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Two-weighted inequalities for maximal commutators in generalized weighted Morrey spaces on spaces of homogeneous type

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Abstract

In this paper we give a characterization of two-weighted inequalities for maximal commutators in generalized weighted Morrey spaces on spaces of homogeneous type $\mathcal{M}_{\omega}^{p,\varphi}(X)$. We prove the boundedness of maximal commutators $[M,b]$ from the spaces $\mathcal{M}_{\omega_1^{\delta}}^{p,\varphi_1}(X)$ to the spaces $\mathcal{M}_{\omega_2^{\delta}}^{p,\varphi_2}(X)$, where $1<p<\infty$, $0<\delta<1$ and $(\omega_1,\omega_2)\in\widetilde{A}_p(X)$.

Key words: Maximal operator; commutator; generalized weighted Morrey space; spaces of homogeneous type; regularity
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R. Coifman and G. Weiss introduced certain topological measure spaces which are equipped with a metric which is compatible with the given measure in a sense in the 1970s. These spaces are called spaces of homogeneous type. In this work, we find necessary and sufficient conditions for the boundedness of Hardy-Littlewood maximal commutators in generalized weighted Morrey spaces on spaces of homogeneous type. As a generalization of $L_p(\mathbb{R}^n)$, the classical Morrey spaces were introduced by Morrey [30] to study the local behavior of solutions to second-order elliptic partial differential equations. Moreover, various Morrey spaces are defined in the process of study. Guliyev, Mizuhara and Nakai [15, 29, 32] introduced generalized Morrey spaces $M^{p,\varphi}(\mathbb{R}^n)$ (see, also [16, 17, 38]).

Recently, Komori and Shirai [27] defined the weighted Morrey spaces $L_w^{p,\kappa}(\mathbb{R}^n)$ and studied the boundedness of some classical operators such as the Hardy- Littlewood maximal operator, the Calderón-Zygmund operator on these spaces. Also, Guliyev in [18] first introduced the generalized weighted Morrey spaces $M_w^{p,\varphi}(\mathbb{R}^n)$ and studied the boundedness of the sublinear

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operators and their higher order commutators generated by Calderón-Zygmund operators and Riesz potentials in these spaces (see, also [21, 25]). Note that, Guliyev [18] gave the concept of generalized weighted Morrey space which could be viewed as an extension of both $M_w^{p,\varphi}(\mathbb{R}^n)$ and $L_w^{p,\kappa}(\mathbb{R}^n)$.

We say that $X = (X, d, \mu)$ is a space of homogeneous type in the sense of Coifman and Weiss [10] if d is a quasi-metric on X and μ is a positive measure satisfying the doubling condition, i.e. X is a topological space endowed with a quasi-metric d and a positive measure μ such that

$$\begin{aligned} d(x,y) &= d(y,x) \geq 0 \text{ for all } x,y \in X, \\ d(x,y) &= 0 \text{ if and only if } x = y\,, \\ d(x,y) &\leq C_k\left[d(x,z) + d(z,y)\right] \text{ for all } x,y,z \in X, \end{aligned}$$

the balls $B(x,r) = \{y \in X : d(x,y) < r\}\,, r > 0$, form a basis of neighborhoods of point x, μ is defined on a σ-algebra of subsets of X which contains the balls, and

$$0 < \mu(B(x,2r)) < C_\mu \mu(B(x,r)) < \infty,$$

where $C_k, C_\mu \geq 1$ are constants independent of $x,y,z \in X$ and $r > 0$. As usual, the dilation of a ball $B = B(x,r)$ will be denoted by $\lambda B = B(x, \lambda r)$ for every $\lambda > 0$.

Throughout this paper we always assume that $\mu(X) = \infty$, the space of compactly supported continuous function is dense in $L_1(X,\mu)$ and that X is N-homogeneous $(N > 0)$, i.e.

$$C_1 r^N \leq \mu(B(x,r)) \leq C_2 r^N,$$

where $C_i \geq 1\,(i = 1,2)$ are constants independent of x and r. The n-dimensional Euclidean space $\mathbb{R}^n$ is n-homogeneous.

