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Weak Pullback Mean Attractor for p-Laplacian Selkov Lattice Systems with Locally Lipschitz Delay Diffusion Terms

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Abstract

This paper is concerned with the dynamics of a wide class of nonlinear, reversible, stochastic p-Laplacian Selkov delay lattice systems define on $\mathbb{Z}^d$ driven by locally Lipschitz noise. We first establish the global well-posedness of the systems with local Lipschitz delay diffusion terms. Under certain conditions, we prove the existence and uniqueness of weak pullback mean random attractors for the mean random dynamical system associated with the stochastic equation in a product Hilbert space $L^2(\Omega, \mathcal{F}_\tau; \ell^2 \times \ell^2) \times L^2(\Omega, \mathcal{F}_\tau; L^2((-\rho, 0), \ell^2 \times \ell^2))$. The mean random dynamical systems theory proposed by Wang (J. Differ. Equ., 31:2177-2204, 2019) is used to deal with the difficulty caused by the nonlinear noise. The results of this paper are new even when the discrete p-Laplacian is replaced by the usual discrete Laplacian.

Key words: Nonlinear p-Laplacian; Weak pullback mean attractor; Selkov systems; Dalay time; Lipschitze noise
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1 Introduction

For the past ten years, lattice dynamical systems have received much attention owing to their wide applications in the sciences and engineering (see, e.g., [1, 2] and the references therein). The asymptotic behavior of lattice systems has been discussed in [3, 4] for deterministic lattice systems, and in [5–7] for stochastic lattice systems. Recently, the existence of mean random attractors has been studied for stochastic lattice systems with nonlinear noises in [8–10].

In this article, we study the random attractors of the nonlinear, reversible, p-Laplacian stochastic lattice Selkov systems with time delay defined on $\mathbb{Z}^d$ driven by locally Lipschitz

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noise:

$$\begin{cases} du_i(t) = \Big(-d_1 \sum_{j=1}^{d} \big(|u_i(t) - u_{i^{j-1}}(t)|^{p_1-2}(u_i(t) - u_{i^{j-1}}(t)) \\ \qquad - |u_{i^{j+1}}(t) - u_i(t)|^{p_1-2}(u_{i^{j+1}}(t) - u_i(t)) \big) \\ \qquad - a_1 u_i(t) + b_1 u_i^{2q}(t) v_i(t) - b_2 u_i^{2q+1}(t) + f_{1i}(t) \Big) dt \\ \qquad + \epsilon_1 \sum_{k=1}^{\infty} \big(h_{k,i}(t) + \widehat{\sigma}_{k,i}(u_i(t-\rho)) \big) dW_k(t), \\ dv_i(t) = \Big(-d_2 \sum_{j=1}^{d} \big(|v_i(t) - v_{i^{j-1}}(t)|^{p_2-2}(v_i(t) - v_{i^{j-1}}(t)) \\ \qquad - |v_{i^{j+1}}(t) - v_i(t)|^{p_2-2}(v_{i^{j+1}}(t) - v_i(t)) \big) \\ \qquad - a_2 v_i(t) - b_1 u_i^{2q}(t) v_i(t) + b_2 u_i^{2q+1}(t) + f_{2i}(t) \Big) dt \\ \qquad + \epsilon_2 \sum_{k=1}^{\infty} \big(h_{k,i}(t) + \widehat{\sigma}_{k,i}(v_i(t-\rho)) \big) dW_k(t), \end{cases} \tag{1}$$

with initial conditions

$$u(\tau) = u_0, \;\; u(s) = \phi(s-\tau), \quad v(\tau) = v_0, \;\; v(s) = \xi(s-\tau), \quad s \in (\tau-\rho, \tau), \tag{2}$$

where $t > \tau$, $\tau \in \mathbb{R}$, $i = (i_1, \ldots, i_j, \ldots, i_d) \in \mathbb{Z}^d$, $i^{j+1} := (i_1, \ldots, i_j + 1, \ldots, i_d)$, $i^{j-1} := (i_1, \ldots, i_j - 1, \ldots, i_d)$, $u = (u_i)_{i\in\mathbb{Z}^d}, v = (v_i)_{i\in\mathbb{Z}^d} \in \ell^2 := \{u = (u_i)_{i\in\mathbb{Z}^d} : \sum_{i\in\mathbb{Z}^d} |u_i|^2 < +\infty\}$, $\epsilon_1, \epsilon_2 > 0$, $p_1, p_2 \geq 2$, $q \geq 1$, $\rho \in [0,1]$ is a time delay parameter, $d_1, d_2, a_1, a_2, b_1, b_2$ are positive constants, $f_1(t) = (f_{1i}(t))_{i\in\mathbb{Z}^d}, f_2(t) = (f_{2i}(t))_{i\in\mathbb{Z}^d}, h(t) = (h_{k,i}(t))_{k\in\mathbb{N}, i\in\mathbb{Z}^d} \in \ell^2$ are time independent random sequences, $(W_k)_{k\in\mathbb{N}}$ is a sequence of independent two-sided real-valued Wiener process which is defined on a complete filtered probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\in\mathbb{R}}, \mathbb{P})$, and $\widehat{\sigma}_{k,i} : \mathbb{R} \longrightarrow \mathbb{R}$ is a sequence locally Lipschitz continuous functions (uniformly in k), which satisfies, for any $s \in \mathbb{R}$, $i \in \mathbb{Z}^d$ and $k \in \mathbb{N}$, there exist constants $\delta_{k,i} > 0$ and $\alpha_k > 0$ such that

$$|\widehat{\sigma}_{k,i}(s)| \leq \delta_{k,i} + \alpha_k |s|, \tag{3}$$

where $\|\delta\|^2 = \sum_{k\in\mathbb{N}} \sum_{i\in\mathbb{Z}^d} |\delta_{k,i}|^2 < \infty$, $\|\alpha\|^2 = \sum_{k\in\mathbb{N}} |\alpha_k|^2 < \infty$.

In this work, we study the stochastic lattice reversible Selkov system; that is, a widely influential and classic mathematical model of this feedback of ATP(Adenosine triphosphate) on PFK (Adenosine diphosphate), which has been widely investigated in [11–16]. The attractors of stochastic Selkov systems have been studied by many scholars. For example, for stochastic lattice reversible Selkov equations with additive noises, the existence of a random attractor was proved by Li in [17, 18]. For the Selkov equations on a bounded domain space with dimension $n < 3$, the existence of a global attractor for the solution semiflow was explored by You [19]. For non-autonomous stochastic reversible Selkov systems with multiplicative noise, the existence of random attractors and upper semi-continuous of an attractor as noise approaches zero have been established by Guo et al. [20]. According to the results in the existing literature, we find that the existence and uniqueness of weak pullback mean random attractors for nonlinear, reversible, p-Laplacian stochastic lattice Selkov equations (1)-(2) are not studied. So, in this paper, we will investigate the existence and uniqueness of weak pullback mean random attractors of the system (1)-(2) with the local Lipschitz noise and delay terms.

The theory of attractors is an effective way to study long time behavior and qualitative properties of dynamic systems. The concept of pathwise pullback random attractor was first proposed by Crauel, Flandoli, and Schmalfuss in [21, 22]. Then the pathwise pullback random attractors are widely discussed and studied by many authors, such as literature [23–51]. The classical theory of random dynamical systems is pathwise. However, when the stochastic equation is driven by additive or linear multiplicative noise, the pathwise random attractors can be explored by using the idea of a deterministic pathwise equation; that is, the stochastic equation can be transformed into a pathwise deterministic system. However, we don't know how to transform the stochastic equation into a pathwise deterministic equation when the noise is nonlinear, so we have no way to study the pathwise pullback random attractors of the system driven by nonlinear noise. Therefore, in order to overcome the difficulty caused by the Lipschitz nonlinear diffusion terms σ (see, e.g., [8, 36, 39]). In the present article, we apply the mean random dynamical systems theory proposed by Wang in reference [34] to investigate the weak pullback mean random attractors of (1)-(2).

