EJAM
www.ejamjournal.com

Electronic Journal of Applied Mathematics

Research article

Mersin University Publishing
DOI: 10.61383/ejam.20231237
2023, Vol. 1, No. 2, 60–70

Recovering solution of the Reverse nonlinear time Fractional diffusion equations with fluctuations data

Doan Thi Thanh Xuan[1], Vo Thi Thanh Ha[1,†]

1. Faculty of Fundamental Science, Industrial University of Ho Chi Minh City, Viet Nam

[†]Corresponding author

Abstract

In this study, our focus is on obtaining an estimated solution for the nonlinear fractional time diffusion equation. Specifically, we have utilized the Riemann Liouville fractional derivative. Additionally, we have concerned Gaussian white noise in the input data. As we are aware, this problem is considered ill-posed according to Hadamard's definition. To tackle this problem, we have proposed a regularized solution and demonstrated the convergence between the mild solution and the regularized solution.

Key words: Riemann-Liouville; Regularized solution; Gaussian white noise; Ill-posed
2020 Mathematics Subject Classification: 35K99; 47J06; 47H10; 35K05
Article history: Received 2 Aug 2023; Revised 2 Sep 2023; Accepted 5 Sep 2023; Online 9 Sep 2023

1 Introduction

In this study, we investigate the nonlinear time-fractional diffusion equations subject to Dirichlet boundary conditions

$$\begin{cases} \partial_t u - D_t^{1-\alpha}\Delta u = G(x,t,u), \quad (x,t) \in \Omega \times (0,T) \\ u(x,t) = 0, \quad x \in \partial\Omega, \quad t \in (0,T) \end{cases} \tag{1}$$

and the terminal value condition

$$u(x,T) = \xi(x), \quad x \in \Omega. \tag{2}$$

The domain $\Omega \subset \mathbb{R}^d$ $(d \geq 1)$ is a bounded open set with a sufficiently smooth boundary $\partial\Omega$, and $T > 0$ is the terminal time. The symbol D_t^α represents the Riemann Liouville derivative of order α $(0 < \alpha < 1)$, which will be defined later. Recently, many scientists have become increasingly interested in the problems of fractional diffusion equations. Fractional differential equations

Contact: Doan Thi Thanh Xuan ✉ doanthithanhxuan@iuh.edu.vn;
Vo Thi Thanh Ha ✉ vothithanhthanh@iuh.edu.vn

have practical applications in modeling anomalous diffusion phenomena in various scientific fields, such as physics, chemistry, engineering, and more [1, 2]. There are several definitions of fractional derivatives, including Caputo, Riemann-Liouville, Caputo-Riesz, and others, which are nonlocal operators in contrast to the local operators of integer-order derivatives. Depending on the specific application and their experience, researchers may use a certain type of derivative. One of the main differences between fractional derivatives and classical derivatives is the non-local property of fractional derivatives. This property reflects the fact that the change at a specific location in the environment is affected by the state of the entire region.

The main objective of this article is to propose a regularized solution that can approximate the solution of (1)–(2). It is crucial to mention that our problem of reverse time is considered ill-posed according to the Hadamard definition. Hence, a regularization method is required to recover an accurate approximation. Assuming that the final value h is observed as ξ^ϵ, it is well-known that observations are susceptible to random errors that arise due to the limitations of the measuring device (measurement errors). Consequently, it is common to encounter data that is perturbed or noisy. This work focuses on situations where such perturbations manifest in the form of additive stochastic white noise.

$$\xi^\epsilon(x) = \xi(x) + \epsilon W(x). \tag{3}$$

Let's examine a situation where ϵ indicates the amplitude of the noise, and $W(x)$ represents a process of Gaussian white noise. Additionally, suppose that the observations given in (3) can not be accurately measured but can only be discretized and therefore observed in a sampled format.

$$\langle \xi^\epsilon, \phi_p \rangle = \langle \xi, \phi_p \rangle + \epsilon \langle W, \phi_p \rangle, \quad p = 1, \dots, n. \tag{4}$$

where $\{\phi_p\}$ is a orthonormal basic of Hilbert space $L^2(\Omega)$; $\langle \cdot, \cdot \rangle$ denotes the inner product in $L^2(\Omega)$.

The direct problem of time-fractional diffusion equations featuring various types of fractional derivatives has been widely investigated in recent years. Inverse problems for time-fractional diffusion equations seek to retrieve initial data, source function, diffusion coefficient, and other parameters through additional data. However, such problems have received little attention recently [3–5]. As far as we know, no previous studies have focused on (1)–(2) concerning random noise as depicted in (4).

The structure of this paper is structured as follows. Section 2 presents some preliminary materials. An example of Ill-Posed is provided in Section 3.1. The primary results of this paper are presented in Section 3.3 which is followed by an analysis of the convergent estimates between a mild solution and a regularized solution under some prior assumptions on the exact solution.

2 Preliminaries

Definition 2.1 (Podlubny [6]). Let $\alpha > 0$ and $\beta \in \mathbb{R}$, the Mittag-Leffler function is

$$E_{\alpha,\beta}(\omega) = \sum_{j=1}^{\infty} \frac{\omega^j}{\Gamma(\alpha j + \beta)}, \quad \omega \in \mathbb{C}. \tag{5}$$

Definition 2.2 (Podlubny [6]). Let α be a real number such that $\alpha \in (0,1)$. The Riemann-Liouville derivative of fractional order α with power-law of function $u(t)$ is defined as

$$D_t^{1-\alpha} u(t) = \frac{\mathrm{d}}{\mathrm{d}t} \left(\frac{1}{\Gamma(1-\alpha)} \int_0^t (t-s)^{\alpha-1} u(\tau) \mathrm{d}\tau \right) \tag{6}$$

where $\Gamma(\cdot)$ is the gamma function.

