{max} > 0$ is the terminal time and $\mathbb{S}^1 = \mathbb{R}/2\pi\mathbb{Z}$ is a unit circle. Then the unit arc-length parametrization s is given by $ds = g du$, where $g = |\partial_u \mathbf{X}|$. The unit tangent vector is given by $\mathbf{T} = \partial_s \mathbf{X}$. In the case when the curvature $\kappa = |\mathbf{T} \times \partial_s \mathbf{T}| > 0$ is strictly positive, we can define the so-called the Frenet frame. It means that the unit normal and binormal vectors $\mathbf{N}$ and $\mathbf{B}$ can be uniquely defined as follows:

$$\mathbf{N} = \kappa^{-1}\partial_s \mathbf{T},\ \ \mathbf{B} = \mathbf{T} \times \mathbf{N},$$

respectively. These unit vectors satisfy the following identities:

$$\mathbf{B} = \mathbf{T} \times \mathbf{N}, \qquad \mathbf{T} = \mathbf{N} \times \mathbf{B}, \qquad \mathbf{N} = \mathbf{B} \times \mathbf{T},$$

and the Frenet-Serret formulae:

$$\frac{d}{ds}\begin{pmatrix} \mathbf{T} \\ \mathbf{N} \\ \mathbf{B} \end{pmatrix} = \begin{pmatrix} 0 & \kappa & 0 \\ -\kappa & 0 & \tau \\ 0 & -\tau & 0 \end{pmatrix} \begin{pmatrix} \mathbf{T} \\ \mathbf{N} \\ \mathbf{B} \end{pmatrix},$$

where τ is the torsion of a curve. For $\kappa > 0$, the torsion τ is given by

$$\tau = \kappa^{-2}(\mathbf{T} \times \partial_s \mathbf{T}) \cdot \partial_s^2 \mathbf{T} = \kappa^{-2}(\partial_s \mathbf{X} \times \partial_s^2 \mathbf{X}) \cdot \partial_s^3 \mathbf{X}.$$

The time evolution of $\{\Gamma_t\}_{t\in[0,t_{max})}$ is given by the geometric flow in the form of the following initial-value problem for the parametrization $\mathbf{X} = \mathbf{X}(u,t)$ in $\mathbb{R}^3$,

$$\begin{aligned} \partial_t \mathbf{X} &= v_N \mathbf{N} + v_B \mathbf{B} + v_T \mathbf{T}, && \text{in } \mathbb{S}^1 \times [0,t_{max}), \\ \mathbf{X}|_{t=0} &= \mathbf{X}_0 && \text{in } \mathbb{S}^1, \end{aligned} \tag{1}$$

where $\mathbf{X}_0$ is the parametrization for the initial curve Γ_0 .

Firstly, we recall that the dynamic equations for unit vectors $\mathbf{N}, \mathbf{B}, \mathbf{T}$ and the local geometric quantities (i.e., the curvature κ, the torsion τ) during the motion given by (1) in the following (see [27] for details).

- The unit vectors $\mathbf{N}, \mathbf{B}, \mathbf{T}$ forming the Frenet frame satisfy the evolution equations:

$$\begin{aligned} \partial_t \mathbf{T} &= (\partial_s v_N + \kappa v_T - \tau v_B)\,\mathbf{N} + (\partial_s v_B + \tau v_N)\,\mathbf{B}, \\ \kappa \partial_t \mathbf{N} &= -\kappa\,(\partial_s v_N + \kappa v_T - \tau v_B)\,\mathbf{T} \\ &\quad + \left(\partial_s^2 v_B + v_N \partial_s \tau + 2\tau \partial_s v_N + \tau\,(\kappa v_T - \tau v_B)\right)\mathbf{B}, \\ \kappa \partial_t \mathbf{B} &= -\kappa\,(\partial_s v_B + \tau v_N)\,\mathbf{T} - \left(\partial_s^2 v_B + v_N \partial_s \tau + 2\tau \partial_s v_N + \tau\,(\kappa v_T - \tau v_B)\right)\mathbf{N}. \end{aligned} \tag{2}$$

- The local length element $g = |\partial_u \mathbf{X}|$ satisfies $\partial_t g = (-\kappa v_N + \partial_s v_T) g$.
- The curvature κ and the torsion τ (for $\kappa(s,t) > 0$) satisfy the following system of evolution partial differential equations:

$$\partial_t \kappa = \partial_s^2 v_N + \kappa^2 v_N + v_T \partial_s \kappa - \partial_s(\tau v_B) - \tau \partial_s v_B - \tau^2 v_N, \tag{3}$$

$$\begin{aligned} \partial_t \tau &= \kappa\,(\partial_s v_B + \tau v_N) + \partial_s \left(\kappa^{-1}\left(\partial_s^2 v_B + v_N \partial_s \tau + 2\tau \partial_s v_N + \tau\,(\kappa v_T - \tau v_B)\right)\right) \\ &\quad + \tau(\kappa v_N - \partial_s v_T). \end{aligned} \tag{4}$$

Definition 2.1. [21, 26] (Trajectory surface) For given velocities v_T, v_N and v_B, an initial curve Γ_0 and terminal time t_{max}, we define the trajectory surface $\sum_{t_{max}}$ as $\sum_{t_{max}} := \bigcup_{t \in [0, t_{max})} \Gamma_t$.