However, the mean random dynamic systems theory is established on reflexive Banach spaces (see [34, 36]). We usually choose the Banach spaces as a phase space to study the dynamics of delayed systems, see [52–54]. Note that the Banach space $C([-\rho,0],\ell^2\times\ell^2)$ is not reflexive, and hence can not be used as a phase space to study mean random attractors of (1)-(2). In order to solve this problem, we choose the product Hilbert space $L^2(\Omega,\mathcal{F}_\tau;\ell^2\times\ell^2)\times L^2\big(\Omega,\mathcal{F}_\tau;L^2((-\rho,0),\ell^2\times\ell^2)\big)$ as a phase space to study the existence and uniqueness of the weak pullback mean random attractors for the stochastic nonlinear p-Laplacian Selkov system (1)-(2) with time delay on $\mathbb{Z}^d$ driven by locally Lipschitz noise.

The lattice systems have many applications in physics, chemistry, biology, and other disciplines, such as propagation of nerve pulses, electric circuits, and image processing [11, 55–57]. In addition, lattice systems come from the spatial discretizations of partial differential equations [58]. Random attractors of lattice systems have been studied by many scholars, see, e.g., [59–64] for the pathwise pullback random attractors. [7, 34, 65–68] for the pullback mean random attractors.

This paper is organized as follows. In Section 2, in order to prove the existence and uniqueness of the solutions of (1)-(2), we transform the system (1)-(2) with time delay into system (22)-(23) without time delay. In Section 3, we first define the concept of a mean random dynamical system in a product Hilbert space $L^2(\Omega,\mathcal{F}_\tau;\ell^2\times\ell^2)\times L^2\big(\Omega,\mathcal{F}_\tau;L^2((-\rho,0),\ell^2\times\ell^2)\big)$, then prove the existence and uniqueness of weak pullback mean random attractors of (1)-(2) under condition $(\epsilon_1^2+\epsilon_2^2)<\frac{\lambda}{2\|\alpha\|^2}$.

2 Existence and uniqueness of solutions

In this section, we prove the existence and uniqueness of solutions to (1)-(2).

Let $\ell^r:=\{u=(u_i)_{i\in\mathbb{Z}^d}:\sum_{i\in\mathbb{Z}^d}|u_i|^r<+\infty\}$ for $r\geq 1$. The norm of ℓ^r is denoted by $\|\cdot\|_r$. The norm and inner product of ℓ^2 are written as $\|\cdot\|$ and $\langle\cdot,\cdot\rangle$, respectively.

In this paper we will use the following inequalities many times:

$$|x^r-y^r|\leq C_r|x-y||x^{r-1}+y^{r-1}|,\ \forall\, x,y\in\mathbb{R},\ r\geq 1, \tag{4}$$

$$\Big||x_1|^{r-2}x_1-|x_2|^{r-2}x_2\Big|\leq C_r(|x_1|^{r-2}+|x_2|^{r-2})|x_1-x_2|,\ \forall\, x_1,x_2\in\mathbb{R},\ r\geq 2, \tag{5}$$

$$b_1b_2x^{2q+1}y-b_2^2x^{2q+2}-b_1^2x^{2q}y^2+b_2b_1x^{2q+1}y\leq 0,\quad \forall x,y\in\mathbb{R},\ p\geq 1. \tag{6}$$

Assume $a\geq 0$, $k_1,k_2\geq 0$ are constants, if $k_1>k_2$, there exists a constant $\gamma>0$ such that

$$\gamma-k_1+k_2e^{\gamma a}<0. \tag{7}$$

We define four operators $B_j, B_j^* : \ell^2 \to \ell^2$, $F : \ell^2 \times \ell^2 \to \ell^2$ and $G : \ell^2 \to \ell^2$ by $(B_j u)_i = u_{i_{j+1}} - u_i$, $(B_j^* u)_i = u_{i_{j-1}} - u_i$, $F(u,v) = (u_i^{2q} v_i)_{i\in\mathbb{Z}^d}$ and $G(u) = (u_i^{2q+1})_{i\in\mathbb{Z}^d}$ for $j \in [1,d] \cap \mathbb{N}$, $u = (u_i)_{i\in\mathbb{Z}^d} \in \ell^2$,$v = (v_i)_{i\in\mathbb{Z}^d} \in \ell^2$.

Then the discrete d-dimensional p-Laplace operator $A_p : \ell^2(\mathbb{Z}^d) \to \ell^2(\mathbb{Z}^d)$ with $p > 2$ is defined by

$$(A_p u)_i = -\sum_{j=1}^{d} \Big(|(B_j^* u)_i|^{p-2} \times (B_j^* u)_i + |(B_j u)_i|^{p-2} \times (B_j u)_i \Big),$$

which indicates that

$$\langle A_p u, u\rangle = \sum_{j=1}^{d} \|B_j u\|_p^p \geq 0. \tag{8}$$

By (5) and [69, 70], for $p \geq 2$, $u, v \in \ell^2$, we obtain

$$\begin{aligned}
\|A_p u - A_p v\|^2 = & \sum_{i\in\mathbb{Z}^d} \sum_{j=1}^{d} \Big(|(B_j^* v)_i|^{p-2} \times (B_j^* v)_i - |(B_j^* u)_i|^{p-2} \times (B_j^* u)_i \\
& + |(B_j v)_i|^{p-2} \times (B_j v)_i - |(B_j u)_i|^{p-2} \times (B_j u)_i \Big) \\
& \times \Big(|(B_j^* v)_i|^{p-2} \times (B_j^* v)_i - |(B_j^* u)_i|^{p-2} \times (B_j^* u)_i \\
& + |(B_j v)_i|^{p-2} \times (B_j v)_i - |(B_j u)_i|^{p-2} \times (B_j u)_i \Big) \\
= & \sum_{i\in\mathbb{Z}^d} \sum_{j=1}^{d} \Big| |(B_j^* v)_i|^{p-2} \times (B_j^* v)_i - |(B_j^* u)_i|^{p-2} \times (B_j^* u)_i \\
& + |(B_j v)_i|^{p-2} \times (B_j v)_i - |(B_j u)_i|^{p-2} \times (B_j u)_i \Big|^2 \\
\leq & \, 2 \sum_{i\in\mathbb{Z}^d} \sum_{j=1}^{d} \Big| |(B_j^* v)_i|^{p-2} \times (B_j^* v)_i - |(B_j^* u)_i|^{p-2} \times (B_j^* u)_i \Big|^2 \\
& + 2 \sum_{i\in\mathbb{Z}^d} \sum_{j=1}^{d} \Big| |(B_j v)_i|^{p-2} \times (B_j v)_i - |(B_j u)_i|^{p-2} \times (B_j u)_i \Big|^2 \\
\leq & \, 4 \sum_{i\in\mathbb{Z}^d} \sum_{j=1}^{d} \Big| |(B_j v)_i|^{p-2} \times (B_j v)_i - |(B_j u)_i|^{p-2} \times (B_j u)_i \Big|^2 \\
\leq & \, 8C_p^2 \sum_{i\in\mathbb{Z}^d} \sum_{j=1}^{d} \left(|(B_j v)_i|^{2p-4} + |(B_j u)_i|^{2p-4} \right) |(B_j v)_i - (B_j u)_i|^2 \\
\leq & \, 2^{2p+1} d C_p^2 (\|u\|^{2p-4} + \|v\|^{2p-4}) \|u - v\|^2.
\end{aligned} \tag{9}$$

By (4), Young's inequality and [16], we find that for any $q \geq 1$, $u, u_1, u_2, v, v_1, v_2 \in \ell^2$ and $\|u\| \leq n, \|u_1\| \leq n, \|u_2\| \leq n, \|v\| \leq n, \|v_1\| \leq n, \|v_2\| \leq n$,

$$\begin{aligned}
& \|F(u_1, v_1) - F(u_2, v_2)\|^2 \\
& \quad \leq 2d(1 + 4C_{2q}^2)(\|u_1\|_{4q}^{4q} + \|u_2\|_{4q}^{4q} + \|v_2\|_{4q}^{4q})(\|u_1 - u_2\|^2 + \|v_1 - v_2\|^2),
\end{aligned} \tag{10}$$

and

$$\|G(u) - G(v)\| \leq 2dC_{2q+1}^2 (\|u\|_{4q}^{4q} + \|v\|_{4q}^{4q}) \|u - v\|^2. \tag{11}$$