Let (X, d, μ) be a homogeneous space, $1 \leq p < \infty$, φ be a positive measurable function on $(0,\infty)$ and ω be a non-negative measurable function on X. We denote by $\mathcal{M}_\omega^{p,\varphi}$ the generalized weighted Morrey space on spaces of homogeneous type, the space of all functions $f \in L_{p,\omega}^{loc}(X)$ with finite norm

$$\|f\|_{\mathcal{M}_\omega^{p,\varphi}} = \sup_{x\in X, r>0} \frac{1}{\varphi(x,r)\|\omega\|_{L_p(B(x,r))}} \|f\|_{L_{p,\omega}(B(x,r))},$$

where the supremum is taken over all balls $B(x,r)$ in X and $L_{p,\omega}(B(x,r))$ denotes the weighted L_p-space of measurable functions f for which

$$\|f\|_{L_{p,\omega}(B(x,r))} \equiv \|f\chi_{B(x,r)}\|_{L_{p,\omega}(X)} = \left(\int\limits_{B(x,r)} |f(y)|^p \omega(y) d\mu(y) \right)^{\frac{1}{p}}.$$

Moreover, by $W\mathcal{M}_\omega^{p,\varphi}$ we denote the weak generalized weighted Morrey space on spaces of homogeneous type of all functions $f \in WL_{p,\omega}^{loc}(X)$ with finite norm

$$\|f\|_{W\mathcal{M}_\omega^{p,\varphi}} = \sup_{x\in X, r>0} \frac{1}{\varphi(x,r)\|\omega\|_{L_p(B(x,r))}} \|f\|_{WL_{p,\omega}(B(x,r))},$$

where $WL_{p,\omega}(B(x,r))$ denotes the weak weighted L_p-space of measurable functions f for which

$$\|f\|_{WL_{p,\omega}(B(x,r))} \equiv \|f\chi_{B(x,r)}\|_{WL_{p,\omega}(X)} = \sup_{t>0} t \left(\int\limits_{\{y\in B(x,r): |f(y)|>t\}} |f(y)|^p \omega(y) d\mu(y) \right)^{\frac{1}{p}}.$$

Note that if $\omega(x) = \chi_{B(x,r)}$, then $\mathcal{M}^{p,\varphi}_{\omega}(X) = \mathcal{M}^{p,\varphi}(X)$ is the generalized Morrey space and if $\varphi(x,r) = \left(\frac{r^{\lambda}}{\mu(B(x,r))}\right)^{\frac{1}{p}}$, then $\mathcal{M}^{p,\varphi}_{\omega}(X) = L_{p,\lambda}(X)$ is the classic Morrey space.

We now recall the definition of Hardy-Littlewood operator on space of homogeneous type.

Let f be a locally integrable function on X. The so-called of Hardy-Littlewood maximal function is defined by the formula

$$Mf(x) = \sup_{r>0} \frac{1}{\mu\left(B\left(x,r\right)\right)} \int\limits_{B(x,r)} |f(y)| d\mu\left(y\right),$$

where $\mu\left(B\left(x,r\right)\right)$ is measure of the ball $B(x,r)$.

The study of maximal operators is one of the most important topics in harmonic analysis. These significant non-linear operators, whose behavior are very informative in particular in differentiation theory, provided the understanding and the inspiration for the development of the general class of singular and potential operators.

In this paper we aim to give a characterization of two-weighted inequalities for maximal commutators in generalized weighted Morrey spaces on spaces of homogeneous type. Two-weight norm inequalities for fractional maximal operators and singular integrals on Lebesgue spaces were widely studied (see, for example [11, 12, 14, 26, 28]). The weighted norm inequalities with different types of weights on Morrey spaces were also studied (see, for example [22, 33, 37]). The two-weight norm inequalities for maximal operator, fractional maximal operator, singular integral operators, Riesz potential and fractional integral operators on Morrey-type spaces were studied by many authors (see, for example [3, 7, 8, 35, 39]). Also, two-weighted inequalities for maximal operator in generalized weighted Morrey spaces on spaces of homogeneous type were studied in [4].