Trajectory surfaces have been studied in [1, 21, 23, 26] for the following cases, such as the binormal flow (i.e., $v_T = v_N = 0, v_B = \kappa$), the curve shortening flow (i.e., $v_T = v_B = 0, v_N = \kappa$), minimal surface generating flow (i.e., $v_T = v_B = 0, v_N = \tau^{-\frac{1}{2}}$), inextensible flows (i.e., geometric flows with $\frac{\partial g}{\partial t} = 0$).

(i) A Hasimoto surface $\sum_1$ is the surface traced out by a curve $\mathbf{X}(u,t)$ in $\mathbb{R}^3$ as it evolves over time according to this evolution equation:

$$\partial_t \mathbf{X} = \kappa \mathbf{B}, \tag{5}$$

which motion law is the binormal flow, also known as the localized induction approximation of the vortex filament flow, which has applications in incompressible flows (e.g. see [1]).

(ii) A shortening trajectory surface $\sum_2$ is the surface traced out by a family of curves $\{\Gamma_t\}_{t \in [0, t_{max})}$ in $\mathbb{R}^3$ as it evolves over time according to the curve shortening flow:

$$\begin{aligned} \partial_t \mathbf{X} &= \kappa \mathbf{N}, && \text{in } \mathbb{S}^1 \times [0, t_{max}), \\ \mathbf{X}|_{t=0} &= \mathbf{X}_0, && \text{in } \mathbb{S}^1, \end{aligned} \tag{6}$$

where $\mathbf{X}_0$ is the parametrization for the initial curve Γ_0 in $\mathbb{R}^3$. Moreover, if the curve approaches singularity as t goes to t_{max}, the shortening trajectory surface will be an embedded disc without a point in the center of the original sphere (e.g. see [23]).

(iii) A minimal trajectory surface $\sum_3$ is the surface traced out by a family of curves $\{\Gamma_t\}_{t \in [0, t_{max})}$ in $\mathbb{R}^3$ as it evolves over time according to the minimal surface generating flow (see [26]):

$$\begin{aligned} \partial_t \mathbf{X} &= \tau^{-\frac{1}{2}} \mathbf{N}, && \text{in } \mathbb{S}^1 \times [0, t_{max}), \\ \mathbf{X}|_{t=0} &= \mathbf{X}_0, && \text{in } \mathbb{S}^1, \end{aligned} \tag{7}$$

where $\mathbf{X}_0$ is the parametrization for the initial curve Γ_0 in $\mathbb{R}^3$ with positive curvature and torsion.

(iv) A $\sqrt{\tau}$-normal trajectory surface $\sum_4$ is the surface traced out by a family of curves $\{\Gamma_t\}_{t \in [0, t_{max})}$ in $\mathbb{R}^3$ as it evolves over time according to the $\sqrt{\tau}$-normal geometric flow:

$$\begin{aligned} \partial_t \mathbf{X} &= \sqrt{\tau} \mathbf{N}, && \text{in } \mathbb{S}^1 \times [0, t_{max}), \\ \mathbf{X}|_{t=0} &= \mathbf{X}_0, && \text{in } \mathbb{S}^1, \end{aligned} \tag{8}$$

where $\mathbf{X}_0$ is the parametrization for the initial curve Γ_0 in $\mathbb{R}^3$ with positive curvature and nonnegative torsion.

The first and the second fundamental forms of the surface $\sum_{t_{max}}$ generated by (1) are given by

$$\begin{aligned}
\mathrm{I} \equiv \begin{pmatrix} \mathcal{E} & \mathcal{F} \\ \mathcal{F} & \mathcal{G} \end{pmatrix} &= \begin{pmatrix} ||\partial_u \mathbf{X}||^2 & \langle \partial_u \mathbf{X}, \partial_t \mathbf{X}\rangle \\ \langle \partial_t \mathbf{X}, \partial_u \mathbf{X}\rangle & ||\partial_t \mathbf{X}||^2 \end{pmatrix}, \\
\mathrm{II} \equiv \begin{pmatrix} \mathcal{L} & \mathcal{M} \\ \mathcal{M} & \mathcal{N} \end{pmatrix} &= \begin{pmatrix} \langle \partial_u^2 \mathbf{X}, n\rangle & \langle \partial_u\partial_t \mathbf{X}, n\rangle \\ \langle \partial_t\partial_u \mathbf{X}, n\rangle & \langle \partial_t^2 \mathbf{X}, n\rangle \end{pmatrix},
\end{aligned}$$

respectively, where the unit normal vector n of the trajectory surface $\sum_{t_{max}}$ can be expressed as

$$n = \frac{\partial_u \mathbf{X} \times \partial_t \mathbf{X}}{||\partial_u \mathbf{X} \times \partial_t \mathbf{X}||} = \frac{-v_B \mathbf{N} + v_N \mathbf{B}}{\sqrt{v_B^2 + v_N^2}}.$$