By (9)-(11), we infer that the operators $A_p u$, $F(u,v)$, $G(u)$ are locally Lipschitz continuous; that is, for every $n \in \mathbb{N}$, exists $c_1(n), c_2(n), c_3(n) > 0$, such that for any $u, v, u_1, u_2, v_1, v_2 \in \ell^2$, $p \geq 2$, and $\|u\| \leq n$, $\|u_1\| \leq n$, $\|u_2\| \leq n$, $\|v\| \leq n$, $\|v_1\| \leq n$, $\|v_2\| \leq n$,

$$\begin{aligned}
&\|A_p u - A_p v\|^2 \leq c_1(n)\|u - v\|^2, \\
&\big|\langle A_p u_1 - A_p u_2, u_1 - u_2\rangle\big| \leq c_1(n)\|u_1 - u_2\|^2, \\
&\big|\langle A_p u_1 - A_p u_2, v_1 - v_2\rangle\big| \leq c_1(n)(\|u_1 - u_2\|^2 + \|v_1 - v_2\|^2),
\end{aligned} \tag{12}$$

$$\begin{aligned}
&\|F(u_1,v_1) - F(u_2,v_2)\|^2 \leq c_2(n)(\|u_1 - u_2\|^2 + \|v_1 - v_2\|^2), \\
&\big|\langle F(u_1,v_1) - F(u_2,v_2), u_1 - u_2\rangle\big| \leq c_2(n)(\|u_1 - u_2\|^2 + \|v_1 - v_2\|^2), \\
&\big|\langle F(u_1,v_1) - F(u_2,v_2), v_1 - v_2\rangle\big| \leq c_2(n)(\|u_1 - u_2\|^2 + \|v_1 - v_2\|^2),
\end{aligned} \tag{13}$$

and

$$\begin{aligned}
&\|G(u) - G(v)\|^2 \leq c_3(n)\|u - v\|^2, \\
&\big|\langle G(u_1) - G(u_2), u_1 - u_2\rangle\big| \leq c_3(n)\|u_1 - u_2\|^2, \\
&\big|\langle G(u_1) - G(u_2), v_1 - v_2\rangle\big| \leq c_3(n)(\|u_1 - u_2\|^2 + \|v_1 - v_2\|^2).
\end{aligned} \tag{14}$$

For every $k \in \mathbb{N}$, we write $\sigma_k(u) = (\widehat{\sigma}_{k,i}(u_i))_{i \in \mathbb{Z}^d}$ for all $u \in \ell^2$. Then we find

$$\sum_{k \in \mathbb{N}} \|\sigma_k(u)\|^2 \leq 2\|\delta\|^2 + 2\|\alpha\|^2\|u\|^2. \tag{15}$$

Note that $\sigma_k : \ell^2 \to \ell^2$ is also locally Lipschitz continuous; that is, for every $n \in \mathbb{N}$, there exists $c_4(n) > 0$, for every $u, v \in \ell^2$, $\|u\| \leq n$, $\|v\| \leq n$,

$$\sum_{k \in \mathbb{N}} \|\sigma_k(u) - \sigma_k(v)\|^2 \leq c_4(n)\|u - v\|^2. \tag{16}$$

In order to explore the existence and uniqueness of the solutions to the problem (1)-(2), we assume f_1, f_2, h_k are ℓ^2-valued progressively measurable processes satisfy the following condition, for all $\tau \in \mathbb{R}$ and $T > 0$,

$$\int_\tau^{\tau+T} \mathbb{E}\big(\|f_1(t)\|^2 + \|f_2(t)\|^2 + \sum_{k \in \mathbb{N}} \|h_k(t)\|^2\big)dt < \infty. \tag{17}$$

For convenience, we can rewrite system (1) as the following system in $\ell^2 \times \ell^2$ for $t \geq \tau \in \mathbb{R}$:

$$\begin{cases}
du(t) = \Big(-d_1 A_{p_1} u(t) - a_1 u(t) + b_1 u^{2q}(t)v(t) - b_2 u^{2q+1}(t) + f_1(t)\Big)dt \\
\qquad + \epsilon_1 \sum_{k=1}^{\infty} \big(h_k(t) + \sigma_k(u(t-\rho))\big)dW_k(t), \\
dv(t) = \Big(-d_2 A_{p_2} v(t) - a_2 v(t) - b_1 u^{2q}(t)v(t) + b_2 u^{2q+1}(t) + f_2(t)\Big)dt \\
\qquad + \epsilon_2 \sum_{k=1}^{\infty} \big(h_k(t) + \sigma_k(v(t-\rho))\big)dW_k(t),
\end{cases} \tag{18}$$

with initial date

$$\psi(\tau) = \psi_0 = (u_0, v_0), \quad \psi(s) = (u(s), v(s)) = \vartheta(s - \tau) = (\phi(s - \tau), \xi(s - \tau)), \quad s \in (\tau - \rho, \tau),$$

where $\psi(t) = (u(t), v(t))^T$, $\vartheta(s - \tau) = (\phi(s - \tau), \xi(s - \tau))^T$.

Next, we prove the existence and uniqueness of the solutions for the problem (18) in the following sense.

Definition 2.1. Let $\tau \in \mathbb{R}$, $T > 0$, $\psi_0 \in L^2(\Omega, \mathcal{F}_\tau; \ell^2 \times \ell^2)$, $\vartheta \in L^2\big(\Omega, \mathcal{F}_\tau; L^2((-\rho, 0), \ell^2 \times \ell^2)\big)$. A $\ell^2 \times \ell^2$-valued stochastic process $\psi(t) = (u(t), v(t))^T$, $t > \tau - \rho$, is called a solution of system (18) if

(i) $\psi \in L^2(\Omega, \mathcal{F}_\tau; L^2\big((\tau - \rho, \tau), \ell^2 \times \ell^2)\big)$ and $u_\tau = \phi$, $v_\tau = \xi$.
(ii) ψ is pathwise continuous on $[\tau, \infty)$, $\mathcal{F}_\tau$-adapted for all $t \geq \tau$, $\psi(\tau) = \psi_0$, $\psi \in L^2(\Omega, C[\tau, \tau + T], \ell^2 \times \ell^2)$ for all $T > 0$.
(iii) For system (1) in $\ell^2 \times \ell^2$,

$$\begin{cases} u(t) = u_0 + \displaystyle\int_\tau^t \Big(- d_1 A_{p_1} u(s) - a_1 u(s) + b_1 u^{2q}(s) v(s) - b_2 u^{2q+1}(s) + f_1(s) \Big) ds \\ \qquad + \epsilon_1 \displaystyle\sum_{k=1}^{\infty} \int_\tau^t (h_k(s) + \sigma_k(u(s-\rho))) dW_k(s), \\ v(t) = v_0 + \displaystyle\int_\tau^t \Big(- d_2 A_{p_2} v(s) - a_2 v(s) - b_1 u^{2q}(s) v(s) + b_2 u^{2q+1}(s) + f_2(s) \Big) ds \\ \qquad + \epsilon_2 \displaystyle\sum_{k=1}^{\infty} \int_\tau^t (h_k(s) + \sigma_k(v(s-\rho))) dW_k(s), \ \ \mathbb{P}\text{-a.s..} \end{cases} \tag{19}$$

Theorem 2.2. *Assume (3) and (17) hold. Then for all* $\psi_0 \in L^2(\Omega, \mathcal{F}_\tau; \ell^2 \times \ell^2)$, $\vartheta \in L^2\big(\Omega, \mathcal{F}_\tau; L^2((-\rho, 0), \ell^2 \times \ell^2)\big)$, *system (18) has a unique solution* ψ *in the meaning of Definition 2.1,*

$$\mathbb{E}\Big(\sup_{\tau \leq s \leq \tau + T} \|\psi(s)\|^2_{\ell^2 \times \ell^2} \Big) \leq K \quad \forall\, T > 0,$$

where

$$K = K_1 e^{K_1 T} \Big(T + \mathbb{E}(\|\psi_0\|^2_{\ell^2 \times \ell^2}) + \int_{-\rho}^0 \mathbb{E}(\|\phi(s)\|^2_{\ell^2 \times \ell^2} + \|\xi(s)\|^2_{\ell^2 \times \ell^2}) ds$$
$$+ \int_\tau^{\tau+T} \mathbb{E}(\|f_1(s)\|^2 + \|f_2(s)\|^2 + \sum_{k=1}^{\infty} \|h(s)\|^2) ds \Big),$$

and $K_1 > 0$ *is a constant independent of* ψ_0, ϑ, ρ, τ *and* T.