In the sequel we use the letter C for a positive constant, independent of appropriate parameters and not necessary the same at each occurrence. For every $p \in [1,\infty]$, we denote p' the conjugate of p, i.e., $\frac{1}{p} + \frac{1}{p'} = 1$. $\mathfrak{M}(\mathbb{R}_+)$, $\mathfrak{M}^+(\mathbb{R}_+)$ and $\mathfrak{M}^+(\mathbb{R}_+;\uparrow)$
stand for the set of Lebesgue-measurable functions on $\mathbb{R}_+$, and its subspaces of nonnegative and nonnegative non-decreasing functions, respectively.

1 Preliminaries

Let (X,d,μ) be space of N-homogeneous type as mentioned in Section 1. We now recall the definition of A_p weight functions.

Definition 1.1. The weight function ω belongs to the class $A_p(X)$ for $1 \le p < \infty$, if

$$\sup_{x\in X, r>0} \left(\mu\left(B\left(x,r\right)\right)^{-1} \int\limits_{B(x,r)} \omega^{p}(y) d\mu\left(y\right)\right)^{\frac{1}{p}} \left(\mu\left(B\left(x,r\right)\right)^{-1} \int\limits_{B(x,r)} \omega^{-p'}(y) d\mu\left(y\right)\right)^{\frac{1}{p'}}$$

is finite and ω belongs to $A_1(X)$, if there exists a positive constant C such that for any $x \in X$ and $r > 0$

$$\mu\left(B\left(x,r\right)\right)^{-1} \int\limits_{B(x,r)} \omega(y) d\mu\left(y\right) \le C \operatorname*{ess\,sup}_{y\in B(x,r)} \frac{1}{\omega(y)}.$$

The weight function (ω_1, ω_2) belongs to the class $\widetilde{A}_p(X)$ for $1 < p < \infty$, if

$$\sup_{x \in X, r>0} \left(\mu\left(B\left(x,r\right)\right)^{-1} \int\limits_{B(x,r)} \omega_2^p(y) d\mu\left(y\right) \right)^{\frac{1}{p}} \left(\mu\left(B\left(x,r\right)\right)^{-1} \int\limits_{B(x,r)} \omega_1^{-p'}(y) d\mu\left(y\right) \right)^{\frac{1}{p'}}$$

is finite.

The following theorem was proved in [31](Theorem 1, p. 209 and Theorem 2, p. 215).

Theorem 1.2. *Let $1 \leq p < \infty$.*
1) Then the operator M is bounded in $L_{p,\omega}(X)$ if and only if $\omega \in A_p(X)$.
2) Then the operator M is bounded from $L_{1,\omega}(X)$ to $WL_{1,\omega}(X)$ if and only if $\omega \in A_1(X)$.

Theorem 1.3. *[4] Let $1 < p < \infty$, $0 < \delta < 1$ and $(\omega_1, \omega_2) \in \widetilde{A}_p(X)$. Then*

$$\|Mf\|_{L_{p,\omega_2^\delta}(B(x,r))} \leq C \|\omega_2^\delta\|_{L_p(B(x,r))} \sup_{t>r} \|f\|_{L_{p,\omega_1^\delta}(B(x,t))} \|\omega_2^\delta\|^{-1}_{L_p(B(x,t))}$$

for every $f \in L_{p,\omega_1^\delta}(X)$, where C does not depend on f, x and r.

Theorem 1.4. *[4] Let $1 < p < \infty$, $0 < \delta < 1$, $(\omega_1, \omega_2) \in \widetilde{A}_p(X)$ and the function $\varphi_1(x,r)$ and $\varphi_2(x,r)$ satisfy the condition*

$$\sup_{t>r} \frac{\operatorname{ess\,inf}_{t<s<\infty} \varphi_1(x,s) \|\omega_1^\delta\|_{L_p(B(x,s))}}{\|\omega_2^\delta\|_{L_p(B(x,t))}} \leq C \varphi_2(x,r),$$

where C does not depend on x and t.
Then the operator M is bounded from the space $\mathcal{M}^{p,\varphi_1}_{\omega_1^\delta}(X)$ to the space $\mathcal{M}^{p,\varphi_2}_{\omega_2^\delta}(X)$.