Using evolution equations (2)-(4), the elements of the first fundamental form I read

$$\mathcal{E} = g^2,\ \mathcal{F} = g v_T,\ \mathcal{G} = v_N^2 + v_B^2 + v_T^2, \tag{9}$$

and we give the elements of I:

$$\begin{aligned}
\mathcal{L} &= -\frac{g^2 \kappa v_B}{\sqrt{v_B^2+v_N^2}}, \\
\mathcal{M} &= g\left(\frac{-v_B\partial_s v_N + v_N \partial_s v_B - \kappa v_T v_B}{\sqrt{v_B^2+v_N^2}} + \tau\sqrt{v_B^2+v_N^2}\right), \\
\mathcal{N} &= \frac{v_N\partial_t v_B - v_B \partial_t v_N + v_T[v_N(\partial_s v_B + \tau v_N) - v_B(\partial_s v_N + \kappa v_T - \tau v_B)]}{\sqrt{v_B^2+v_N^2}} \\
&\quad + \kappa^{-1}\sqrt{v_B^2+v_N^2}\left(\partial_s^2 v_B + v_N \partial_s \tau + 2\tau\partial_s v_N + \kappa\tau v_T - \tau^2 v_B\right).
\end{aligned} \tag{10}$$

The Gaussian curvature K, the mean curvature H, and the principal curvatures $P_i, i = 1, 2$ of the trajectory surface $\sum_{t_{max}}$ as

$$\begin{aligned}
K &= \frac{\det\ \mathrm{II}}{\det\ \mathrm{I}} = \frac{\mathcal{L}\mathcal{N} - \mathcal{M}^2}{\mathcal{E}\mathcal{G} - \mathcal{F}^2},\ H = \frac{1}{2}\mathrm{Tr}(II(I)^{-1}) = \frac{\mathcal{L}\mathcal{G} - 2\mathcal{M}\mathcal{F} + \mathcal{N}\mathcal{E}}{2(\mathcal{E}\mathcal{G} - \mathcal{F}^2)}, \\
P_1 &= H + \sqrt{H^2 - K},\ P_2 = H - \sqrt{H^2 - K},
\end{aligned}$$

respectively. By Eqs. (9) and Eqs. (10), we can write K and H in terms of the functions v_T, v_N, v_B, κ and τ as

$$\begin{aligned}
K &= \frac{\kappa v_B\left(v_B\partial_t v_N - v_N \partial_t v_B + v_T(-v_N(\partial_s v_B + \tau v_N) + v_B(\partial_s v_N + \kappa v_T - \tau v_B))\right)}{(v_B^2+v_N^2)^2} \\
&\quad - \frac{v_B(\partial_s^2 v_B + 2\tau\partial_s v_N + \partial_s\tau\partial_s v_N + \kappa\tau v_T - \tau^2 v_B)}{v_B^2+v_N^2} - \frac{(v_N\partial_s v_B - v_B\partial_s v_N - \kappa v_T v_B)^2}{(v_B^2+v_N^2)^2} \\
&\quad - \frac{2\tau(v_N\partial_s v_B - v_B\partial_s v_N - \kappa v_T v_B)}{v_B^2+v_N^2} - \tau^2, \\
H &= -\frac{\kappa v_B(v_N^2+v_B^2+v_T^2)}{2(v_N^2+v_B^2)^{\frac{3}{2}}} - v_T\left(\frac{v_N\partial_s v_B - v_B\partial_s v_N - \kappa v_B v_T}{(v_N^2+v_B^2)^{\frac{3}{2}}} + \frac{\tau}{\sqrt{v_N^2+v_B^2}}\right) \\
&\quad + \frac{v_N\partial_t v_B - v_B\partial_t v_N + v_T(v_N(\partial_s v_B + \tau v_N) - v_B(\partial_s v_N + \kappa v_T - \tau v_B))}{2(v_N^2+v_B^2)^{\frac{3}{2}}} \\
&\quad + \frac{\partial_s^2 v_B + 2\tau\partial_s v_N + v_N\partial_s\tau + \kappa\tau v_T - \tau^2 v_B}{2\kappa\sqrt{v_B^2+v_N^2}}.
\end{aligned}$$