Proof. We first prove the existence of solutions for (18) on $[\tau, \tau + \rho]$. By (15), BDG inequality we get for $\phi, \xi \in L^2(\Omega, \mathcal{F}_\tau; L^2((-\rho, 0), \ell^2 \times \ell^2))$,

$$\begin{aligned} &\mathbb{E}\Big(\sup_{\tau \leq t \leq \tau + \rho} \Big| \int_\tau^{\tau+\rho} \sum_{k=1}^{\infty} (\sigma_k(u(t-\rho)) dW_k(t) \Big| \Big) \\ &= \mathbb{E}\Big(\sup_{\tau \leq t \leq \tau + \rho} \Big| \int_\tau^{\tau+\rho} \sum_{k=1}^{\infty} (\sigma_k(\phi(t - \rho - \tau)) dW_k(t) \Big| \Big) \\ &\leq 3\mathbb{E}\Big(\Big(\int_\tau^{\tau+\rho} \|\sigma_k(\phi(t - \rho - \tau)\|^2 dt \Big)^{\frac{1}{2}} \Big) \\ &\leq \frac{1}{2} + \frac{9}{2} \mathbb{E}\Big(\int_\tau^{\tau+\rho} \|\sigma_k(\phi(t - \rho - \tau)\|^2 dt \Big) \\ &\leq \frac{1}{2} + 9\rho\|\delta\|^2 + 9\|\alpha\|^2 \int_\tau^{\tau+\rho} \mathbb{E}(\|\phi(t - \rho - \tau)\|^2) dt \\ &= \frac{1}{2} + 9\rho\|\delta\|^2 + 9\|\alpha\|^2 \int_{-\rho}^0 \mathbb{E}(\|\phi(t)\|^2) dt < \infty, \end{aligned} \tag{20}$$

similarly, we have

$$\begin{aligned}\mathbb{E}\Big(\sup_{\tau\leq t\leq\tau+\rho}\Big|\int_{\tau}^{\tau+\rho}\sum_{k=1}^{\infty}(\sigma_k(v(t-\rho))dW_k(t)\Big|\Big)\\ &=\mathbb{E}\Big(\sup_{\tau\leq t\leq\tau+\rho}\Big|\int_{\tau}^{\tau+\rho}\sum_{k=1}^{\infty}(\sigma_k(\xi(t-\rho-\tau))dW_k(t)\Big|\Big)\\ &\leq\frac{1}{2}+9\rho\|\delta\|^2+9\|\alpha\|^2\int_{-\rho}^{0}\mathbb{E}(\|\xi(t)\|^2)dt<\infty.\end{aligned}\tag{21}$$

By (20)-(21), problem (18) on $[\tau,\tau+\rho]$ is transformed into the following equations without delay:

$$\begin{cases}du(t)=\Big(-d_1A_{p_1}u(t)-a_1u(t)+b_1u^{2q}(t)v(t)-b_2u^{2q+1}(t)+f_1(t)\Big)dt\\ \qquad+\epsilon_1\sum_{k=1}^{\infty}\big(h_k(t)+\sigma_k(\phi(t-\rho-\tau))\big)dW_k(t),\\ dv(t)=\Big(-d_2A_{p_2}v(t)-a_2v(t)-b_1u^{2q}(t)v(t)+b_2u^{2q+1}(t)+f_2(t)\Big)dt\\ \qquad+\epsilon_2\sum_{k=1}^{\infty}\big(h_k(t)+\sigma_k(\xi(t-\rho-\tau))\big)dW_k(t),\ \ t\in(\tau,\tau+\rho],\end{cases}\tag{22}$$

with initial conditions

$$u(\tau)=u_0,\ \ v(\tau)=v_0.\tag{23}$$

Then by Theorem 1.1 in [16], by conditions (3) and (17), the system (22)-(23) has a unique solution ψ defined on $[\tau,\tau+\rho]$ such that $\psi\in L^2(\Omega,C([\tau,\tau+\rho],\ell^2\times\ell^2))$. Repeating this discussion, we can extend the solution ψ to the interval $[\tau,\infty)$ such that $\psi\in L^2(\Omega,C([\tau,\tau+T],\ell^2\times\ell^2))$ for all $T>0$.

Next, we conduct uniform estimates of the solutions. Applying Ito's formula to (18), we get

$$\begin{aligned}&b_2\|u(t)\|^2+\int_{\tau}^{t}\Big[2d_1b_2\langle A_{p_1}u(s),u(s)\rangle+2a_1b_2\|u(s)\|^2+2b_2^2\langle G(u(s)),u(s)\rangle\Big]ds\\ =&b_2\|u(\tau)\|^2+2b_2\int_{\tau}^{t}\langle f_1(s),u(s)\rangle ds+2b_1b_2\int_{\tau}^{t}\langle F(u(s),v(s)),u(s)\rangle ds\\ &+b_2\epsilon_1^2\sum_{k=1}^{\infty}\int_{\tau}^{t}\|h_k(s)+\sigma_k(u(s-\rho))\|^2ds\\ &+2b_2\epsilon_1\sum_{k=1}^{\infty}\int_{\tau}^{t}\langle h_k(s)+\sigma_k(u(s-\rho)),u(s)\rangle dW_k(s),\end{aligned}\tag{24}$$

and

$$\begin{aligned}&b_1\|v(t)\|^2+\int_{\tau}^{t}\Big[2d_2b_1\langle A_{p_2}v(s),v(s)\rangle+2a_2b_1\|v(s)\|^2+2b_1^2\langle F(u(s),v(s)),v(s)\rangle\Big]ds\\ =&b_1\|v(\tau)\|^2+2b_1\int_{\tau}^{t}\langle f_2(s),v(s)\rangle ds+2b_2b_1\int_{\tau}^{t}\langle G(u(s)),v(s)\rangle ds\\ &+b_1\epsilon_2^2\sum_{k=1}^{\infty}\int_{\tau}^{t}\|h_k(s)+\sigma_k(v(s-\rho))\|^2ds\\ &+2b_1\epsilon_2\sum_{k=1}^{\infty}\int_{\tau}^{t}\langle h_k(s)+\sigma_k(v(s-\rho)),v(s)\rangle dW_k(s).\end{aligned}\tag{25}$$

By (6) and (8), we know

$$\begin{aligned}
&\mathbb{E}\Big(\sup_{\tau\le r\le t}\Big(b_2\|u(r)\|^2+b_1\|v(r)\|^2\Big)\Big)\le \mathbb{E}(b_2\|u(\tau)\|^2+b_1\|v(\tau)\|^2)\\
&+a_1^{-1}b_2\int_\tau^t \mathbb{E}(\|f_1(s)\|^2)ds+a_2^{-1}b_1\int_\tau^t \mathbb{E}(\|f_2(s)\|^2)ds\\
&+b_2\epsilon_1^2\sum_{k=1}^{\infty}\int_\tau^t \mathbb{E}(\|h_k(s)+\sigma_k(u(s-\rho))\|^2)ds+b_1\epsilon_2^2\sum_{k=1}^{\infty}\int_\tau^t \mathbb{E}(\|h_k(s)+\sigma_k(v(s-\rho))\|^2)ds\\
&+\underbrace{2b_2\epsilon_1\mathbb{E}\Big(\sup_{\tau\le r\le t}\Big|\sum_{k=1}^{\infty}\int_\tau^r\langle h_k(s)+\sigma_k(u(s-\rho)),u(s)\rangle dW_k(s)\Big|\Big)}_{I_1}\\
&+\underbrace{2b_1\epsilon_2\mathbb{E}\Big(\sup_{\tau\le r\le t}\Big|\sum_{k=1}^{\infty}\int_\tau^r\langle h_k(s)+\sigma_k(v(s-\rho)),v(s)\rangle dW_k(s)\Big|\Big)}_{I_2}. \qquad (26)
\end{aligned}$$

For the terms I_1 and I_2 in (26), by the BDG inequality, we get

$$\begin{aligned}
I_1&\le 6b_2\epsilon_1\mathbb{E}\Big(\int_\tau^t\sum_{k=1}^{\infty}\|h_k(s)+\sigma_k(u(s-\rho))\|^2\|u(s)\|^2dW_k(s)ds\Big)^{\frac12}\\
&\le\frac13\mathbb{E}\Big(\sup_{\tau\le s\le t}b_2\|u(s)\|^2\Big)+27b_2\epsilon_1^2\sum_{k=1}^{\infty}\int_\tau^t\mathbb{E}(\|h_k(s)+\sigma_k(u(s-\rho))\|^2)ds, \qquad (27)
\end{aligned}$$

and

$$I_2\le\frac13\mathbb{E}\Big(\sup_{\tau\le s\le t}b_1\|v(s)\|^2\Big)+27b_1\epsilon_2^2\sum_{k=1}^{\infty}\int_\tau^t\mathbb{E}(\|h_k(s)+\sigma_k(v(s-\rho))\|^2)ds. \qquad (28)$$

