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Classification of Blow-up and Global Existence of Solutions to a System of Petrovsky Equations

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Abstract

In this paper, we investigate global existence, uniform decay, and blow-up of solutions for a class of system of Petrovsky equations containing nonlinear damping and sources. By introducing a family of potential wells, we not only obtain the invariant sets and vacuum isolating of solutions but also give some threshold results of global existence and nonexistence of solutions. Furthermore, by using energy techniques, we also establish certain qualitative estimates for solution.

Key words: Global existence; Decay rate; Blow-up in finite time; Petrovsky systems
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1 Introduction

Let $\Omega \subset \mathbb{R}^n$ be an open bounded domain with a smooth boundary $\partial\Omega$. In this paper, depending on suitable conditions of the initial datum $(u_0, v_0, u_1, v_1) \in \left[H_0^2(\Omega)\right]^2 \times \left[L^2(\Omega)\right]^2$, we are interested in both finite time blow-up solutions and solutions which exist globally in time of the following system of Petrovsky equations

$$u_{tt} + \Delta^2 u + |u_t|^{q_1-2} u_t = f_1(u,v),\ (x,t) \in \Omega \times (0,\infty), \tag{1}$$

$$v_{tt} + \Delta^2 v + |v_t|^{q_2-2} v_t = f_2(u,v),\ (x,t) \in \Omega \times (0,\infty), \tag{2}$$

associated with homogeneous Dirichlet boundary conditions

$$u(x,t) = v(x,t) = \partial_\nu u(x,t) = \partial_\nu v(x,t) = 0,\ (x,t) \in \partial\Omega \times (0,\infty), \tag{3}$$

and supplemented with the following initial conditions

$$u(x,0) = u_0(x),\ u_t(x,0) = u_1(x),\ v(x,0) = v_0(x),\ v_t(x,0) = v_1(x),\ x \in \Omega, \tag{4}$$

where

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i. the operator Δ^2 is a bi-Laplace operator, which given by

$$\Delta^2 u := \Delta(\Delta u) = \sum_{\ell\in\mathbb{Z}_+^n, |\ell|=2} \frac{m!}{\ell!}\partial^{2\ell} u = \sum_{\ell\in\mathbb{Z}_+^n, |\ell|=2} \frac{m!}{\ell_1!\cdots\ell_n!}\frac{\partial^4 u}{\partial x_1^{2\ell_1}\cdots\partial x_n^{2\ell_n}};$$

ii. the vector ν denotes the unit normal vector pointing toward the exterior of Ω, and ∂_ν stands by for normal derivative;

iii. the nonlinearities f_1, f_2 are interior sources, which satisfy certain conditions specified later.

Let us begin by introducing the physical significance and mathematical meaning of the so-called bi-Laplace operator Δ^2. This operator appears naturally for the vibration of a clamped plate in the classical elastic mechanics. In order to study the long time behavior of the roadway of a suspension bridge, in [1], Runzhang Xu et al. considered the following nonlinear wave equation with bi-Laplace operator

$$u_{tt} + \Delta^2 u + au + \mu|u_t|^{q-2}u_t = |u|^{p-2}u,\ (x,t)\in(0,\pi)\times(-\ell,\ell),\ t\in(0,\infty),$$

associated with free-type boundary conditions

$$\begin{cases} u(0,y,t) = u_{xx}(0,y,t) = u(\pi,y,t) = u_{xx}(\pi,y,t),\ y\in(-\ell,\ell), \\ u_{yy}(x,\pm\ell,t) + \sigma u_{xx}(x,\pm\ell,t) = u_{yyy}(x,\pm\ell,t) + (2-\sigma)u_{xxy}(x,\pm\ell,t) = 0,\ x\in(0,\pi), \end{cases}$$

and initial data

$$u(x,y,0) = u_0(x),\ u_t(x,y,t) = u_1(x),$$

where $\mu > 0$, $2 < q < p < \infty$, $\sigma \in \left(0, \frac{1}{2}\right)$. By using potential well method, the authors gave a threshold result for the global existence and blow-up in finite time. Furthermore, a certain decay estimate also obtained by using Nakao inequality. Not only appear in evolution equations, but also for steady state equation. For example, in [2], Xiaotian Hao considered the eigenvalue problem

$$\begin{cases} \mathfrak{L}_\nu^2 u = \Gamma u,\ x\in\Omega, \\ u = \dfrac{\partial u}{\partial \mathbf{n}} = 0,\ x\in\partial\Omega, \end{cases}$$

where $\mathbf{n}$ denotes the outward unit normal to the boundary, and $\mathfrak{L}_\nu$ is Xin-Laplacian operator. First, the authors proved the existence of the eigenvalues of Xin-Laplacian operator and then estimated the spectrum gap of these eigenvalues.