For a regular curve $\gamma(x)$ on trajectory surface $\sum_{t_{max}}$ in $\mathbb{R}^3$, the geodesic torsion $\mathcal{T}_g$, the geodesic curvature $\mathcal{K}_g$, and the normal curvature $\mathcal{K}_n$ are given by

$$\mathcal{T}_g = \frac{\langle \dot{\gamma}, n \times \dot{n} \rangle}{||\dot{\gamma}||^2}, \ \mathcal{K}_g = \frac{\langle \ddot{\gamma}, n \times \dot{\gamma} \rangle}{||\dot{\gamma}||^2}, \ \mathcal{K}_n = \frac{\langle \ddot{\gamma}, n \rangle}{||\dot{\gamma}||^2},$$

respectively, where $\dot{\gamma} = \frac{\partial \gamma}{\partial x}, \ddot{\gamma} = \frac{\partial^2 \gamma}{\partial x^2}, \dot{n} = \frac{\partial n}{\partial x}$.

Now, we can write the following important definitions:

Definition 2.2. [28]

- A regular surface $\sum$ is a developable (flat) surface if its Gaussian curvature $K \equiv 0$, whereas it is a minimal surface if its mean curvature $H \equiv 0$;
- For a regular curve $\gamma(t)$ on a trajectory surface $\sum_{t_{max}}$, the following facts are well-known:
  1) $\gamma(t)$ is a principal line if and only if the geodesic torsion $\mathcal{T}_g \equiv 0$;
  2) $\gamma(t)$ is an asymptotic line if and only if the normal curvature $\mathcal{K}_n \equiv 0$;
  3) $\gamma(t)$ is a geodesic curve if and only if the geodesic curvature $\mathcal{K}_g \equiv 0$.

Definition 2.3. The family of all u-parameter curves and the family of all t-parameter curves of a trajectory surface $\sum_{t_{max}}$ in $\mathbb{R}^3$ are denoted by Λ^u and Ω^t, respectively.

Using evolution equations (2)-(4), we obtain the geodesic torsion ${}^t\mathcal{T}_g$, the geodesic curvature ${}^t\mathcal{K}_g$, and the normal curvature ${}^t\mathcal{K}_n$ of the t parameter curves $\mathbf{X}(u,t)$ on trajectory surface $\sum_{t_{max}}$ as

$$\begin{aligned} {}^t\mathcal{T}_g &= \frac{\langle \dot{\mathbf{X}}, n \times \dot{n} \rangle}{||\dot{\mathbf{X}}||^2} = \frac{\rho[(v_N^2 + v_B^2)n_1 - (v_B n_3 + v_N n_2)v_T]}{v_N^2 + v_B^2 + v_T^2}, && (11) \\ {}^t\mathcal{K}_g &= \frac{\langle \ddot{\mathbf{X}}, n \times \dot{\mathbf{X}} \rangle}{||\dot{\mathbf{X}}||^2} = \frac{\rho[-(v_N^2 + v_B^2)\xi_1 + (v_B \xi_2 + v_N \xi_3)v_T]}{v_N^2 + v_B^2 + v_T^2}, && (12) \\ {}^t\mathcal{K}_n &= \frac{\langle \ddot{\mathbf{X}}, n \rangle}{||\dot{\mathbf{X}}||^2} = \frac{\rho(v_N \xi_3 - v_B \xi_2)}{v_N^2 + v_B^2 + v_T^2}, && (13) \end{aligned}$$

respectively, where

$$\begin{aligned} \xi_1 &= \partial_t v_T - v_N(\partial_s v_N + \kappa v_T - \tau v_B) - v_B(\partial_s v_B + \tau v_N), \\ \xi_2 &= \partial_t v_N - v_B \kappa^{-1}(\partial_s^2 v_B + 2\tau \partial_s v_N + v_N \partial_s \tau + \tau \kappa v_T - \tau^2 v_B) + v_T(\partial_s v_N + \kappa v_T - \tau v_B), \\ \xi_3 &= \partial_t v_B + v_N \kappa^{-1}(\partial_s^2 v_B + 2\tau \partial_s v_N + v_N \partial_s \tau + \tau \kappa v_T - \tau^2 v_B) + v_T(\partial_s v_B + \tau v_N), \\ n_1 &= \rho[v_B(\partial_s v_N + \kappa v_T - \tau v_B) - v_N(\partial_s v_B + \tau v_N)], \\ n_2 &= -\rho \partial_t v_B - \kappa^{-1} \rho v_N(\partial_s^2 v_B + 2\tau \partial_s v_N + v_N \partial_s \tau + \tau \kappa v_T - \tau^2 v_B) - v_B \partial_t \rho, \\ n_3 &= \rho \partial_t v_N - \kappa^{-1} \rho v_B(\partial_s^2 v_B + 2\tau \partial_s v_N + v_N \partial_s \tau + \tau \kappa v_T - \tau^2 v_B) + v_N \partial_t \rho, \\ \dot{\mathbf{X}} &= \frac{\partial \mathbf{X}}{\partial t}, \ddot{\mathbf{X}} = \frac{\partial^2 \mathbf{X}}{\partial t^2}, \dot{n} = \frac{\partial n}{\partial t}, \rho = \frac{1}{\sqrt{v_B^2 + v_N^2}}. \end{aligned}$$