Lemma 1.5. *[34] Let $1 < p < \infty$ and $(\omega_1, \omega_2) \in \widetilde{A}_p(X)$, then $(\omega_2^{-1}, \omega_1^{-1}) \in \widetilde{A}_{p'}(X)$, with $\frac{1}{p} + \frac{1}{p'} = 1$.*

Lemma 1.6. *[34] Let $1 < p < \infty$, $0 < \delta < 1$ and $(\omega_1, \omega_2) \in \widetilde{A}_p(X)$. If $\frac{q-1}{p-1} = \delta$, then $(\omega_1, \omega_2) \in \widetilde{A}_{q'}(X)$, with $\frac{1}{p} + \frac{1}{p'} = 1$.*

Corollary 1.7. *[34] Let $1 < p < \infty$, $0 < \delta < 1$ and $(\omega_1, \omega_2) \in \widetilde{A}_p(X)$, then the operator M is bounded from $L_{p,\omega_1^\delta}(X)$ to $L_{p,\omega_2^\delta}(X)$.*

Let $M^\sharp$ be the sharp maximal function defined by

$$M^\sharp f(x) = \sup_{r>0} \mu\left(B\left(x,r\right)\right)^{-1} \int\limits_{B(x,r)} |f(y) - f_{B(x,r)}| d\mu\left(y\right),$$

where $f_{B(x,r)}(x) = \mu\left(B\left(x,r\right)\right)^{-1} \int\limits_{B(x,r)} f(y) d\mu\left(y\right)$.

Lemma 1.8. *Let $1 < p < \infty$ and $\omega \in A_p(X)$. Then*

$$\|f\omega\|_{L_p} \leq C \|\omega M^\sharp f\|_{L_p}$$

with a constant $C > 0$ not depending on f.

Definition 1.9. We define the $BMO(X)$ space as the set of all locally integrable functions f such that

$$\|f\|_{BMO} = \sup_{x\in X,\, r>0} \mu\left(B\left(x,r\right)\right)^{-1} \int\limits_{B(x,r)} |f(y) - f_{B(x,r)}| d\mu\left(y\right) < \infty$$

or

$$\|f\|_{BMO} = \inf_{C} \sup_{x\in X,\, r>0} \mu\left(B\left(x,r\right)\right)^{-1} \int\limits_{B(x,r)} |f(y) - C| d\mu\left(y\right) < \infty.$$

Definition 1.10. Given a measurable function b the maximal commutator is defined by

$$M_b(f)(x) = \sup_{r>0} \mu\left(B\left(x,r\right)\right)^{-1} \int\limits_{B(x,r)} |b(x) - b(y)||f(y)| d\mu\left(y\right)$$

for all $x \in X$.

This operator plays an important role in the study of commutators of singular integral operators with BMO symbols.

Definition 1.11. Given a measurable function b the commutator of the Hardy-Littlewood maximal operator M and b is defined by

$$[M,b]f(x) = M(bf)(x) - b(x)Mf(x)$$

for all $x \in X$.

This operator arises, for example, when one tries to give a meaning to the product of a function in H^1 and a function in BMO (which may not be a locally integrable function). Operators M_b and $[M,b]$ essentially differ from each other. For example, M_b is a positive and sublinear operator, but $[M,b]$ is neither positive nor sublinear. However, if b satisfies some additional conditions, then operator M_b controls $[M,b]$. The maximal operators and their commutators were widely investigated in various Morrey-type spaces (see, for example, [2, 5, 6, 13, 36, 40, 41]).

Definition 1.12. We define the $BMO_{p,\omega}(X)$ $(1 \le p < \infty)$ space as the set of all locally integrable functions f such that

$$\|f\|_{BMO_{p,\omega}} = \sup_{x\in X,\, r>0} \frac{\|(f(\cdot) - f_{B(x,r)})\chi_{B(x,r)}\|_{L_{p,\omega}(X)}}{\|\omega\|_{L_p(B(x,r))}}$$

or

$$\|f\|_{BMO_{p,\omega}} = \sup_{x\in X,\, r>0} \frac{1}{\mu\left(B\left(x,r\right)\right)} \|(f(\cdot) - f_{B(x,r)})\chi_{B(x,r)}\|_{L_{p,\omega}(X)} \|\omega^{-1}\|_{L_{p'}(B(x,r))} < \infty.$$

The proof of the following theorem can be obtained similarly with the Corollary 4.5 in [24].