On the other hand, it is well known that nonlinear wave equations can be used to describe a variety of problems in physics, engineering, chemistry, material science and other sciences. The study of nonlinear wave equations has also great significance in mathematical analysis [3–8]. In [5], Messaoudi consider the fourth-order wave equation

$$u_{tt} + \Delta^2 u + |u_t|^{q-2}u_t = |u|^{p-2}u,\ (x,t)\in\Omega\times(0,\infty).$$

First, by using standard Faedo-Galerkin method, the author established the local existence and uniqueness of the weak solution. Later, he showed the solution blows-up in finite time if $q \geqslant p$, and the solution exists globally if $q < p$. That means if the damping term dominates the source term, then the solution may exist globally, and vice versa. However, no decay rate of the global solution is given and no blow-up result is discussed for the initial energy being nonnegative. In [6], the authors considered the following system of Petrovsky equations

$$\begin{cases} u_{tt} + \Delta^2 u + |u_t|^{q-2}u_t = f_1(u,v),\ (x,t)\in\Omega\times(0,\infty), \\ v_{tt} + \Delta^2 v + |v_t|^{q-2}v_t = f_2(u,v),\ (x,t)\in\Omega\times(0,\infty), \end{cases}$$

where

$$f_1(u,v) = \alpha_1 |u+v|^{p-2}(u+v) + \alpha_2 |u|^{\frac{p-4}{2}} u |v|^{\frac{p}{2}},\ f_2(u,v)$$
$$= \alpha_1 |u+v|^{p-2}(u+v) + \alpha_2 |v|^{\frac{p-4}{2}} v |u|^{\frac{p}{2}}.$$

By using potential well method as well as some useful differential inequality, the authors in this paper the global existence, decay and blow-up of solutions of this problem. To our best knowledge, despite the fact that we have numerous literature that consider the system of wave or heat equations, the previous authors just considered some specific nonlinear terms. The interested reader is refered to Remark 2.4 for more examples. We are also aware of the important role of considering the system of two equations rather than a single equation, especially considering the coupling effects and interactions of the nonlinear terms, at least, from mathematical point of view. But so far we have no standard method to circumvent this difficulty, the main reason is we cannot simply solve this problem by parallelizing the method for a single equation due to the interactions in the nonlinearities. At least, we need to answer some fundamental questions. First of all, we realize that we are still unable to handle all but the most important coupling nonlinearities. So we need decide which nonlinearities can be prioritized. Although these nonlinear features have a very clear physical and applied background, we must be honest to say that these particular nonlinear cases also bring a lot of convenience to us for constructing the variational structure and conducting corresponding analysis. Recently, in [9], the authors introduced a new assumption to dealing with more nonlinearities which considered before. We will introduce and discuss this assumption in Section 2. We also note that, in this manuscript, the nonlinear terms also contain two nonlinear weak damping terms. These terms usually describe the friction during the process of motion. The interested reader is referred to [10] for more details. The rest of this paper is organized as follows.

i. In Section 2, we prepare some notations, preliminaries. We also introduce a new assumption for source terms;
ii. In Section 3, we investigate the stationary problem under the new assumption. We also introduce a family of potential wells and consider the vacuum isolating phenomena of solution for Problem (1)-(4);
iii. In Section 4, by applying the concept of the family of potential wells, we prove the global existence and finite time blow up of solutions with subcritical initial energy, i.e., $E(0) < d$, which is shown in Theorems 4.1, 4.2, 4.3 and 4.8;
iv. In Section 5, we extend in parallel the results in subcritical initial energy level to critical initial energy level, i.e., $E(0) = d$, which are shown in Theorems 5.3 and 5.4;
v. In Section 6, at arbitrarily high initial energy level, i.e., $E(0) > d$, we give the sufficient conditions for blow-up in finite time. The result is displayed in Theorems 6.4.