Similarly, the geodesic torsion ${}^u\mathcal{T}_g$, the geodesic curvature ${}^u\mathcal{K}_g$, the normal curvature ${}^u\mathcal{K}_n$ of the u parameter curves $\mathbf{X}(u,t)$ on trajectory surface $\sum_{t_{max}}$ as

$$\begin{aligned} {}^u\mathcal{T}_g &= \frac{\langle \mathbf{X}_u, n \times n_u \rangle}{||\mathbf{X}_u||^2} = \frac{\rho(v_B(\rho \tau v_B - (\rho v_N)_s) + v_N((\rho v_B)_s + \rho v_N \tau))}{g}, && (14) \\ {}^u\mathcal{K}_g &= \frac{\langle \mathbf{X}_{uu}, n \times \mathbf{X}_u \rangle}{||\mathbf{X}_u||^2} = \kappa g \rho v_N, \ {}^u\mathcal{K}_n = \frac{\langle \mathbf{X}_{uu}, n \rangle}{||\mathbf{X}_u||^2} = -\kappa \rho v_B, && (15) \end{aligned}$$

respectively, where

$$\mathbf{X}_u = \frac{\partial \mathbf{X}}{\partial u}(u,t), \mathbf{X}_{uu} = \frac{\partial^2 \mathbf{X}}{\partial u^2}(u,t), n_u = \frac{\partial n}{\partial u}(u,t), \rho = \frac{1}{\sqrt{v_B^2 + v_N^2}}.$$

For the sake of simplicity, we denote

$$\begin{aligned}
\Omega^t_{\mathcal{T}_g} &= \{\mathbf{X} = \mathbf{X}(u,t) \in \Omega^t : \, {}^t\mathcal{T}_g = 0\}, \; \Lambda^u_{\mathcal{T}_g} &= \{\mathbf{X} = \mathbf{X}(u,t) \in \Lambda^u : \, {}^u\mathcal{T}_g = 0\}, \\
\Omega^t_{\mathcal{K}_g} &= \{\mathbf{X} = \mathbf{X}(u,t) \in \Omega^t : \, {}^t\mathcal{K}_g = 0\}, \; \Lambda^u_{\mathcal{K}_g} &= \{\mathbf{X} = \mathbf{X}(u,t) \in \Lambda^u : \, {}^u\mathcal{K}_g = 0\}, \\
\Omega^t_{\mathcal{K}_n} &= \{\mathbf{X} = \mathbf{X}(u,t) \in \Omega^t : \, {}^t\mathcal{K}_n = 0\}, \; \Lambda^u_{\mathcal{K}_n} &= \{\mathbf{X} = \mathbf{X}(u,t) \in \Lambda^u : \, {}^u\mathcal{K}_n = 0\}.
\end{aligned}$$

3 Some Geometric Properties of solutions of the geometric curve flows in space

In this section, we obtain the Gaussian curvature, the mean curvature, and the principal curvatures of some trajectory surfaces including the Hasimoto surface, the shortening trajectory surface, the minimal trajectory surface, the $\sqrt{\tau}$-normal trajectory surface in $\mathbb{R}^3$ and give necessary and sufficient conditions for t-parameter and u-parameter curves of some trajectory surfaces in $\mathbb{R}^3$ to be geodesics, asymptotic lines, and principal lines.

Some geometric properties and some characterizations of parameter curves of the Hasimoto surfaces in $\mathbb{R}^3$ are given [1] and are briefly presented here for completeness.

Theorem 3.1. *(see [1]) Let $\mathbf{X} = \mathbf{X}(u,t)$ be a Hasimoto surface $\sum_1$ in $\mathbb{R}^3$.*

(1) Then the Gaussian curvature K, the mean curvature H, the principal curvatures P_i (i=1,2) of the Hasimoto surface $\sum_1$ are

$$K = -\frac{\kappa_{ss}}{\kappa}, \; H = \frac{1}{2\kappa}(\frac{\kappa_{ss}}{\kappa} - \kappa^2 - \tau^2),$$

$$P_1 = \frac{1}{2\kappa}(\frac{\kappa_{ss}}{\kappa} - \kappa^2 - \tau^2) + \frac{\sqrt{(\kappa^{-1}\kappa_{ss} - \kappa^2 - \tau^2)^2 + 4\kappa\kappa_{ss}}}{2\kappa},$$

$$P_2 = \frac{1}{2\kappa}(\frac{\kappa_{ss}}{\kappa} - \kappa^2 - \tau^2) - \frac{\sqrt{(\kappa^{-1}\kappa_{ss} - \kappa^2 - \tau^2)^2 + 4\kappa\kappa_{ss}}}{2\kappa},$$

respectively;