The Riemann-Liouville definition conforms to all the mathematical principles within the domain of fractional calculus, especially when utilizing Laplace transform.

Property 2.1. *The Laplace transform of the Riemann-Liouville fractional integral operator of order* $\alpha \in (0,1)$ *can be obtained in the form of:*

$$\mathcal{L}\left\{D_t^{1-\alpha}u(t)\right\} = \omega^{1-\alpha}\widehat{u} - D_t^{-\alpha}u(0)$$

where $\widehat{u}$ *is the Laplace transform of* u*,*

$$\widehat{u} = \mathcal{L}\{u(t)\} = \int_0^{+\infty} \mathrm{e}^{-\omega s}u(s)\mathrm{d}s, \quad s \in \mathbb{C}.$$

Property 2.2 (Podlubny [6])**.** *Given reals number* $\alpha \in (0,1)$ *and* $\lambda > 0$*. There always exists positive constants* $B_1(\alpha)$ *and* $B_2(\alpha)$ *such that*

$$\frac{B_1(\alpha)}{1+\lambda} \leq E_{\alpha,1}(-\lambda) \leq \frac{B_2(\alpha)}{1+\lambda}. \tag{7}$$

Moreover, the following identity is applicable for $t > 0$

$$\int_0^{\infty} \mathrm{e}^{-\omega t}E_{\alpha,1}(-\lambda t^{\alpha})\mathrm{d}t = \frac{\omega^{\alpha-1}}{\omega^{\alpha}+\lambda}, \quad \omega \in \mathbb{C}, \quad \mathrm{Re}(\omega) > \lambda^{1/\alpha}.$$

2.1 Some necessary spaces

We have $\Omega \subset \mathbb{R}^d$ be an open bounded domain and let $\langle\cdot,\cdot\rangle$ be the inner product of $L^2(\Omega)$. Then, there exists an orthonormal basis $\{\phi_p\}_{p=1}^{\infty}$ ($\phi_p \in \mathcal{H}1_0(\Omega) \cap C^{\infty}(\Omega)$) of $L^2(\Omega)$ consisting of eigenfunctions $0 < \lambda_1 \leq \lambda_2 \leq \ldots \leq \lim_{p\to\infty} \lambda_p = +\infty$ of the Laplacian operator $-\Delta$ in Ω such that $-\Delta\phi_p(x) = \lambda_p\phi_p(x)$ for $x \in \Omega$ and $\phi_p(x) = 0$ for $x \in \partial\Omega$. For $\mu > 0$, the Sobolev class of function is defined as follows

$$\mathcal{H}^{\mu}(\Omega) = \left\{u \in L^2(\Omega) : \sum_{p=1}^{\infty} \lambda_p^{\mu}\left\langle u,\phi_p\right\rangle^2 < \infty\right\},$$

which is a Hilbert space and endowed with the norm $\|u\|_{\mathcal{H}^{\mu}}^2 = \sum_{p=1}^{\infty} \lambda_p^{\mu}\left\langle u,\phi_p\right\rangle^2$.

Definition 2.3 (Zou and Wang [7])**.** Given a measure probability space $(D,\mathcal{F},\mu)$ and Banach space B. The function $u : (0,T) \to B$ measurable such that $\operatorname{ess\,sup}_{t\in(0,T)} \|u(t)\|_B < \infty$. The Bochner space $\mathcal{L}^2(D,B) \equiv \mathcal{L}^2((D,\mathcal{F},\mu);B)$ is defined with a norm

$$\|u(t)\|_{\mathcal{L}^2(D,B)} := \left(\int_D \|u(t)\|_B\mathrm{d}\mu\right)^{1/2} = \left(\mathbb{E}\|u(t)\|_B^2\right)^{1/2} < +\infty. \tag{8}$$

The space of B-valued predictable processes u such that

$$\|u\|_{\mathbf{Z}(0,T;B)} = \sup_{0\leq t\leq T} \|u(t)\|_{\mathcal{L}^2(\Omega,B)}. \tag{9}$$

2.2 Integral form of the solution

We denote the inner product $u_p(u) = \langle u(\cdot,t),\phi_p\rangle$, $\xi_p = \langle g,\phi_p\rangle$ and $G_p(u)(t) = \langle G(\cdot,t,u(\cdot,t)),\phi_p\rangle$. The first equation of system (1) is transformed into

$$\langle\partial_t u,\phi_p\rangle - \langle D_t^{1-\alpha}\Delta u,\phi_p\rangle = \langle G(\cdot,t,u(\cdot,t)),\phi_p\rangle.$$

Since $-\Delta\phi_p = \lambda_p\phi_p$, so $\partial_t u_p + \lambda_p D_t^{1-\alpha} u_p = G_p(u)$. Utilize Laplace transformation to obtain

$$\mathcal{L}\left\{\partial_t u_p\right\} + \mathcal{L}\left\{\lambda_p D_t^{1-\alpha} u_p\right\} = \mathcal{L}\left\{G_p(u)\right\},$$

Applying the properties of Laplace transformations

$$\omega\widehat{u}_p - u_p(0) + \omega^{1-\alpha}\widehat{u}_p - D_t^{-\alpha} u_p(0) = \widehat{G_p}(u).$$