By (26)-(28) and (15), we know that

$$\begin{aligned}
&28b_2\epsilon_1^2\sum_{k=1}^{\infty}\int_\tau^t\mathbb{E}(\|h_k(s)+\sigma_k(u(s-\rho))\|^2)ds\\
&\le 112b_2\epsilon_1^2\|\delta\|^2(t-\tau)+56b_2\epsilon_1^2\sum_{k=1}^{\infty}\int_\tau^t\mathbb{E}(\|h_k(s)\|^2)ds\\
&\quad+112b_2\epsilon_1^2\|\alpha\|^2\int_{-\rho}^0\mathbb{E}\|\phi(s)\|^2ds+112b_2\epsilon_1^2\|\alpha\|^2\int_\tau^t\mathbb{E}(\|u(s)\|^2)ds, \qquad (29)
\end{aligned}$$

and

$$\begin{aligned}
&28b_1\epsilon_2^2\sum_{k=1}^{\infty}\int_\tau^t\mathbb{E}(\|h_k(s)+\sigma_k(v(s-\rho))\|^2)ds\\
&\le 112b_1\epsilon_2^2\|\delta\|^2(t-\tau)+56b_1\epsilon_2^2\sum_{k=1}^{\infty}\int_\tau^t\mathbb{E}(\|h_k(s)\|^2)ds\\
&\quad+112b_1\epsilon_2^2\|\alpha\|^2\int_{-\rho}^0\mathbb{E}\|\xi(s)\|^2ds+112b_1\epsilon_2^2\|\alpha\|^2\int_\tau^t\mathbb{E}(\|v(s)\|^2)ds. \qquad (30)
\end{aligned}$$

From (26) to (30), we find that there exists a positive number C_1 independent of $u_0, v_0, \phi, \xi, T, \tau$

such that for all $t \in [\tau, \tau + T]$,

$$
\begin{aligned}
&\mathbb{E}\Big(\sup_{\tau \le r \le t} \Big(b_2\|u(r)\|^2 + b_1\|v(r)\|^2\Big)\Big) \\
&\le 3\mathbb{E}(b_2\|u_0\|^2 + b_1\|v_0\|^2) + C_1\mathbb{E}\int_{-\rho}^{0} \Big(\|\phi(s)\|^2 + \|\xi(s)\|^2\Big)ds \\
&+ C_1\mathbb{E}\int_{\tau}^{\tau+T} \Big(\|f_1(s)\|^2 + \|f_2(s)\|^2 + \sum_{k=1}^{\infty}\|h(s)\|^2\Big)ds + C_1T \\
&+ C_1\int_{\tau}^{t}\mathbb{E}\Big(\sup_{\tau \le r \le s} \Big(b_2\|u(r)\|^2 + b_1\|v(r)\|^2\Big)ds\Big) \\
&\le C_2 + C_1\int_{\tau}^{t}\mathbb{E}\Big(\sup_{\tau \le r \le s} \Big(b_2\|u(r)\|^2 + b_1\|v(r)\|^2\Big)ds\Big).
\end{aligned} \tag{31}
$$

where $C_2 > 0$ is a finite number from (17).

By (31) and Gronwall's lemma, we obtain, for all $t \in [\tau, \tau + T]$ with $T > 0$,

$$
\mathbb{E}\Big(\sup_{\tau \le r \le t} \Big(b_2\|u(r)\|^2 + b_1\|v(r)\|^2\Big)\Big) \le C_2 e^{C_1 T},
$$

which completes the proof. □

3 Weak pullback mean random attractors

For any $\tau \in \mathbb{R}$ and $t \in \mathbb{R}^+$, let $\Phi(t, \tau)$ be a mapping from $L^2(\Omega, \mathcal{F}_\tau; \ell^2 \times \ell^2) \times L^2(\Omega, \mathcal{F}_\tau; L^2((-\rho, 0), \ell^2 \times \ell^2))$ to $L^2(\Omega, \mathcal{F}_{t+\tau}; \ell^2 \times \ell^2) \times L^2(\Omega, \mathcal{F}_{t+\tau}; L^2((-\rho, 0), \ell^2 \times \ell^2))$ given by

$$
\Phi(t, \tau)(\psi_0, \vartheta) = \big(\psi(t + \tau; \tau, \psi_0, \vartheta), \psi_{t+\tau}(\cdot; \tau, \psi_0, \vartheta)\big),
$$

for any $(\psi_0, \vartheta) \in L^2(\Omega, \mathcal{F}_\tau; \ell^2 \times \ell^2) \times L^2(\Omega, \mathcal{F}_\tau; L^2((-\rho, 0), \ell^2 \times \ell^2))$, where $\psi(t; \tau, \psi_0, \vartheta) = \big(u(t; \tau, u_0, \phi),$
$v(t; \tau, v_0, \xi)\big)^T$ is the solution of (18), and $\psi_{t+\tau}(\theta; \tau, \psi_0, \vartheta) = \psi(t + \tau + \theta; \tau, \psi_0, \vartheta)$ for $\theta \in (-\rho, 0)$.

Then Φ is a mean random dynamical system on $L^2(\Omega, \mathcal{F}; \ell^2 \times \ell^2) \times L^2(\Omega, \mathcal{F}; L^2((-\rho, 0), \ell^2 \times \ell^2))$ over the filtration $\{\mathcal{F}_t\}_{t\in\mathbb{R}}$, see [34, Definition 2.9].

In this section, we discuss the existence and uniqueness of weak pullback mean random attractors of (18). For convenience, for every $\tau \in \mathbb{R}$, we set

$$
H_\tau = L^2(\Omega, \mathcal{F}_\tau; \ell^2 \times \ell^2) \times L^2\big(\Omega, \mathcal{F}_\tau; L^2((-\rho, 0), \ell^2 \times \ell^2)\big).
$$

And H_τ is a product Hilbert space with inner product and norm

$$
\big\langle(\omega_1, \varsigma_1), (\omega_2, \varsigma_2)\big\rangle_{H_\tau} = \mathbb{E}\langle\omega_1, \omega_2\rangle + \mathbb{E}\Big(\int_{-\rho}^{0}\langle\varsigma_1(s), \varsigma_2(s)\rangle ds\Big), \ \ (\omega_1, \varsigma_1), (\omega_2, \varsigma_2) \in H_\tau,
$$

$$
\|(\omega, \varsigma)\|_{H_\tau} = \Big(\mathbb{E}(\|\omega\|^2_{\ell^2\times\ell^2}) + \int_{-\rho}^{0}\mathbb{E}(\|\varsigma(s)\|^2_{\ell^2\times\ell^2})ds\Big)^{\frac{1}{2}}, \ \ (\omega, \varsigma) \in H_\tau.
$$

In addition, we suppose ϵ_1, ϵ_2 in (1) are small enough such that

$$
(\epsilon_1^2 + \epsilon_2^2) < \frac{\lambda}{2\|\alpha\|^2}. \tag{32}
$$

By (7) and (32), there exist constants $\mu, \eta > 0$ such that

$$\eta + \mu - 2\lambda + 4\|\alpha\|^2(\epsilon_1^2 + \epsilon_2^2)e^{\mu\rho} < 0. \tag{33}$$

Let $B = \{B(\tau) \subseteq H_\tau : \tau \in \mathbb{R}\}$ be a family of nonempty bounded sets such that

$$\lim_{\tau \to -\infty} e^{\mu\tau}\|B(\tau)\|_{H_\tau}^2 = 0, \tag{34}$$

where $\|B(\tau)\|_{H_\tau} = \sup_{(\omega,\varsigma)\in H_\tau} \|(\omega,\varsigma)\|_{H_\tau}$. Denote by

$$\mathcal{D} = \Big\{ B = \{B(\tau) \subseteq H_\tau : \tau \in \mathbb{R}\} : B \text{ satisfies (34)} \Big\}.$$

We will prove that the system (18) has a unique weak $\mathcal{D}$-pullback mean random attractor. Therefore we further suppose that for every $\tau \in \mathbb{R}$,

$$\int_{-\infty}^{\tau} e^{\mu s}\mathbb{E}\big(\|f_1(s)\|^2 + \|f_2(s)\|^2 + \sum_{k=1}^{\infty}\|h_k(s)\|^2\big)ds < \infty, \tag{35}$$

where $\mu > 0$ is the constant as in (33).