(2) We give necessary and sufficient conditions for parameter curves $\mathbf{X}(u,t)$ of a Hasimoto surface $\sum_1$ in $\mathbb{R}^3$ to be principal lines, geodesics, and asymptotic lines, i.e.,

$$\begin{aligned}
\Omega^t_{\mathcal{T}_g} &= \{\mathbf{X} \in \Omega^t : \, \tau = 0\}, & \Lambda^u_{\mathcal{T}_g} &= \{\mathbf{X} \in \Lambda^u : \, \tau = 0\}, \\
\Omega^t_{\mathcal{K}_g} &= \{\mathbf{X} \in \Omega^t : \, \kappa_s = 0\}, & \Lambda^u_{\mathcal{K}_g} &= \Lambda^u, \\
\Omega^t_{\mathcal{K}_n} &= \{\mathbf{X} \in \Omega^t : \, \kappa_{ss} - \kappa\tau^2 = 0\}, & \Lambda^u_{\mathcal{K}_n} &= \{\mathbf{X} \in \Lambda^u : \, \kappa = 0\}.
\end{aligned}$$

(3) The binormal flow (5) is inextensible flows (i.e., geometric flows with $\frac{\partial g}{\partial t} = 0$).

Proof. By using Eqs. (9), Eqs. (10) and Eqs. (11)-(15), we have

$$I = \begin{pmatrix} g^2 & 0 \\ 0 & \kappa^2 \end{pmatrix}, II = \begin{pmatrix} -g^2\kappa & g\tau\kappa \\ g\tau\kappa & \kappa_{ss} - \kappa\tau^2 \end{pmatrix},$$

$${}^t\mathcal{T}_g = -\tau, \; {}^t\mathcal{K}_g = \kappa_s \; {}^t\mathcal{K}_n = \frac{\kappa_{ss} - \kappa\tau^2}{\kappa^2},$$

$${}^u\mathcal{T}_g = \frac{\tau}{g}, \; {}^u\mathcal{K}_g = 0, \; {}^u\mathcal{K}_n = -\kappa, \; \partial_t g = (-\kappa v_N + \partial_s v_T)g = 0.$$

This completes the proof. □

Theorem 3.2. *Let $\mathbf{X} = \mathbf{X}(u,t)$ be a shortening trajectory surface $\sum_2$ in $\mathbb{R}^3$.*

(1) Then the Gaussian curvature K, the mean curvature H, the principal curvatures P_i (i=1,2) of the shortening trajectory surface $\sum_2$ are

$$K = -\tau^2,\ H = \frac{2\tau\kappa_s + \tau_s\kappa}{2\kappa^2},$$
$$P_1 = \frac{2\tau\kappa_s + \tau_s\kappa}{2\kappa^2} + \frac{1}{2\kappa^2}\sqrt{(2\tau\kappa_s + \tau_s\kappa)^2 + 4\kappa^4\tau^2},$$
$$P_2 = \frac{2\tau\kappa_s + \tau_s\kappa}{2\kappa^2} - \frac{1}{2\kappa^2}\sqrt{(2\tau\kappa_s + \tau_s\kappa)^2 + 4\kappa^4\tau^2},$$

respectively;

(2) we give necessary and sufficient conditions for parameter curves $\mathbf{X}(u,t)$ of a shortening trajectory surface $\sum_2$ in $\mathbb{R}^3$ to be principal lines, geodesics, and asymptotic lines, i.e.,

$$\begin{array}{lll} \Omega^t_{\mathcal{T}_g} &=& \{\mathbf{X} \in \Omega^t :\ \tau = 0\}, \\ \Omega^t_{\mathcal{K}_g} &=& \{\mathbf{X} \in \Omega^t :\ \kappa_s = 0\}, \\ \Omega^t_{\mathcal{K}_n} &=& \{\mathbf{X} \in \Omega^t :\ 2\tau\kappa_s + \kappa\tau_s = 0\}, \end{array} \quad \begin{array}{lll} \Lambda^u_{\mathcal{T}_g} &=& \{\mathbf{X} \in \Lambda^u :\ \tau = 0\}, \\ \Lambda^u_{\mathcal{K}_g} &=& \{\mathbf{X} \in \Lambda^u :\ \kappa = 0\}, \\ \Lambda^u_{\mathcal{K}_n} &=& \Lambda^u. \end{array}$$