Use simple transformations we have

$$\widehat{u}_p = \frac{\omega^{\alpha-1}}{\omega^\alpha + \lambda_p}\left(u_p(0) + D_t^{-\alpha} u_p(0)\right) + \frac{\omega^{\alpha-1}}{\omega^\alpha + \lambda_p}\widehat{G_p}(u). \tag{10}$$

Property 2.2–iii implies that

$$\mathcal{L}^{-1}\left\{\frac{\omega^{\alpha-1}}{\omega^\alpha + \lambda_p}\right\} = E_{\alpha,1}(-\lambda_p t^\alpha)$$

and Laplace transform of the convolution integral

$$\mathcal{L}^{-1}\left\{\frac{\omega^{\alpha-1}}{\omega^\alpha + \lambda_p}\widehat{G}_p(u)\right\} = \int_0^t E_{\alpha,1}(-\lambda_p(t-s)^\alpha)G(u)(s)\mathrm{d}s.$$

Taking inverse Laplace transforms the two sides of (10) we get

$$u_p(t) = E_{\alpha,1}(-\lambda_p t^\alpha)\left(u_p(0) + D_t^{-\alpha} u_p(0)\right) + \int_0^t E_{\alpha,1}(-\lambda_p(t-s)^\alpha)G_p(u)(s)\mathrm{d}s,$$

and the terminal condition $u_p(T) = \xi_p$ give us

$$\left(u_p(0) + D_t^{-\alpha} u_p(0)\right) = \frac{1}{E_{\alpha,1}(-\lambda_p t^\alpha)}\left(\xi_p - \int_0^T E_{\alpha,1}(-\lambda_p(T-s)^\alpha)G_p(u)(s)\mathrm{d}s,\right).$$

Thus

$$\begin{aligned} u_p(t) = {} & \frac{E_{\alpha,1}(-\lambda_p t^\alpha)}{E_{\alpha,1}(-\lambda_p T^\alpha)}\left(\xi_p - \int_0^T E_{\alpha,1}(-\lambda_p(T-s)^\alpha)G_p(s)(u)\mathrm{d}s,\right) \\ & + \int_0^t E_{\alpha,1}(-\lambda_p(t-s)^\alpha)G_p(u)(s)\mathrm{d}s. \end{aligned}$$

Definition 2.4 (Mild solution)**.** Suppose that Problem (1)–(2) has a solution $u \in L^2(0,T;\Omega)$, then the mild solution is presented as

$$\begin{aligned} u(x,t) = {} & \sum_{p=1}^\infty \frac{E_{\alpha,1}(-\lambda_p t^\alpha)}{E_{\alpha,1}(-\lambda_p T^\alpha)}\left(\xi_p - \int_0^T E_{\alpha,1}(-\lambda_p(T-s)^\alpha)G_p(u)(s)\mathrm{d}s\right)\phi_p(x) \\ & + \sum_{p=1}^\infty \left(\int_0^t E_{\alpha,1}(-\lambda_p(t-s)^\alpha)G_p(u)(s)\mathrm{d}s\right)\phi_p(x), \end{aligned} \tag{11}$$

2.3 Statistic estimate the terminal function

Definition 2.5 (Phuong et al. [8])**.** Assuming the function $\xi \in \mathcal{H}^\mu$ ($\mu > 0$). The sequences of n observations of ξ are $\langle \xi^\epsilon, \phi_p \rangle$, $p = 1, \ldots, n$. Statistics estimation of ξ is proposed as

$$\tilde{\xi}^\epsilon(x) = \sum_{p=1}^{n} \langle \xi^\epsilon, \phi_p \rangle \phi_p(x). \tag{12}$$

Lemma 2.6. *Let $\tilde{\xi}^\epsilon \in L^2(\Omega)$ and suppose that $\xi \in \mathcal{H}^\mu(\Omega), \mu > 0$. The error is estimated as following*

$$\mathbb{E}\|\tilde{\xi}^\epsilon - \xi\|^2_{L^2(\Omega)} \le \epsilon^2 n + \frac{1}{\lambda_n^\mu} ||\xi||^2_{\mathcal{H}^\mu}. \tag{13}$$

Where $n(\epsilon) := n$ depends on ϵ and satisfies that $\lim_{\epsilon \to 0^+} n(\epsilon) = +\infty$.

Proof. We begin our argument by recognizing that

$$\begin{aligned}
\mathbf{E}||\tilde{\xi}^\epsilon - \xi||^2_{L^2(\Omega)} &= \mathbb{E}\left(\sum_{p=1}^{n} \langle \xi_p^\epsilon - \xi, \phi_p \rangle^2\right) + \sum_{p=n+1}^{\infty} \langle \xi, \phi_p \rangle^2 \\
&= \epsilon^2 \mathbb{E}\left(\sum_{p=1}^{n} W_p^2\right) + \sum_{p=n+1}^{\infty} \lambda_p^{-\mu} \lambda_p^{\mu} \langle \xi, \phi_p \rangle^2 \\
&\le \epsilon^2 \mathbb{E}\left(\sum_{p=1}^{n} W_p^2\right) + \frac{1}{\lambda_p^\mu} \sum_{p=n+1}^{\infty} \lambda_p^{\mu} \langle \xi, \phi_p \rangle^2.
\end{aligned}$$