2 Notation and primary results

First of all, let us recall a few preliminaries about the notations and some useful lemmas. Throughout this paper, we use the conventional notation $L^p(\Omega)$, with $1 \leqslant p \leqslant \infty$, for the usual Lebesgue space equipped with $\|\cdot\|_{L^p(\Omega)}$ norm. For simplicity, we write $\|\cdot\|_p$ for $\|\cdot\|_{L^p(\Omega)}$, we also denote the inner product on the Hilbert space $L^2(\Omega)$ by $\langle \cdot, \cdot \rangle$. We also need to introduce the Sobolev space

$$H_0^2(\Omega) = \left\{ u \in H^2(\Omega) : u = \partial_\nu u = 0 \right\}.$$

By applying Poincaré inequality, we can easily show that $H_0^2(\Omega)$ is a Hilbert space when endowed with the inner product

$$\langle u,v\rangle_{H_0^2} = \langle \Delta u, \Delta v\rangle = \int_\Omega \Delta u(x)\,\Delta v(x)\,\mathrm{d}x,\ \forall u,\ v\in H_0^2(\Omega).$$

This inner product induces a norm

$$\|u\|_{H_0^2} = \sqrt{\langle u,u\rangle_{H_0^2}} = \|\Delta u\|^2,\ \forall u\in H_0^2(\Omega),$$

which is equivalent to the usual norm $\|\cdot\|_{H^2}$. Moreover, we have the following Sobolev Embedding Inequality for this case as follows.

Lemma 2.1. *Let $\Omega\subset\mathbb{R}^n$ be an open bounded domain with a smooth boundary $\partial\Omega$. Suppose that $2<p<2^{**}$, where $2^{**}=\frac{2n}{n-4}$ when $n\geqslant 4$, or $2^{**}=\infty$ if $n\in\{1,2,3\}$. Then, we have the embedding $H_0^2(\Omega)\hookrightarrow L^p(\Omega)$ is compact. Furthermore, we have the following inequality*

$$\|u\|_p \lesssim \|\Delta u\|_2,\ \forall u\in H_0^2(\Omega). \tag{5}$$

Before going further, some assumptions on the nonlinear sources will be needed. We are mainly interested in higher order nonlinearities, as described in the following assumption.

Assumption 2.2. Let $f_1,\ f_2\in C^1(\mathbb{R}^2)$ satisfying the following conditions:

i. There exists a constant $p\in(2,2^{**})$ such that

$$|\nabla f_i(u,v)| \lesssim 1+|u|^{p-2}+|v|^{p-2},\ \forall i\in\{1,2\},\ (u,v)\in\mathbb{R}^2; \tag{6}$$

ii. The vector field $f=(f_1,f_2):\mathbb{R}^2\longrightarrow\mathbb{R}^2$ is a conservative vector field. That means there exists a potential function $F:\mathbb{R}^2\to\mathbb{R}$ such that

$$\nabla F(u,v) = f(u,v) = (f_1(u,v), f_2(u,v)),\ \forall (u,v)\in\mathbb{R}^2; \tag{7}$$

iii. The potential function F is homogeneous degree p. Namely, we have

$$F(\lambda u,\lambda v) = \lambda^p F(u,v),\ \forall (u,v)\in\mathbb{R}^2,\ \lambda\in(0,\infty); \tag{8}$$

iv. The potential function F is positive on $\mathbb{R}^2\setminus\{0\}$.

Remark 2.3. Since the potential function F is homogeneous degree p, then it is easy to check that

$$f(u,v)\cdot(u,v) = \nabla F(u,v)\cdot(u,v) = pF(u,v),\ \forall (u,v)\in\mathbb{R}^2. \tag{9}$$

Furthermore, we also obtain two functions f_1 and f_2 are homogeneous degree $p-1$. On the other hand, for any $(u,v)\in\mathbb{R}^2$, we have

$$\begin{aligned}
|f_i(u,v)| &= \left| f_i(0,0) + \int_0^1 \nabla f_i(su,sv)\cdot(u,v)\,\mathrm{d}s\right| \\
&\lesssim 1+\int_0^1 |\nabla f_i(su,sv)|\,|(u,v)|\,\mathrm{d}s \\
&\lesssim 1+\left(|u|^{p-2}+|v|^{p-2}\right)(|u|+|v|) \\
&\lesssim 1+|u|^{p-1}+|v|^{p-1},
\end{aligned}$$

for all $(u,v) \in \mathbb{R}^2$ and $i \in \{1,2\}$. Hence, the homogeneity of three functions f_1, f_2 and F enable us to verify that there exist two positive constants c_2 and C_2 such that

$$\begin{aligned} |f_i(u,v)| &\leqslant c_2 \left(|u|^{p-1} + |v|^{p-2}\right), \\ F(u,v) &\leqslant C_2 \left(|u|^p + |v|^p\right), \forall i \in \{1,2\}, (u,v) \in \mathbb{R}^2. \end{aligned} \tag{10}$$

Finally, for any $(u,v) \in \mathbb{R}^2$, it follows from (8) that

$$\begin{aligned} F(u,v) &= \left(u^2+v^2\right)^{\frac{p}{2}} F\left(\frac{u}{\sqrt{u^2+v^2}}, \frac{u}{\sqrt{u^2+v^2}}\right) \\ &\gtrsim \left(\min_{(u,v)\in\mathbb{R}^2: u^2+v^2=1} F(u,v)\right)\left(|u|^p+|v|^p\right). \end{aligned}$$