Proof. By using Eqs. (9), Eqs. (10) and Eqs. (11)-(15), we have

$$I = \begin{pmatrix} g^2 & 0 \\ 0 & \kappa^2 \end{pmatrix}, II = \begin{pmatrix} 0 & g\tau\kappa \\ g\tau\kappa & \kappa\tau_s + 2\tau\kappa_s \end{pmatrix},$$
$${}^t\mathcal{T}_g = -\tau,\ {}^t\mathcal{K}_g = \kappa_s\ {}^t\mathcal{K}_n = \frac{2\tau\kappa_s + \kappa\tau_s}{\kappa^2},$$
$${}^u\mathcal{T}_g = \frac{\tau}{g},\ {}^u\mathcal{K}_g = \kappa g,\ {}^u\mathcal{K}_n = 0,\ \partial_t g = (-\kappa v_N + \partial_s v_T)g = -\kappa^2 g.$$

This finishes the proof of Theorem 3.2. □

Theorem 3.3. *Let $\mathbf{X} = \mathbf{X}(u,t)$ be a minimal trajectory surface $\sum_3$ in $\mathbb{R}^3$.*

(1) Then the Gaussian curvature K, the mean curvature H, the principal curvatures P_i (i=1,2) of the minimal trajectory surface $\sum_3$ are (see [26])

$$K = -\tau^2,\ H = 0,\ P_1 = \tau,\ P_2 = -\tau,$$

respectively;

(2) we give necessary and sufficient conditions for parameter curves $\mathbf{X}(u,t)$ of a minimal trajectory surface $\sum_3$ in $\mathbb{R}^3$ to be principal lines, geodesics, and asymptotic lines, i.e.,

$$\begin{array}{lll} \Omega^t_{\mathcal{T}_g} &=& \emptyset, \\ \Omega^t_{\mathcal{K}_g} &=& \{\mathbf{X} \in \Omega^t :\ \tau_s = 0\}, \\ \Omega^t_{\mathcal{K}_n} &=& \Omega^t, \end{array} \quad \begin{array}{lll} \Lambda^u_{\mathcal{T}_g} &=& \emptyset, \\ \Lambda^u_{\mathcal{K}_g} &=& \emptyset, \\ \Lambda^u_{\mathcal{K}_n} &=& \Lambda^u. \end{array}$$

Proof. By using Eqs. (9), Eqs. (10) and Eqs. (11)-(15), we have

$$I = \begin{pmatrix} g^2 & 0 \\ 0 & \frac{1}{\tau} \end{pmatrix}, II = \begin{pmatrix} 0 & g\sqrt{\tau} \\ g\sqrt{\tau} & 0 \end{pmatrix},$$
$${}^t\mathcal{T}_g = -\tau,\ {}^t\mathcal{K}_g = -\frac{\tau_s}{2\tau^{\frac{3}{2}}}\ {}^t\mathcal{K}_n = 0,$$
$${}^u\mathcal{T}_g = \frac{\tau}{g},\ {}^u\mathcal{K}_g = \kappa g,\ {}^u\mathcal{K}_n = 0,\ \partial_t g = \frac{\kappa}{\sqrt{\tau}}.$$

Note that $\kappa \neq 0$, $\tau > 0$, hence this completes the proof. □

Theorem 3.4. *Let $\mathbf{X} = \mathbf{X}(u,t)$ be a $\sqrt{\tau}$-normal trajectory surface $\sum_4$ in $\mathbb{R}^3$.*

(1) Then the Gaussian curvature K, the mean curvature H, the principal curvatures P_i (i=1,2) of $\sqrt{\tau}$-normal trajectory surface $\sum_4$ are

$$K = -\tau^2,\ H = \frac{\tau_s}{\kappa},\ P_1 = \frac{\tau_s + \sqrt{\tau_s^2 + \kappa^2\tau^2}}{\kappa},\ P_2 = \frac{\tau_s - \sqrt{\tau_s^2 + \kappa^2\tau^2}}{\kappa},$$

respectively;

(2) we give necessary and sufficient conditions for parameter curves $\mathbf{X}(u,t)$ of a $\sqrt{\tau}$-normal trajectory surface $\sum_4$ in $\mathbb{R}^3$ to be principal lines, geodesics, and asymptotic lines, i.e.,

$$\begin{array}{lllll} \Omega^t_{\mathcal{T}_g} & = & \{\mathbf{X} \in \Omega^t : \tau = 0\}, & \Lambda^u_{\mathcal{T}_g} & = \{\mathbf{X} \in \Lambda^u : \tau = 0\}, \\ \Omega^t_{\mathcal{K}_g} & = & \{\mathbf{X} \in \Omega^t : \tau_s = 0\}, & \Lambda^u_{\mathcal{K}_g} & = \emptyset, \\ \Omega^t_{\mathcal{K}_n} & = & \{\mathbf{X} \in \Omega^t : \tau_s = 0\}, & \Lambda^u_{\mathcal{K}_n} & = \Lambda^u. \end{array}$$