Since $W_p = \langle W, \phi_p \rangle \overset{i.i.d}{\sim} N(0, 1)$ implies that $\mathbb{E}W_p^2 = 1$. We, therefore, acquire the desired result. □

3 Main results

3.1 Ill-posedness

Suppose the terminal data $\xi(x) = \phi_n(x) \in \mathcal{H}^\mu(\Omega)$ has observation values that follow the random model

$$\langle \xi^\epsilon, \phi_p \rangle = \langle \xi, \phi_p \rangle + \epsilon \langle W, \phi_p \rangle, \quad p = 1, \ldots, n.$$

We have statistics estimate for h and f that are obeyed in the models

$$\tilde{\xi}^\epsilon(x) = \sum_{p=1}^{n} \langle \xi^\epsilon, \phi_p \rangle \phi_p(x).$$

The source

$$G(x, t, u^\epsilon) = \sum_{p=1}^{\infty} \frac{B_1}{1 + \lambda_p T^\alpha} \langle u^\epsilon(\cdot, t), \phi_p \rangle \phi_p(x). \tag{14}$$

Problem (1) associated with the terminal function $\tilde{\xi}^\epsilon(x)$ and the source $f^\epsilon(x, t)$ has the mild solution

$$\begin{aligned}
u^\epsilon(x, t) = &\sum_{p=1}^{\infty} \frac{E_{\alpha,1}(-\lambda_p t^\alpha)}{E_{\alpha,1}(-\lambda_p T^\alpha)} \left(\tilde{\xi}_p^\epsilon - \int_0^T E_{\alpha,1}(-\lambda_p (T-s)^\alpha) G_p(u^\epsilon)(s) \mathrm{d}s, \right) \phi_p(x) \\
&+ \sum_{p=1}^{\infty} \left(\int_0^t E_{\alpha,1}(-\lambda_p (t-s)^\alpha) G_p(u^\epsilon)(s) \mathrm{d}s\right) \phi_p(x).
\end{aligned} \tag{15}$$

We consider

$$
\begin{aligned}
u(x,t) - u^{\epsilon}(x,t) &= \sum_{p=1}^{\infty} \frac{E_{\alpha,1}(-\lambda_p t^{\alpha})}{E_{\alpha,1}(-\lambda_p T^{\alpha})}(\xi_p - \tilde{\xi}_p^{\epsilon})\phi_p(x) \\
&- \sum_{p=1}^{\infty} \frac{E_{\alpha,1}(-\lambda_p t^{\alpha})}{E_{\alpha,1}(-\lambda_p T^{\alpha})} \left(\int_0^T E_{\alpha,1}(-\lambda_p (T-s)^{\alpha})(G_p(u)(s) - G_p(u^{\epsilon})(s))\mathrm{d}s, \right) \phi_p(x) \\
&+ \sum_{p=1}^{\infty} \left(\int_0^t E_{\alpha,1}(-\lambda_p (t-s)^{\alpha})(G_p(u)(s) - G_p(u^{\epsilon})(s))\mathrm{d}s \right) \phi_p(x),
\end{aligned}
$$

at the initial time, we have

$$
\begin{aligned}
u(x,0) - u^{\epsilon}(x,0) &= \sum_{p=1}^{\infty} \frac{1}{E_{\alpha,1}(-\lambda_p T^{\alpha})}(\xi_p - \tilde{\xi}_p^{\epsilon})\phi_p(x) \\
&- \sum_{p=1}^{\infty} \frac{1}{E_{\alpha,1}(-\lambda_p T^{\alpha})} \left(\int_0^T E_{\alpha,1}(-\lambda_p (T-s)^{\alpha})(G_p(u)(s) - G_p(u^{\epsilon})(s))\mathrm{d}s, \right) \phi_p(x),
\end{aligned}
$$

yields

$$
\begin{aligned}
&\mathbb{E}\|u(\cdot,0) - u^{\epsilon}(\cdot,0)\|_{L^2(\Omega)} \\
&\quad = \mathbb{E}\left| \frac{1}{E_{\alpha,1}(-\lambda_p T^{\alpha})} \left((\xi_p - \tilde{\xi}_p^{\epsilon}) - \left(\int_0^T E_{\alpha,1}(-\lambda_p (T-s)^{\alpha})(G_p(u)(s) - G_p(u^{\epsilon})(s))\mathrm{d}s \right) \right) \right|^2 .
\end{aligned}
$$

For convenience, we denote

$$
\begin{aligned}
\mathcal{M}_1 &:= \xi_p - \tilde{\xi}_p^{\epsilon} \\
\mathcal{M}_2 &:= \int_0^T E_{\alpha,1}(-\lambda_p (T-s)^{\alpha})(G_p(u)(s) - G_p(u^{\epsilon})(s))\mathrm{d}s
\end{aligned}
$$

then

$$
2\mathbb{E}\|u(\cdot,0) - u^{\epsilon}(\cdot,0)\|_{L^2(\Omega)} \geq \left| \frac{1}{E_{\alpha,1}(-\lambda_p T^{\alpha})} \right|^2 \left(\mathbb{E}\mathcal{M}_1^2 - 2\mathbb{E}\mathcal{M}_2^2 \right). \tag{16}
$$