Lemma 3.1. *Assume (3), (17), (32) and (35) hold. Then for every $\tau \in \mathbb{R}$ and $B = \{B(t)\}_{t\in\mathbb{R}} \in \mathcal{D}$, there exists $T = T(\tau, B) > \rho$ such that for all $t \geq T$,*

$$\begin{aligned}
&\mathbb{E}(b_2\|u(\tau;\tau-t,u_0,\phi)\|^2 + b_1\|v(\tau;\tau-t,v_0,\xi)\|^2) \\
&\qquad + \int_{-\rho}^{0} \mathbb{E}(b_2\|u_\tau(s;\tau-t,u_0,\phi)\|^2 + b_1\|v_\tau(s;\tau-t,v_0,\xi)\|^2)ds \\
&\leq (1+\rho e^{\mu\rho})\Big[1 + (b_2\epsilon_1^2 + b_1\epsilon_2^2)\frac{4\|\delta\|^2}{\mu} \\
&\qquad + c\int_{-\infty}^{\tau} e^{\mu(s-\tau)}\mathbb{E}(\|f_1(s)\|^2 + \|f_2(s)\|^2 + \sum_{k=1}^{\infty}\|h_k(s)\|^2)ds\Big],
\end{aligned} \tag{36}$$

where c is a positive constant which is independent of $\epsilon_1, \epsilon_2, t, \tau, u_0, v_0, \phi, \xi$.

Proof. For all $t > 0$ and $r \in (\tau - t, \tau]$, by (24)-(25) we have

$$\begin{aligned}
&e^{\mu r}\mathbb{E}(b_2\|u(r)\|^2 + b_1\|v(r)\|^2) \\
&= \int_{\tau-t}^{r} e^{\mu s}\mathbb{E}\Big[-2b_2d_1\langle A_{p_1}u(s),u(s)\rangle - 2b_1d_2\langle A_{p_2}v(s),v(s)\rangle - 2a_1b_2\|u(s)\|^2 \\
&\quad - 2a_2b_1\|v(s)\|^2 - 2b_2^2\langle G(u(s)),u(s)\rangle + 2b_1b_2\langle F(u(s),v(s)),u(s)\rangle \\
&\quad + 2b_2b_1\langle G(u(s)),v(s)\rangle - 2b_1^2\langle F(u(s),v(s)),v(s)\rangle\Big]ds \\
&\quad + 2b_2\int_{\tau-t}^{r} e^{\mu s}\mathbb{E}\langle f_1(s),u(s)\rangle ds + 2b_1\int_{\tau-t}^{r} e^{\mu s}\mathbb{E}\langle f_2(s),v(s)\rangle ds \\
&\quad + b_2\epsilon_1^2\sum_{k=1}^{\infty}\int_{\tau-t}^{r} e^{\mu s}\mathbb{E}(\|h_k(s) + \sigma_k(u(s-\rho))\|^2)ds \\
&\quad + b_1\epsilon_2^2\sum_{k=1}^{\infty}\int_{\tau-t}^{r} e^{\mu s}\mathbb{E}(\|h_k(s) + \sigma_k(v(s-\rho))\|^2)ds \\
&\quad + e^{\mu(\tau-t)}\mathbb{E}(b_2\|u_0\|^2 + b_1\|v_0\|^2) + \mu\int_{\tau-t}^{r} e^{\mu s}\mathbb{E}(b_2\|u(s)\|^2 + b_1\|v(s)\|^2)ds.
\end{aligned} \tag{37}$$

By (6), (8) and (37), we know that

$$\int_{\tau-t}^{r} e^{\mu s}\mathbb{E}\Big[-2a_1b_2\|u(s)\|^2 - 2a_2b_1\|v(s)\|^2 - 2b_2^2\langle G(u(s)),u(s)\rangle + 2b_1b_2\langle F(u(s),v(s)),u(s)\rangle$$
$$+2b_2b_1\langle G(u(s)),v(s)\rangle - 2b_1^2\langle F(u(s),v(s)),v(s)\rangle\Big]ds$$
$$\le -2\lambda\int_{\tau-t}^{r} e^{\mu s}\mathbb{E}(b_2\|u(s)\|^2 + b_1\|v(s)\|^2)ds, \tag{38}$$

where $\lambda = (a_1 \wedge a_2) > 0$.
By (37), we have

$$\begin{aligned}
&2b_2\int_{\tau-t}^{r} e^{\mu s}\mathbb{E}\langle f_1(s),u(s)\rangle ds\\
&\le 2b_2\int_{\tau-t}^{r} e^{\mu s}\mathbb{E}(\|f_1(s)\|\|u(s)\|)ds\\
&\le \eta\int_{\tau-t}^{r} e^{\mu s}\mathbb{E}(b_2\|u(s)\|^2)ds + \frac{1}{\eta}\int_{\tau-t}^{r} b_2e^{\mu s}\mathbb{E}(\|f_1(s)\|^2)ds,
\end{aligned}\tag{39}$$

and

$$\begin{aligned}
&2b_1\int_{\tau-t}^{r} e^{\mu s}\mathbb{E}\langle f_2(s),v(s)\rangle ds\\
&\le \eta\int_{\tau-t}^{r} e^{\mu s}\mathbb{E}(b_1\|v(s)\|^2)ds + \frac{1}{\eta}\int_{\tau-t}^{r} b_1e^{\mu s}\mathbb{E}(\|f_2(s)\|^2)ds.
\end{aligned}\tag{40}$$

By (15) and (37), we obtain

$$\begin{aligned}
&b_2\epsilon_1^2\sum_{k=1}^{\infty}\int_{\tau-t}^{r} e^{\mu s}\mathbb{E}(\|h_k(s)+\sigma_k(u(s-\rho))\|^2)ds\\
&\le 2b_2\epsilon_1^2\sum_{k=1}^{\infty}\int_{\tau-t}^{r} e^{\mu s}\mathbb{E}(\|h_k(s)\|^2)ds + 4b_2\epsilon_1^2\int_{\tau-t}^{r} e^{\mu s}\|\delta\|^2ds + 4b_2\epsilon_1^2\|\alpha\|^2\int_{\tau-t}^{r} e^{\mu s}\mathbb{E}(\|u(s-\rho)\|^2)ds\\
&\le 2b_2\epsilon_1^2\sum_{k=1}^{\infty}\int_{\tau-t}^{r} e^{\mu s}\mathbb{E}(\|h_k(s)\|^2)ds + \frac{4b_2\epsilon_1^2\|\delta\|^2}{\mu}(e^{\mu r}-e^{\mu(\tau-t)}) + 4b_2\epsilon_1^2\|\alpha\|^2e^{\mu\rho}\int_{\tau-t-\rho}^{r-\rho} e^{\mu s}\mathbb{E}(\|u(s)\|^2)ds\\
&\le 2b_2\epsilon_1^2\sum_{k=1}^{\infty}\int_{\tau-t}^{r} e^{\mu s}\mathbb{E}(\|h_k(s)\|^2)ds + \frac{4b_2\epsilon_1^2\|\delta\|^2}{\mu}(e^{\mu r}-e^{\mu(\tau-t)})\\
&+4b_2\epsilon_1^2\|\alpha\|^2e^{\mu\rho}\int_{\tau-t}^{r} e^{\mu s}\mathbb{E}(\|u(s)\|^2)ds + 4b_2\epsilon_1^2\|\alpha\|^2e^{\mu\rho}\int_{\tau-t-\rho}^{\tau-t} e^{\mu s}\mathbb{E}(\|u(s)\|^2)ds\\
&= 2b_2\epsilon_1^2\sum_{k=1}^{\infty}\int_{\tau-t}^{r} e^{\mu s}\mathbb{E}(\|h_k(s)\|^2)ds + \frac{4b_2\epsilon_1^2\|\delta\|^2}{\mu}(e^{\mu r}-e^{\mu(\tau-t)})\\
&+4\epsilon_1^2\|\alpha\|^2e^{\mu\rho}\int_{\tau-t}^{r} e^{\mu s}\mathbb{E}(b_2\|u(s)\|^2)ds + 4b_2\epsilon_1^2\|\alpha\|^2e^{\mu\rho}e^{\mu(\tau-t)}\int_{-\rho}^{0} e^{\mu s}\mathbb{E}(\|\phi(s)\|^2)ds,
\end{aligned}\tag{41}$$