Theorem 1.13. *Let $1 \le p < \infty$ and ω be a Lebesgue measurable function. If $\omega \in A_p(X)$, then the norms $\|\cdot\|_{BMO_{p,\omega}}$ and $\|\cdot\|_{BMO}$ are mutually equivalent.*

We will need the following lemma while proving our main theorems.

Lemma 1.14. *[23] Let $b \in BMO(X)$. Then there is a constant $C > 0$ such that*

$$\left|b_{B(x,r)} - b_{B(x,t)}\right| \le C\|b\|_{BMO} \ln\frac{t}{r} \quad \textit{for} \quad 0 < 2r < t,$$

where C is independent of b, x, r, and t.

Let $L_{\infty,\omega}(\mathbb{R}_+)$ be the weighted L_∞-space with the norm

$$\|g\|_{L_{\infty,\omega}(\mathbb{R}_+)} = \operatorname*{ess\,sup}_{t>0} \omega(t)g(t).$$

We denote

$$\mathbb{A} = \left\{ \varphi \in \mathfrak{M}^+(\mathbb{R}_+;\uparrow) : \lim_{t\to 0+} \varphi(t) = 0 \right\}.$$

Let u be a continuous and non-negative function on $\mathbb{R}_+$. We define the supremal operator $\overline{S}_u$ by

$$(\overline{S}_u g)(t) := \|u\, g\|_{L_\infty(0,t)}, \ \ t \in (0,\infty).$$

The following theorem was proved in [9].

Theorem 1.15. *[9] Suppose that v_1 and v_2 are nonnegative measurable functions such that $0 < \|v_1\|_{L_\infty(0,t)} < \infty$ for every $t > 0$. Let u be a continuous nonnegative function on $\mathbb{R}$. Then the operator $\overline{S}_u$ is bounded from $L_{\infty,v_1}(\mathbb{R}_+)$ to $L_{\infty,v_2}(\mathbb{R}_+)$ on the cone $\mathbb{A}$ if and only if*

$$\left\| v_2 \overline{S}_u \left(\|v_1\|^{-1}_{L_\infty(0,\cdot)} \right) \right\|_{L_\infty(\mathbb{R}_+)} < \infty.$$

We will use the following statement on the boundedness of the weighted Hardy operator

$$H_w g(t) := \int_0^t g(s)w(s)ds, \ \ H_w^* g(t) := \int_t^\infty g(s)w(s)ds, \quad 0 < t < \infty,$$

where w is a weight.

The following theorem was proved in [19].

Theorem 1.16. *[19] Let v_1, v_2 and w be weights on $(0,\infty)$ and $v_1(t)$ be bounded outside a neighborhood of the origin. The inequality*

$$\sup_{t>0} v_2(t) H_w^* g(t) \le C \sup_{t>0} v_1(t) g(t)$$

holds for some $C > 0$ for all non-negative and non-decreasing g on $(0,\infty)$ if and only if

$$B := \sup_{t>0} v_2(t) \int_t^\infty \frac{w(s)ds}{\operatorname*{ess\,sup}_{s<\tau<\infty} v_1(\tau)} < \infty.$$

Theorem 1.17. *[19, 20] Let v_1, v_2 and w be weights on $(0,\infty)$ and $v_1(t)$ be bounded outside a neighborhood of the origin. The inequality*

$$\sup_{t>0} v_2(t) H_w g(t) \le C \sup_{t>0} v_1(t) g(t) \tag{1}$$

holds for some $C > 0$ for all non-negative and non-decreasing g on $(0,\infty)$ if and only if

$$B := \sup_{t>0} v_2(t) \int_0^t \frac{w(s)ds}{\sup_{0<\tau<s} v_1(\tau)} < \infty.$$

Moreover, the value $C = B$ is the best constant for (1).