Since Lemma 2.2, there exist a constant B_1 such that

$$
E_{\alpha,1}(-\lambda_n (T-s)^{\alpha}) \geq \frac{B_1}{1 + \lambda_n (T-s)^{\alpha}} \geq \frac{B_1}{1 + \lambda_n T^{\alpha}},
$$

we can get

$$
2\mathbb{E}\|u(\cdot,0) - u^{\epsilon}(\cdot,0)\|_{L^2(\Omega)} \geq \left| \frac{1 + \lambda_n T^{\alpha}}{B_1} \right|^2 \left(\mathbb{E}\mathcal{M}_1^2 - 2\mathbb{E}\mathcal{M}_2^2 \right).
$$

Where $\mathbb{E}\mathcal{M}_1^2 = \epsilon^2$ and using Hölder inequality, we have

$$
\begin{aligned}
\mathbb{E}\mathcal{M}_2^2 &= \mathbb{E}\left(\int_0^T E_{\alpha,1}(-\lambda_p (T-s)^{\alpha})(G_p(u)(s) - G_p(u^{\epsilon})(s))\mathrm{d}s \right)^2 \\
&\leq \left(\int_0^T E_{\alpha,1}^2(-\lambda_p (T-s)^{\alpha})\mathrm{d}s \right) \left(\int_0^T \mathbb{E}(G_p(u)(s) - G_p(u)(s))^2 \mathrm{d}s \right).
\end{aligned}
$$

Let B_2 be a constant such that $E_{\alpha,1}(-\lambda_p(T-s)^\alpha) \le B_2$. The function f defined as (14), one obtain

$$\mathbb{E}\mathcal{M}_2^2 \le T^2 B_2^2 \left| \frac{1+\lambda_p T^\alpha}{B_1} \right|^2 \|u(\cdot,0) - u^\epsilon(\cdot,0)\|^2_{L^2(\Omega)}. \tag{17}$$

The inequality (16) will become

$$(2+T^2B_2^2)\mathbb{E}\|u(\cdot,0) - u^\epsilon(\cdot,0)\|_{L^2(\Omega)} \ge \left| \frac{1+\lambda_n T^\alpha}{B_1} \right|^2 \epsilon^2.$$

By choosing $\epsilon = 1/\sqrt[3]{\lambda_n}$, and according to Lemma 2.6, we have

$$\lim_{n\to\infty} \mathbb{E}\|\tilde{\xi}^\epsilon - \xi\|^2_{L^2(\Omega)} \le \lim_{n\to\infty} \left(\epsilon^2 n + \frac{1}{\lambda_n^\mu} ||\xi||^2_{\mathcal{H}^\mu} \right) = 0,$$

however

$$\lim_{n\to\infty} \mathbb{E}(2+T^2B_2^2)\mathbb{E}\|u(\cdot,0) - u^\epsilon(\cdot,0)\|_{L^2(\Omega)} \ge \lim_{n\to\infty} \left| \frac{1+\lambda_n T^\alpha}{B_1} \right|^2 \epsilon^2 = +\infty.$$

Consequently, we can deduce that the problem is ill-posed in the sense of Hadamard.

3.2 Approximate solution

The regularized solution for our problem is constructed in this section using the truncation Fourier series method. To obtain the regularized solution, we first introduce a truncation operator. $\mathbb{I}_N f = \sum_{p=1}^N \langle f, \phi_p \rangle \phi(x)$ for all $f \in L^2(\Omega)$. We consider the problem

$$\begin{cases} \partial_t u^\epsilon - D_t^{1-\alpha} \Delta u^\epsilon = \mathbb{I}_N f(x,t,u^\epsilon), & (x,t) \in \Omega \times (0,T) \\ u^\epsilon(x,T) = \mathbb{I}_N \tilde{\xi}^\epsilon(x), & x \in \Omega. \\ u^\epsilon(x,t) = 0, & x \in \partial\Omega, \quad t \in (0,T) \end{cases} \tag{18}$$

$N := N(\epsilon)$ and $n := n(\epsilon)$ respectively are the regularized parameter and the sample size. Similarly, there is also a mild solution to this problem

$$\begin{aligned} u^\epsilon(x,t) = &\sum_{p=1}^N \frac{E_{\alpha,1}(-\lambda_p t^\alpha)}{E_{\alpha,1}(-\lambda_p T^\alpha)} \left(\tilde{\xi}_p^\epsilon - \int_0^T E_{\alpha,1}(-\lambda_p(T-s)^\alpha) G_p(u^\epsilon)(s) \mathrm{d}s \right) \phi_p(x) \\ &+ \sum_{p=1}^N \left(\int_0^t E_{\alpha,1}(-\lambda_p(t-s)^\alpha) G_p(u^\epsilon)(s) \mathrm{d}s \right) \phi_p(x). \end{aligned}$$

Definition 3.1 (Regularized solution)**.** Given real numbers $\mu, \nu > 0$ and the integrals $0 < N < n$. Let $h \in \mathcal{H}^\mu(\Omega)$ and $G(x,t) \in L^\infty(0,T;\mathcal{H}^\nu(\Omega))$, we call

$$\begin{aligned} u^\epsilon(x,t) = &\sum_{p=1}^N \frac{E_{\alpha,1}(-\lambda_p t^\alpha)}{E_{\alpha,1}(-\lambda_p T^\alpha)} \left(\tilde{\xi}_p^\epsilon - \int_0^T E_{\alpha,1}(-\lambda_p(T-s)^\alpha) G_p(u^\epsilon)(s) \mathrm{d}s \right) \phi_p(x) \\ &+ \sum_{p=1}^N \left(\int_0^t E_{\alpha,1}(-\lambda_p(t-s)^\alpha) G_p(u^\epsilon)(s) \mathrm{d}s \right) \phi_p(x) \end{aligned} \tag{19}$$

as the regularized solutin of Problem (1)–(2).