and

$$\begin{aligned}
&b_1\epsilon_2^2\sum_{k=1}^{\infty}\int_{\tau-t}^{r} e^{\mu s}\mathbb{E}(\|h_k(s)+\sigma_k(v(s-\rho))\|^2)ds\\
&\le 2b_1\epsilon_2^2\sum_{k=1}^{\infty}\int_{\tau-t}^{r} e^{\mu s}\mathbb{E}(\|h_k(s)\|^2)ds + \frac{4b_1\epsilon_2^2\|\delta\|^2}{\mu}(e^{\mu r}-e^{\mu(\tau-t)})\\
&+4\epsilon_2^2\|\alpha\|^2e^{\mu\rho}\int_{\tau-t}^{r} e^{\mu s}\mathbb{E}(b_1\|v(s)\|^2)ds + 4b_1\epsilon_2^2\|\alpha\|^2e^{\mu\rho}e^{\mu(\tau-t)}\int_{-\rho}^{0} e^{\mu s}\mathbb{E}(\|\xi(s)\|^2)ds.
\end{aligned}\tag{42}$$

From (37) to (42), we have

$$e^{\mu r}\mathbb{E}(b_2\|u(r)\|^2+b_1\|v(r)\|^2)\le e^{\mu(\tau-t)}\mathbb{E}(b_2\|u_0\|^2+b_1\|v_0\|^2)$$
$$+\frac{1}{\eta}\int_{\tau-t}^{r}e^{\mu s}\mathbb{E}\big(b_2\|f_1(s)\|^2+b_1\|f_2(s)\|^2\big)ds+(2b_2\epsilon_1^2+2b_1\epsilon_2^2)\sum_{k=1}^{\infty}\int_{\tau-t}^{r}e^{\mu s}\mathbb{E}(\|h_k(s)\|^2)ds$$
$$+(b_2\epsilon_1^2+b_1\epsilon_2^2)\frac{4\|\delta\|^2}{\mu}(e^{\mu r}-e^{\mu(\tau-t)})+4b_2\epsilon_1^2\|\alpha\|^2e^{\mu\rho}e^{\mu(\tau-t)}\int_{-\rho}^{0}e^{\mu s}\mathbb{E}(\|\phi(s)\|^2)ds$$
$$+4b_1\epsilon_2^2\|\alpha\|^2e^{\mu\rho}e^{\mu(\tau-t)}\int_{-\rho}^{0}e^{\mu s}\mathbb{E}(\|\xi(s)\|^2)ds$$
$$+(\eta+\mu-2\lambda+4\|\alpha\|^2(\epsilon_1^2+\epsilon_2^2)e^{\mu\rho})\int_{\tau-t}^{r}e^{\mu s}\mathbb{E}(b_2\|u(s)\|^2+b_1\|v(s)\|^2)ds. \tag{43}$$

By (33) and (43), for any $r\in(\tau-t,\tau]$, we get

$$e^{\mu r}\mathbb{E}(b_2\|u(r)\|^2+b_1\|v(r)\|^2)\le e^{\mu(\tau-t)}\mathbb{E}(b_2\|u_0\|^2+b_1\|v_0\|^2)$$
$$+\frac{1}{\eta}\int_{\tau-t}^{r}e^{\mu s}\mathbb{E}\big(b_2\|f_1(s)\|^2+b_1\|f_2(s)\|^2\big)ds+(2b_2\epsilon_1^2+2b_1\epsilon_2^2)\sum_{k=1}^{\infty}\int_{\tau-t}^{r}e^{\mu s}\mathbb{E}(\|h_k(s)\|^2)ds$$
$$+(b_2\epsilon_1^2+b_1\epsilon_2^2)\frac{4\|\delta\|^2}{\mu}(e^{\mu r}-e^{\mu(\tau-t)})+4b_2\epsilon_1^2\|\alpha\|^2e^{\mu\rho}e^{\mu(\tau-t)}\int_{-\rho}^{0}e^{\mu s}\mathbb{E}(\|\phi(s)\|^2)ds$$
$$+4b_1\epsilon_2^2\|\alpha\|^2e^{\mu\rho}e^{\mu(\tau-t)}\int_{-\rho}^{0}e^{\mu s}\mathbb{E}(\|\xi(s)\|^2)ds,$$

which implies that for all $r\in(\tau-t,\tau]$,

$$\mathbb{E}(b_2\|u(r;\tau-t,u_0,\phi)\|^2+b_1\|v(r;\tau-t,v_0,\xi)\|^2)$$
$$\le e^{\mu(\tau-t-r)}\mathbb{E}(b_2\|u_0\|^2+b_1\|v_0\|^2)$$
$$+\frac{1}{\eta}\int_{\tau-t}^{r}e^{\mu(s-r)}\mathbb{E}\big(b_2\|f_1(s)\|^2+b_1\|f_2(s)\|^2\big)ds$$
$$+(2b_2\epsilon_1^2+2b_1\epsilon_2^2)\sum_{k=1}^{\infty}\int_{\tau-t}^{r}e^{\mu(s-r)}\mathbb{E}(\|h_k(s)\|^2)ds$$
$$+(b_2\epsilon_1^2+b_1\epsilon_2^2)\frac{4\|\delta\|^2}{\mu}+4b_2\epsilon_1^2\|\alpha\|^2e^{\mu\rho}e^{\mu(\tau-t-r)}\int_{-\rho}^{0}e^{\mu s}\mathbb{E}(\|\phi(s)\|^2)ds$$
$$+4b_1\epsilon_2^2\|\alpha\|^2e^{\mu\rho}e^{\mu(\tau-t-r)}\int_{-\rho}^{0}e^{\mu s}\mathbb{E}(\|\xi(s)\|^2)ds. \tag{44}$$

Form (44), we find

$$\mathbb{E}(b_2\|u(\tau;\tau-t,u_0,\phi)\|^2+b_1\|v(\tau;\tau-t,v_0,\xi)\|^2)$$
$$\le e^{-\mu t}\mathbb{E}(b_2\|u_0\|^2+b_1\|v_0\|^2)$$
$$+\frac{1}{\eta}\int_{\tau-t}^{\tau}e^{\mu(s-\tau)}\mathbb{E}\big(b_2\|f_1(s)\|^2+b_1\|f_2(s)\|^2\big)ds$$
$$+(2b_2\epsilon_1^2+2b_1\epsilon_2^2)\sum_{k=1}^{\infty}\int_{\tau-t}^{\tau}e^{\mu(s-\tau)}\mathbb{E}(\|h_k(s)\|^2)ds$$
$$+(b_2\epsilon_1^2+b_1\epsilon_2^2)\frac{4\|\delta\|^2}{\mu}+4b_2\epsilon_1^2\|\alpha\|^2e^{\mu\rho}e^{-\mu t}\int_{-\rho}^{0}\mathbb{E}(\|\phi(s)\|^2)ds$$
$$+4b_1\epsilon_2^2\|\alpha\|^2e^{\mu\rho}e^{-\mu t}\int_{-\rho}^{0}\mathbb{E}(\|\xi(s)\|^2)ds, \tag{45}$$

and we have, for $t \geq \rho$,

$$\begin{aligned}
&\sup_{\tau-\rho\leq r\leq\tau} \mathbb{E}(b_2\|u(r;\tau-t,u_0,\phi)\|^2 + b_1\|v(r;\tau-t,v_0,\xi)\|^2) \\
&\quad\leq e^{\mu(\rho-t)}\mathbb{E}(b_2\|u_0\|^2 + b_1\|v_0\|^2) \\
&\quad+ \frac{e^{\mu\rho}}{\eta}\int_{\tau-t}^{\tau} e^{\mu(s-\tau)}\mathbb{E}\left(b_2\|f_1(s)\|^2 + b_1\|f_2(s)\|^2\right)ds \\
&\quad+ (2b_2\epsilon_1^2 + 2b_1\epsilon_2^2)e^{\mu\rho}\sum_{k=1}^{\infty}\int_{\tau-t}^{\tau} e^{\mu(s-\tau)}\mathbb{E}(\|h_k(s)\|^2)ds \\
&\quad+ (b_2\epsilon_1^2 + b_1\epsilon_2^2)\frac{4\|\delta\|^2}{\mu} + 4b_2\epsilon_1^2\|\alpha\|^2e^{\mu(2\rho-t)}\int_{-\rho}^{0}\mathbb{E}(\|\phi(s)\|^2)ds \\
&\quad+ 4b_1\epsilon_2^2\|\alpha\|^2e^{\mu(2\rho-t)}\int_{-\rho}^{0}\mathbb{E}(\|\xi(s)\|^2)ds.
\end{aligned} \tag{46}$$