2 Two-weighted inequalities for maximal commutators in generalized weighted Morrey spaces on spaces of homogeneous type

In this section we prove two-weighted inequalities for maximal commutators in generalized weighted Morrey spaces on spaces of homogeneous type.

Lemma 2.1. *[1] Let b be any non-negative locally integrable function. Then*

$$|[M,b]f(x)| \le M_b(f)(x), \ \ x \in X$$

holds for all $f \in L_1^{loc}(X)$.

Theorem 2.2. *[1] Let $b \in BMO(X)$. Suppose that Y is a Banach space of measurable functions defined on X. Assume that M is bounded on Y. Then the operator M_b is bounded on Y, and the inequality*

$$\|M_b f\|_Y \le C\|b\|_{BMO}\|f\|_Y$$

holds with constant C independent of f.

Corollary 2.3. *Let $1 \le p < \infty$, $b \in BMO(X)$ and $\omega \in A_p(X)$, then the operator M_b is bounded in $L_{p,\omega}(X)$.*

Theorem 2.4. *Let $1 < p < \infty$, $0 < \delta < 1$, $b \in BMO(X)$ and $(\omega_1,\omega_2) \in \widetilde{A}_p(X)$, $\omega_1 \in A_p(X)$, then the operator M_b is bounded from $L_{p,\omega_1^\delta}(X)$ to $L_{p,\omega_2^\delta}(X)$.*

Proof. Let $f \in L_{p,\omega_1^\delta}(X)$, $b \in BMO(X)$. The inequality ([1], Corollary 1.11), is valid

$$M_b f(x) \le C\|b\|_{BMO} M^2 f(x).$$

By the this inequality, Corollary 1.7, Corollary 2.3 and $(\omega_1,\omega_2) \in \widetilde{A}_p(X)$, $\omega_1 \in A_p(X)$ conditions, we have

$$\|M_b f\|_{L_{p,\omega_2^\delta}(X)} \le C\|b\|_{BMO} \left\|M^2 f\right\|_{L_{p,\omega_2^\delta}(X)}$$

$$\le C\|b\|_{BMO} \left\|Mf\right\|_{L_{p,\omega_1^\delta}(X)} \le C_1\|b\|_{BMO} \left\|f\right\|_{L_{p,\omega_1^\delta}(X)},$$

where $M^2 f(x) = M(Mf(x))$.

Theorem 2.5. *Let $1 < p < \infty$, $0 < \delta < 1$, $b \in BMO(X)$ and $(\omega_1,\omega_2) \in \widetilde{A}_p(X)$, $\omega_1,\omega_2 \in A_p(X)$, then*

$$\|M_b f\|_{L_{p,\omega_2^\delta}(B(x,r))} \le C\|b\|_{BMO}\|\omega_2^\delta\|_{L_p(B(x,r))} \sup_{t>r}\left(1+\ln\frac{t}{r}\right)\frac{\|f\|_{L_{p,\omega_1^\delta}(B(x,t))}}{\|\omega_2^\delta\|_{L_p(B(x,t))}}, \tag{1}$$

for every $f \in L_{p,\omega_1^\delta}(X)$, where C does not depend on f, x and r.

Proof. We represent function f as

$$f = f_1 + f_2, \quad f_1(y) = f(y)\chi_{B(x,2kr)}(y), \quad f_2(y) = f(y)\chi_{X\setminus B(x,2kr)}(y), \ r>0,$$

where k is the constant from the quasi-triangle inequality and have

$$\|M_b f\|_{L_{p,\omega_2^\delta}(B(x,r))} \le \|M_b f_1\|_{L_{p,\omega_2^\delta}(B(x,r))} + \|M_b f_2\|_{L_{p,\omega_2^\delta}(B(x,r))}.$$