Theorem 3.2 (Existence and uniqueness). *Let B_1 and B_2 be a constant such that*

$$\max\left\{E^2_{\alpha,1}(-\lambda_p(t-s)^\alpha), E^2_{\alpha,1}(-\lambda_p(t-s)^\alpha)\right\} \le B_1, \quad \left(\frac{E_{\alpha,1}(-\lambda_p t^\alpha)}{E_{\alpha,1}(-\lambda_p T^\alpha)}\right)^2 \le B_2.$$

Suppose further that G is global Lipschitz,

$$G\|(\cdot,t,u_1(\cdot,t)) - G(\cdot,t,u_1(\cdot,t))\|^2_{L^2(\Omega)} \le L\|u_1-u_2\|^2_{L^2(\Omega)}.$$

If $B_1T(B_2+1)L < 1$ and Problem (1)–(2) has a solution $u \in L^\infty(0,T;L^2(\Omega))$ then the integral equation (19) existence and uniqueness solution.

Proof theorem 3.2. To establish the existence and uniqueness of the solution to the integral equation, we make use of the Banach fixed point theorem. First, we reformulate the operator $\Phi(u)(x,t)$ in the form:

$$\Phi(u^\epsilon)(x,t) = \sum_{p=1}^N \frac{E_{\alpha,1}(-\lambda_p t^\alpha)}{E_{\alpha,1}(-\lambda_p T^\alpha)}\left(\tilde{\xi}^\epsilon_p - \int_0^T E_{\alpha,1}(-\lambda_p(T-s)^\alpha)G_p(u^\epsilon)(s)ds\right)\phi_p(x)$$
$$+ \sum_{p=1}^N \left(\int_0^t E_{\alpha,1}(-\lambda_p(t-s)^\alpha)G_p(u^\epsilon)(s)ds\right)\phi_p(x). \tag{20}$$

For any $u^\epsilon_1, u^\epsilon_2 \in \mathbf{Z}(0,T;H)$, the Parseval identity leads to

$$\|\Phi(u^\epsilon_1)(\cdot,t) - \Phi(u^\epsilon_2)(\cdot,t)\|^2_{L^2(\Omega)}$$
$$= \sum_{p=1}^N \left(\frac{E_{\alpha,1}(-\lambda_p t^\alpha)}{E_{\alpha,1}(-\lambda_p T^\alpha)}\right)^2 \left(\int_0^T E_{\alpha,1}(-\lambda_p(T-s)^\alpha)(G_p(u^\epsilon_1)(s) - G_p(u^\epsilon_2)(s))ds\right)^2$$
$$+ \sum_{p=1}^N \left(\int_0^t E_{\alpha,1}(-\lambda_p(t-s)^\alpha)(G_p(u^\epsilon_1)(s) - G_p(u^\epsilon_2)(s))ds\right)^2,$$

First, note that there exists a constant B_1 and B_2 such that

$$\max\left\{E^2_{\alpha,1}(-\lambda_p(t-s)^\alpha), E^2_{\alpha,1}(-\lambda_p(t-s)^\alpha)\right\} \le B_1, \quad \left(\frac{E_{\alpha,1}(-\lambda_p t^\alpha)}{E_{\alpha,1}(-\lambda_p T^\alpha)}\right)^2 \le B_2$$

and using the Hölder inequality, we arrive at

$$\|\Phi(u^\epsilon_1)(\cdot,t) - \Phi(u^\epsilon_2)(\cdot,t)\|^2_{L^2(\Omega)}$$
$$\le B_1B_2T\sum_{p=1}^N \int_0^T \left|G_p(u^\epsilon_1)(s) - G_p(u^\epsilon_2)(s)\right|^2 ds$$
$$+ B_1T\sum_{p=1}^N \int_0^t \left|G_p(u^\epsilon_1)(s) - G_p(u^\epsilon_2)(s)\right|^2 ds,$$

we find that

$$\|\Phi(u^\epsilon_1)(\cdot,t) - \Phi(u^\epsilon_2)(\cdot,t)\|^2_{L^2(\Omega)}$$
$$\le B_1T(B_2+1)\int_0^T \|G(\cdot,t,u^\epsilon_1(\cdot,t))(s) - G_p(\cdot,t,u^\epsilon_1(\cdot,t))(s)\|^2_{L^2(\Omega)}ds.$$

As for the global Lipschitz property of f, we can get

$$\|\Phi(u_1^\epsilon)(\cdot,t)-\Phi(u_2^\epsilon)(\cdot,t)\|^2_{L^2(\Omega)} \leq B_1T(B_2+1)L\int_0^T \|u_1^\epsilon(\cdot,t)-u_2^\epsilon(\cdot,t)\|^2_{L^2(\Omega)}\mathrm{d}s.$$

One can confirm that

$$\|\Phi(u_1^\epsilon)(\cdot,t)-\Phi(u_2^\epsilon)(\cdot,t)\|^2_{L^2(\Omega)} \leq B_1T^2(B_2+1)L\|u_1^\epsilon-u_2^\epsilon\|^2_{L^\infty(0,T;L^2(\Omega))}.$$

Since $B_1T^2(B_2+1)L$, using Banach fixed point theorem, we show that $\Phi(u^\epsilon)=u^\epsilon$ has a unique solution. □