By (45)-(46), we get that for $t \geq \rho$,

$$\begin{aligned}
&\mathbb{E}(b_2\|u(\tau;\tau-t,u_0,\phi)\|^2 + b_1\|v(\tau;\tau-t,v_0,\xi)\|^2) \\
&\quad+ \int_{\tau-\rho}^{\tau}\mathbb{E}(b_2\|u(s;\tau-t,u_0,\phi)\|^2 + b_1\|v(s;\tau-t,v_0,\xi)\|^2)ds \\
&\quad\leq (1+\rho e^{\mu\rho})\Big[e^{-\mu t}\mathbb{E}(b_2\|u_0\|^2 + b_1\|v_0\|^2) \\
&\quad+ \frac{1}{\eta}\int_{\tau-t}^{\tau} e^{\mu(s-\tau)}\mathbb{E}\left(b_2\|f_1(s)\|^2 + b_1\|f_2(s)\|^2\right)ds \\
&\quad+ (2b_2\epsilon_1^2 + 2b_1\epsilon_2^2)\sum_{k=1}^{\infty}\int_{\tau-t}^{\tau} e^{\mu(s-\tau)}\mathbb{E}(\|h_k(s)\|^2)ds + (b_2\epsilon_1^2 + b_1\epsilon_2^2)\frac{4\|\delta\|^2}{\mu} \\
&\quad+ 4b_2\epsilon_1^2\|\alpha\|^2e^{\mu(\rho-t)}\int_{-\rho}^{0}\mathbb{E}(\|\phi(s)\|^2)ds + 4b_1\epsilon_2^2\|\alpha\|^2e^{\mu(\rho-t)}\int_{-\rho}^{0}\mathbb{E}(\|\xi(s)\|^2)ds\Big].
\end{aligned} \tag{47}$$

For the first term and the last two terms on the right-hand side of (47), by $(u_0,\phi),(v_0,\xi) \in B(\tau-t)$ we have

$$\begin{aligned}
&e^{-\mu t}\mathbb{E}(b_2\|u_0\|^2 + b_1\|v_0\|^2) + 4b_2\epsilon_1^2\|\alpha\|^2e^{\mu(\rho-t)}\int_{-\rho}^{0}\mathbb{E}(\|\phi(s)\|^2)ds \\
&\quad+ 4b_1\epsilon_2^2\|\alpha\|^2e^{\mu(\rho-t)}\int_{-\rho}^{0}\mathbb{E}(\|\xi(s)\|^2)ds \\
&\quad\leq \left(\max\{b_1,b_2\}e^{-\mu\tau} + (b_2\epsilon_1^2 + b_1\epsilon_2^2)4\|\alpha\|^2e^{\mu(\rho-\tau)}\right)e^{\mu(\tau-t)}\|B(\tau-t)\|^2_{H_{\tau-t}} \to 0,
\end{aligned} \tag{48}$$

as $t \to \infty$. Therefore, there exist $T = T(\tau, B) \geq \rho$ and a positive number c such that for all $t \geq T$,

$$\begin{aligned}
&\mathbb{E}(b_2\|u(\tau;\tau-t,u_0,\phi)\|^2 + b_1\|v(\tau;\tau-t,v_0,\xi)\|^2) \\
&\qquad+ \int_{\tau-\rho}^{\tau}\mathbb{E}(b_2\|u(s;\tau-t,u_0,\phi)\|^2 + b_1\|v(s;\tau-t,v_0,\xi)\|^2)ds \\
&\qquad\leq (1+\rho e^{\mu\rho})\Big[1 + (b_2\epsilon_1^2 + b_1\epsilon_2^2)\frac{4\|\delta\|^2}{\mu} \\
&\qquad+ c\int_{-\infty}^{\tau} e^{\mu(s-\tau)}\mathbb{E}(\|f_1(s)\|^2 + \|f_2(s)\|^2 + \sum_{k=1}^{\infty}\|h_k(s)\|^2)ds\Big],
\end{aligned}$$

which completes the proof. □

Next, we show the main result of this section.

Theorem 3.2. *Assume* (3), (17), (32) *and* (35) *hold. Then the mean random dynamical system* Φ *related to* (18) *has a unique weak* $\mathcal{D}$*-pullback mean random attractor* $\mathcal{A} = \{\mathcal{A}(\tau) : \tau \in \mathbb{R}\} \in \mathcal{D}$ *in* $L^2(\Omega, \mathcal{F}; \ell^2 \times \ell^2) \times L^2(\Omega, \mathcal{F}; L^2((-\rho, 0), \ell^2 \times \ell^2))$ *over* $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\in\mathbb{R}}, \mathbb{P})$*; this is, (i)* $\mathcal{A}(\ll)$ *is a weakly compact subset of* $L^2(\Omega, \mathcal{F}_\tau; \ell^2 \times \ell^2) \times L^2(\Omega, \mathcal{F}_\tau; L^2((-\rho, 0), \ell^2 \times \ell^2))$ *for every* $\tau \in \mathbb{R}$*. (ii)* $\mathcal{A}$ *is a weakly* $\mathcal{D}$*-pullback attracting set of* Φ*. (iii)* $\mathcal{A}$ *is the minimal element of* $\mathcal{D}$ *with properties (i) and (ii); this is, if* $K = \{K(\tau) : \tau \in \mathbb{R}\} \in \mathcal{D}$ *satisfies (i) and (ii), then* $\mathcal{A}(\tau) \subseteq K(\tau)$ *for any* $\tau \in \mathbb{R}$*.*

Proof. For every $\tau \in \mathbb{R}$, let us define

$$M(\tau) = \Big\{(\omega, \varsigma) \in H_\tau : \|(\omega, \varsigma)\|^2_{H_\tau} \le R(\tau)\Big\},$$

where

$$\begin{aligned} R(\tau) = (1 + \rho e^{\mu\rho})\Big[1 + (b_2\epsilon_1^2 + b_1\epsilon_2^2)\frac{4\|\delta\|^2}{\mu} \\ + c\int_{-\infty}^{\tau} e^{\mu(s-\tau)}\mathbb{E}(\|f_1(s)\|^2 + \|f_2(s)\|^2 + \sum_{k=1}^{\infty}\|h_k(s)\|^2)ds\Big], \end{aligned}$$

where $\mu > 0$ is the same number as in (33). Because $M(\tau)$ is a bounded closed convex subset of H_τ, it is weakly compact in H_τ.

Furthermore, by (35) we obtain

$$\lim_{\tau\to-\infty} e^{\mu\tau}\|M(\tau)\|^2_{H_\tau} = \lim_{\tau\to-\infty} e^{\mu\tau}R(\tau) = 0,$$

consequently, $M = \{M(\tau) : \tau \in \mathbb{R}\} \in \mathcal{D}$.

In addition, by Lemma 3.1, we find that for each $\tau \in \mathbb{R}$ and $B = \{B(t)\}_{t\in\mathbb{R}} \in \mathcal{D}$, there exists $T = T(\tau, B) \ge \rho$ such that for any $t \ge T$,

$$\Phi(t, \tau - t)(B(\tau - t)) \subseteq M(\tau).$$

Therefore, M is a weakly compact $\mathcal{D}$-pullback absorbing set of Φ. Then by [34, Theorem 2.13], Φ has a unique weak $\mathcal{D}$-pullback mean random attractor $\mathcal{A} \in \mathcal{D}$ in $L^2(\Omega, \mathcal{F}; \ell^2 \times \ell^2) \times L^2(\Omega, \mathcal{F}; L^2((-\rho, 0), \ell^2 \times \ell^2))$.

□

4 Declarations

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The authors declare no conflict of interest.

Ethical Approval

Not applicable.

Author's Contributions

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Not applicable.

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