By Theorem 2.4 we obtain

$$\|M_b f_1\|_{L_{p,\omega_2^\delta}(B(x,r))} \le \|M_b f_1\|_{L_{p,\omega_2^\delta}(X)}$$

$$\le C\|b\|_{BMO}\|f_1\|_{L_{p,\omega_1^\delta}(X)} = C\|b\|_{BMO}\|f\|_{L_{p,\omega_1^\delta}(B(x,2kr))}, \tag{2}$$

where C does not depend on f. From (2) we get

$$\|M_b f_1\|_{L_{p,\omega_2^\delta}(B(x,r))} \le C\|b\|_{BMO}\|\omega_2^\delta\|_{L_p(B(x,r))} \sup_{t>r}\left(1+\ln\frac{t}{r}\right)\frac{\|f\|_{L_{p,\omega_1^\delta}(B(x,t))}}{\|\omega_2^\delta\|_{L_p(B(x,t))}}, \tag{3}$$

which is easily obtained from the fact that $\|f\|_{L_{p,\omega_1^\delta}(B(x,2kr))}$ is non-decreasing in r, therefore $\|f\|_{L_{p,\omega_1^\delta}(B(x,2kr))}$ on the right-hand side of (2) is dominated by the right-hand side of (3).

For $y \in B(x,r)$ we get

$$\begin{aligned}
M_b f_2(y) &= \sup_{t>0} \mu\left(B\left(y,t\right)\right)^{-1} \int\limits_{B(y,t)} |b(y)-b(z)||f_2(z)|d\mu\left(z\right) \\
&= \sup_{t>0} \mu\left(B\left(y,t\right)\right)^{-1} \int\limits_{\complement_{B(x,2kr)}\cap B(y,t)} |b(y)-b(z)||f(z)|d\mu\left(z\right) \\
&\le C \sup_{t>2kr} \mu\left(B\left(y,t\right)\right)^{-1} \int\limits_{B(x,t)} |b(y)-b(z)||f(z)|d\mu\left(z\right) \\
&\le \sup_{t>r} \mu\left(B\left(y,t\right)\right)^{-1} \int\limits_{B(x,t)} |b(z)-b_{B(x,t)}||f(z)|d\mu\left(z\right) \\
&+ \sup_{t>r} \mu\left(B\left(y,t\right)\right)^{-1} \int\limits_{B(x,t)} |b(y)-b_{B(x,t)}||f(z)|d\mu\left(z\right) = I_1 + I_2.
\end{aligned}$$

By Hölder inequality and Theorem 1.13 we obtain

$$\begin{aligned}
I_1 &= \sup_{t>r} \mu\left(B\left(y,t\right)\right)^{-1} \int\limits_{B(x,t)} |b(z)-b_{B(x,t)}||f(z)|d\mu\left(z\right) \\
&\le C \sup_{t>r} \mu\left(B\left(y,t\right)\right)^{-1} \|f\|_{L_{p,\omega_1^\delta}(B(x,t))}\|b(\cdot)-b_{B(x,t)}\|_{L_{p',\omega_1^{-\delta}}(B(x,t))} \\
&\le C\|b\|_{BMO} \sup_{t>r} t^{-N}\|f\|_{L_{p,\omega_1^\delta}(B(x,t))}\|\omega_1^{-\delta}\|_{L_{p'}(B(x,t))}.
\end{aligned}$$

To estimate I_2, by Lemma 1.14 we get

$$\begin{aligned}
I_2 &= \sup_{t>r} \mu\left(B\left(y,t\right)\right)^{-1} |b(y)-b_{B(x,t)}| \int\limits_{B(x,t)} |f(z)|d\mu\left(z\right) \\
&\le C M_b \chi_{B(x,r)}(y) \sup_{t>r} t^{-N}\|f\|_{L_{p,\omega_1^\delta}(B(x,t))}\|\omega_1^{-\delta}\|_{L_{p'}(B(x,t))} \\
&+ C\|b\|_{BMO} \sup_{t>r} t^{-N} \ln\frac{t}{r}\|f\|_{L_{p,\omega_1^\delta}(B(x,t))}\|\omega_1^{-\delta}\|_{L_{p'}(B(x,t))}.
\end{aligned}$$