3.3 Estimation of errors

Theorem 3.3. *Given real numbers $\mu,\nu,\tau>0$ and the integrals $0<N<n$. Let $h\in\mathcal{H}^\mu(\Omega)$ and G is a global Lipschitz. If Problem (1)–(2) has the solution $u\in\mathbf{Z}(0,T;\mathcal{H}^\tau(\Omega))$ and B_1, B_2 are constants which are choose as Theorem 3.2 and satisfies $(B_2L+1)B_1T^2<1$. Suppose further that G is global Lipschitz, then*

$$\|u(\cdot,t)-u^\epsilon(\cdot,t)\|^2_{\mathbf{Z}(0,T;L^2(\Omega))} \leq \frac{4}{1-(B_1B_2T^2L+B_1T^2)}\left[B_1\left(\epsilon^2n+\frac{1}{\lambda_n^\mu}||\xi||^2_{\mathcal{H}^\mu(\Omega)}\right)+\frac{1}{\lambda_N^\tau}\|u\|^2_{\mathbf{Z}_{\mathcal{H}^\tau(\Omega)}}\right]$$

The regularization parameter N and the sample size n are chosen such that

$$\lim_{\epsilon\to0^+}N(\epsilon)=\lim_{\epsilon\to0^+}n(\epsilon)=+\infty,\quad \lim_{\epsilon\to0}\epsilon^2n=0,\quad 0<N(\epsilon)<n(\epsilon).$$

Proof theorem 3.3. We have

$$\begin{aligned}u(x,t)-u^\epsilon(x,t)=&\sum_{p=1}^N\frac{E_{\alpha,1}(-\lambda_pt^\alpha)}{E_{\alpha,1}(-\lambda_pT^\alpha)}(\xi_p-\tilde{\xi}_p^\epsilon)\phi_p(x)\\&-\sum_{p=1}^N\frac{E_{\alpha,1}(-\lambda_pt^\alpha)}{E_{\alpha,1}(-\lambda_pT^\alpha)}\left(\int_0^T E_{\alpha,1}(-\lambda_p(T-s)^\alpha)(G_p(u)(s)-G_p(u^\epsilon)(s))\mathrm{d}s\right)\phi_p(x)\\&+\sum_{p=1}^N\left(\int_0^t E_{\alpha,1}(-\lambda_p(t-s)^\alpha)(G_p(u)(s)-G_p(u^\epsilon)(s))\mathrm{d}s\right)\phi_p(x)\\&+\sum_{p=N+1}^\infty u_p(t)\phi(x)=:\mathcal{Q}_1(x,t)-\mathcal{Q}_2(x,t)+\mathcal{Q}_3(x,t)+\mathcal{Q}_4(x,t).\end{aligned}\tag{21}$$

First, we realize that from Lemma (2.2), there exists a constant B_2 such that

$$\left|\frac{E_{\alpha,1}(-\lambda_pt^\alpha)}{E_{\alpha,1}(-\lambda_pT^\alpha)}\right|^2\leq B_2.$$

To facilitate the reader, we divided proof into 4 steps:

Step 1. We have

$$\mathbb{E}\|\mathcal{Q}_1(\cdot,t)\|^2_{L^2(\Omega)}=\sum_{p=1}^N\left|\frac{E_{\alpha,1}(-\lambda_pt^\alpha)}{E_{\alpha,1}(-\lambda_pT^\alpha)}\right|^2(\xi_p-\tilde{\xi}_p^\epsilon)^2\leq B_1\|h-\tilde{\xi}^\epsilon\|^2_{L^2(\Omega)}.$$

Lemma 2.6 leads to

$$\mathbb{E}\|\mathcal{Q}_1(\cdot,t)\|^2_{L^2(\Omega)} \le B_2\left(\epsilon^2 n + \frac{1}{\lambda_n^{\mu}}||\xi||^2_{\mathcal{H}^{\mu}(\Omega)}\right). \tag{22}$$

Step 2. We also have

$$\mathbb{E}\|\mathcal{Q}_2(\cdot,t)\|^2_{L^2(\Omega)} = \sum_{p=1}^{N}\left|\frac{E_{\alpha,1}(-\lambda_p t^{\alpha})}{E_{\alpha,1}(-\lambda_p T^{\alpha})}\right|^2\left(\int_0^T E_{\alpha,1}(-\lambda_p(T-s)^{\alpha})(G_p(u)(s)-G_p(u^{\epsilon})(s))\mathrm{d}s\right)^2.$$

Using Hölder inequality, we obtain

$$\begin{aligned}&\mathbb{E}\left(\int_0^T E_{\alpha,1}(-\lambda_p(T-s)^{\alpha})(G_p(u)(s)-G_p(u^{\epsilon})(s))\mathrm{d}s\right)^2\\ &\le\left(\int_0^T E^2_{\alpha,1}(-\lambda_p(T-s)^{\alpha})\mathrm{d}s\right)\left(\int_0^T \mathbb{E}(G_p(u)(s)-G_p(u^{\epsilon})(s))^2\mathrm{d}s\right).\end{aligned}$$

Since Lipschitz property of the source, we have

$$\mathbb{E}\|\mathcal{Q}_2(\cdot,t)\|^2_{L^2(\Omega)} \le B_1B_2T^2L\|u-u^{\epsilon}\|^2_{\mathbf{Z}(0,T;L^2(\Omega))}.$$