Proof. By using Eqs. (9), Eqs. (10) and Eqs. (11)-(15), we have

$$I = \begin{pmatrix} g^2 & 0 \\ 0 & \tau \end{pmatrix}, II = \begin{pmatrix} 0 & g\tau^{\frac{3}{2}} \\ g\tau^{\frac{3}{2}} & \frac{2\tau\tau_s}{\kappa} \end{pmatrix},$$

$${}^t\mathcal{T}_g = -\tau,\ {}^t\mathcal{K}_g = \frac{\tau_s}{2\sqrt{\tau}}\ {}^t\mathcal{K}_n = \frac{2\tau_s}{\kappa},$$

$${}^u\mathcal{T}_g = \frac{\tau}{g},\ {}^u\mathcal{K}_g = \kappa g,\ {}^u\mathcal{K}_n = 0,\ \partial_t g = -\kappa\sqrt{\tau} g.$$

Note that $\kappa \neq 0$, $\tau \geq 0$, this finishes the proof of Theorem 3.4. □

Remark 3.5. There are the other types of geometric curve flows (such as the geometric KdV curve flow also called Fukumoto-Miyazaki model, the generalized bi-Schrödinger flow or also called Fukumoto-Moffatt model) in [29, 30], as following

(i) A Fukumoto-Miyazaki surface is the surface traced out by a family of curves $\{\Gamma_t\}_{t\in[0,t_{max})}$ in $\mathbb{R}^3$ as it evolves over time according to the geometric KdV curve flow:

$$\begin{array}{lll} \partial_t \mathbf{X} & = \frac{1}{2}\kappa \mathbf{T} + \kappa_s \mathbf{N} + \kappa\tau\mathbf{B}, & \text{in } \mathbb{S}^1 \times [0, t_{max}), \\ \mathbf{X}|_{t=0} & = \mathbf{X}_0, & \text{in } \mathbb{S}^1, \end{array} \tag{16}$$

where $\mathbf{X}_0$ is the parametrization for the initial curve Γ_0 in $\mathbb{R}^3$ (see e.g. [3, 29]). Note that the geometric KdV curve flow (16) is inextensible flows, in fact,

$$\partial_t g = (-\kappa v_N + \partial_s v_T)g = (-\kappa\kappa_s + (\frac{1}{2}\kappa^2))g = 0.$$

(ii) A Fukumoto-Moffatt surface is the surface traced out by a family of curves $\{\Gamma_t\}_{t\in[0,t_{max})}$ in $\mathbb{R}^3$ as it evolves over time according to the generalized bi-Schrödinger flow:

$$\begin{array}{ll} \partial_t \mathbf{X} & = \lambda\left\{\kappa\mathbf{B} + \nu[\kappa^2\tau\mathbf{T} + (2\kappa_s\tau + \kappa\tau_s)\mathbf{N} + (\kappa\tau^2 - \kappa_{ss})\mathbf{B}] + \mu\kappa^3\mathbf{B}\right\}, \\ \mathbf{X}|_{t=0} & = \mathbf{X}_0, \end{array} \tag{17}$$

where $\mathbf{X}_0$ is the parametrization for the initial curve Γ_0 in $\mathbb{R}^3$, λ, μ and ν are three real parameters (see e.g. [4, 30]). Notice again that the generalized bi-Schrödinger flow (17) is inextensible flows, this is because

$$\partial_t g = (-\kappa v_N + \partial_s v_T)g = (-\kappa\lambda\nu(2\kappa_s\tau + \kappa\tau_s) + \lambda\nu(\kappa^2\tau)_s)g = 0.$$

Similarly, we also give some geometric properties and some characterizations of $u-$parameter curves and $t-$parameter curves of Fukumoto-Miyazaki surface and Fukumoto-Moffatt surface. The details are omitted here.

4 Conclusion

In this paper we investigate the geometric properties of solutions of the geometric curve flows in Euclidean 3-space $\mathbb{R}^3$ with respect to the Frenet frame. This trajectory surfaces including the Hasimoto surface, the shortening trajectory surface, the minimal trajectory surface, the $\sqrt{\tau}$-normal trajectory surface may be useful for some specific applications in theoretical physics and fluid dynamics. It is of interest to analyze some properties (such as the long term behavior, upper bound for the area of the $\sqrt{\tau}$-normal trajectory surface and terminating time) of space curves evolved by the $\sqrt{\tau}$-normal geometric flow (8). These topics are beyond the scope of this paper but may be considered in future work.

5 Declarations

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Competing Interests

The authors declare that they have no competing interests.

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