From Theorem 2.2 we have

$$
\begin{aligned}
&\|M_b f_2\|_{L_{p,\omega_2^\delta}(B(x,r))} \le \|I_1\|_{L_{p,\omega_2^\delta}(B(x,r))} + \|I_2\|_{L_{p,\omega_2^\delta}(B(x,r))} \\
&\le C\|b\|_{BMO}\|\omega_2^\delta\|_{L_p(B(x,r))} \sup_{t>r}\left(1+\ln\frac{t}{r}\right)\frac{\|f\|_{L_{p,\omega_1^\delta}(B(x,t))}}{\|\omega_2^\delta\|_{L_p(B(x,t))}} \\
&+ C\|M_b\chi_{B(x,r)}\|_{L_{p,\omega_2^\delta}(B(x,r))} \sup_{t>r}\left(1+\ln\frac{t}{r}\right)\frac{\|f\|_{L_{p,\omega_1^\delta}(B(x,t))}}{\|\omega_2^\delta\|_{L_p(B(x,t))}} \\
&\le C\|b\|_{BMO}\|\omega_2^\delta\|_{L_p(B(x,r))} \sup_{t>r}\left(1+\ln\frac{t}{r}\right)\frac{\|f\|_{L_{p,\omega_1^\delta}(B(x,t))}}{\|\omega_2^\delta\|_{L_p(B(x,t))}}. \qquad (4)
\end{aligned}
$$

Then from (3) and (4) we obtain (1).

Theorem 2.6. *Let $1 < p < \infty$, $0 < \delta < 1$, $b \in BMO(X)$, $(\omega_1, \omega_2) \in \widetilde{A}_p(X)$, $\omega_1, \omega_2 \in A_p(X)$ and the functions $\varphi_1(x,r)$, $\varphi_2(x,r)$ satisfy the condition*

$$
\sup_{t>r}\left(1+\ln\frac{t}{r}\right)\frac{\operatorname*{ess\,inf}_{t<s<\infty}\varphi_1(x,s)\|\omega_1^\delta\|_{L_p(B(x,s))}}{\|\omega_2^\delta\|_{L_p(B(x,t))}} \le C\varphi_2(x,r), \qquad (5)
$$

where C does not depend on x and t.

Then the operator M_b is bounded from the space $\mathcal{M}^{p,\varphi_1}_{\omega_1^\delta}(X)$ to the space $\mathcal{M}^{p,\varphi_2}_{\omega_2^\delta}(X)$.

Proof. Let $f \in \mathcal{M}^{p,\varphi_1}_{\omega_1^\delta}(X)$. Let us take $v_2 = \frac{1}{\varphi_2(x,t)}$, $g = \|f\|_{L_{p,\omega_1^\delta}(B(x,t))}$, $u = \left(1+\ln\frac{t}{r}\right)\|\omega_2^\delta\|^{-1}_{L_p(B(x,t))}$, $v_1 = \frac{1}{\varphi_1(x,t)\|\omega_1^\delta\|_{L_p(B(x,t))}}$ in Theorem 1.16. Also by using Theorem 2.5 and the inequality (5) we get

$$
\begin{aligned}
&\|M_b f\|_{\mathcal{M}^{p,\varphi_2}_{\omega_2^\delta}(X)} \\
&\le C\|b\|_{BMO} \sup_{x\in X,\, r>0}\frac{\|\omega_2^\delta\|_{L_p(B(x,r))}}{\varphi_2(x,r)\|\omega_2^\delta\|_{L_p(B(x,r))}} \sup_{t>r}\left(1+\ln\frac{t}{r}\right)\frac{\|f\|_{L_{p,\omega_1^\delta}(B(x,t))}}{\|\omega_2^\delta\|_{L_p(B(x,t))}} \\
&\le C\|b\|_{BMO} \sup_{x\in X,\, r>0}\frac{1}{\varphi_1(x,r)\|\omega_1^\delta\|_{L_p(B(x,r))}}\|f\|_{L_{p,\omega_1^\delta}(B(x,r))} = C\|b\|_{BMO}\|f\|_{\mathcal{M}^{p,\varphi_1}_{\omega_1^\delta}(X)},
\end{aligned}
$$

which completes the proof.

3 Declarations

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Competing Interests

The authors declare that they have no competing interests.

Ethical Approval

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