Step 3. Similarly, we have

$$\mathbb{E}\|\mathcal{Q}_3(\cdot,t)\|^2_{L^2(\Omega)} = \sum_{p=1}^{N}\mathbb{E}\left(\int_0^t E_{\alpha,1}(-\lambda_p(t-s)^{\alpha})(G_p(u)(s)-G_p(u^{\epsilon})(s))\mathrm{d}s\right)^2.$$

Note that $E^2_{\alpha,1}(-\lambda_p(t-s)^{\alpha}) \le B_1$. Thus

$$\mathbb{E}\|\mathcal{Q}_3(\cdot,t)\|^2_{L^2(\Omega)} \le B_1T^2\|u-u^{\epsilon}\|^2_{\mathbf{Z}(0,T;L^2(\Omega))} \tag{23}$$

Step 4.

$$\mathcal{Q}_4(x,t) = \sum_{p=N+1}^{\infty}\frac{1}{\lambda_n}\lambda_p u_p(t)\phi(x)$$

then

$$\|\mathcal{Q}_4(\cdot,t)\|^2_{L^2(\Omega)} = \sum_{p=N+1}^{\infty}\frac{1}{\lambda_p^{\tau}}\lambda_p^{\tau}u_p^2(t)$$

so we have

$$\mathbb{E}\|\mathcal{Q}_4(\cdot,t)\|^2_{L^2(\Omega)} \le \frac{1}{\lambda_N^{\tau}}\|u\|^2_{\mathbf{Z}(0,T;\mathcal{H}^{\tau}(\Omega))}.$$

Taking (22)–(3.3) into account, we have

$$\begin{aligned}&\frac{1}{4}\mathbb{E}\|u(\cdot,t)-u^{\epsilon}(\cdot,t)\|^2_{L^2(\Omega)}\\ &\le\|\mathcal{Q}_1(\cdot,t)\|^2_{L^2(\Omega)}+\|\mathcal{Q}_2(\cdot,t)\|^2_{L^2(\Omega)}+\|\mathcal{Q}_3(\cdot,t)\|^2_{L^2(\Omega)}+\|\mathcal{Q}_4(\cdot,t)\|^2_{L^2(\Omega)}\\ &\le B_1\left(\epsilon^2 n+\frac{1}{\lambda_n^{\mu}}||\xi||^2_{\mathcal{H}^{\mu}(\Omega)}\right)+(B_1B_2T^2L+B_1T^2)\|u-u^{\epsilon}\|^2_{\mathbf{Z}(0,T;L^2(\Omega))}\\ &\quad+\frac{1}{\lambda_N^{\tau}}\|u\|^2_{\mathbf{Z}(0,T;\mathcal{H}^{\tau}(\Omega))}.\end{aligned}$$

By rearranging, we get the result of the theorem. □

4 Declarations

Funding

Not applicable.

Competing Interests

The author(s) declare that they have no competing interests.

Ethical Approval

Not applicable.

Authors' Contributions

All authors contributed equally. All the authors read and approved the final manuscript.

Availability Data and Materials

Not applicable.

Acknowledgements

The authors thank Dr. Nguyen Duc Phuong for helping us complete this research. The authors The authors would like to thank the anonymous reviewer for his/her comments.

References

[1] Nguyen Huy Tuan, Nguyen Hoang Tuan, Dumitru Baleanu, and Tran Ngoc Thach, *On a backward problem for fractional diffusion equation with riemann-liouville derivative*, Mathematical Methods in the Applied Sciences **43** (2019), no. 3, 1292–1312, doi 10.1002/mma.5943.

[2] Nguyen Anh Triet, Tran Thanh Binh, Nguyen Duc Phuong, Dumitru Baleanu, and Nguyen Huu Can, *Recovering the initial value for a system of nonlocal diffusion equations with random noise on the measurements*, Mathematical Methods in the Applied Sciences **44** (2020), no. 6, 5188–5209, doi 10.1002/mma.7102.

[3] Tran Bao Ngoc, Yong Zhou, Donal O'Regan, and Nguyen Huy Tuan, *On a terminal value problem for pseudoparabolic equations involving riemann–liouville fractional derivatives*, Applied Mathematics Letters **106** (2020), 106373, doi 10.1016/j.aml.2020.106373.

[4] Nguyen Duc Phuong, Dumitru Baleanu, Tran Thanh Phong, and Le Dinh Long, *Recovering the source term for parabolic equation with nonlocal integral condition*, Mathematical Methods in the Applied Sciences **44** (2021), no. 11, 9026–9041, doi 10.1002/mma.7331.

[5] Nguyen Duc Phuong, Tran Binh, and Nguyen Luc, *On a final value problem for parabolic equation on the sphere with linear and nonlinear source*, Advances in the Theory of Nonlinear Analysis and its Application **4** (2020), no. 3, 143–151, doi 10.31197/atnaa.753458.

[6] Igor Podlubny, *Fractional differential equations*, Academic Press, New York, 1999.

[7] Guang An Zou and Bo Wang, *Stochastic burgers' equation with fractional derivative driven by multiplicative noise*, Computers & Mathematics with Applications **74** (2017), no. 12, 3195–3208, doi 10.1016/j.camwa.2017.08.023.

[8] Nguyen Duc Phuong, Nguyen Huy Tuan, Zakia Hammouch, and Rathinasamy Sakthivel, *On a pseudo-parabolic equations with a non-local term of the kirchhoff type with random gaussian white noise*, Chaos, Solitons and Fractals **145** (2021), 110771, doi 10.1016/j.chaos.2021.110771.