Therefore, there exists a positive constant C_1 such that

$$C_1\left(|u|^p+|v|^p\right) \leqslant F(u,v), \forall (u,v) \in \mathbb{R}^2. \tag{11}$$

Remark 2.4. We also note that Assumption 2.2 contains a large class of nonlinear source, which considered in the previous paper [5, 6, 11–13]. Indeed, by direct calculation, it is easy to verify that the following nonlinear sources

$$f_1(u,v) = |u|^{p-2}u + |v|^{\frac{p}{2}}|u|^{\frac{p}{2}-2}, \; f_2(u,v) = |v|^{q-2}v + |u|^{\frac{p}{2}}|v|^{\frac{p}{2}-2}, \tag{12}$$

or

$$\begin{aligned} f_1(u,v) &= \alpha_1|u+v|^{p-2}(u+v) + \alpha_2|u|^{\frac{p-4}{2}}u|v|^{\frac{p}{2}}, \; f_2(u,v) \\ &= \alpha_1|u+v|^{p-2}(u+v) + \alpha_2|v|^{\frac{p-4}{2}}v|u|^{\frac{p}{2}}, \end{aligned} \tag{13}$$

or

$$\begin{aligned} f_1(u,v) &= |u+v|^{p-2}(u+v) + |u-v|^{p-2}(u-v), \; f_2(u,v) \\ &= |u+v|^{p-2}(u+v) - |u-v|^{p-2}(u-v), \end{aligned} \tag{14}$$

where p, α_1, α_2 are given parameters satisfying some restrictions.

Next, we give the precise definition of a weak solution of Problem (1)-(4).

Definition 2.5. A coupled function (u,v) is called a weak solution of the Problem (1)-(4) on $(0,T)$ if

$$(u,v) \in C\left([0,T]; \left[H_0^2(\Omega)\right]^2\right) \cap C^1\left([0,T]; \left[L^2(\Omega)\right]^2\right), \tag{15}$$

with

$$u_t \in L^{q_1}(0,T; L^{q_1}(\Omega)), \; v_t \in L^{q_2}(0,T; L^{q_2}(\Omega)), \tag{16}$$

and (u,v) satisfies (1)-(4) in the following sense:

i. (u,v) is a distributional solution, i.e.,

$$\frac{\mathrm{d}}{\mathrm{d}t}\langle u'(t), w_1\rangle + \langle \Delta u(t), \Delta w_1\rangle + \left\langle |u'(t)|^{q_1}u'(t), w_1\right\rangle = \langle f_1(u(t), v(t)), w_1\rangle, \tag{17}$$

and

$$\frac{\mathrm{d}}{\mathrm{d}t}\langle v'(t), w_2\rangle + \langle \Delta v(t), \Delta w_2\rangle + \left\langle |v'(t)|^{q_2}v'(t), w_2\right\rangle = \langle f_2(u(t), v(t)), w_2\rangle, \tag{18}$$

for all test function $(w_1, w_2) \in \left[H_0^2(\Omega)\right]^2$ and $t \in (0,T)$;

ii. (u, v) fulfills the initial datum

$$u(0) = u_0,\ v(0) = v_0,\ u'(0) = u_1,\ v'(0) = v_0. \tag{19}$$

We have the following theorem about the local existence and uniqueness of a weak solution.

Theorem 2.6. *Let Assumption 2.2 be in force, and $2 < p < 2^{**}$, $q_1 \geqslant 2$, $q_2 \geqslant 2$. Then, for any $(u_0, v_0, u_1, v_1) \in \left[H_0^2(\Omega)\right]^2 \times \left[L^2(\Omega)\right]^2$, Problem (1)-(4) admits a unique weak solution (u, v) on $(0, T_*)$ with $T_* > 0$ small enough.*

Moreover, if we denote T_∞ is a maximal existence time of the solution (u, v) for the Problem (1)-(4), the following alternatives hold

i. If $T_\infty = \infty$, then we say that the solution of Problem (1)-(4) is global;
ii. If $T_\infty < \infty$, then

$$\lim_{t \uparrow T_\infty} \left(\|u'(t)\|_2^2 + \|v'(t)\|_2^2 + \|\Delta u(t)\|_2^2 + \|\Delta v(t)\|_2^2 \right) = \infty, \tag{20}$$

and we say that the solution of Problem (1)-(4) blows up in finite time, and that $T_\infty = \infty$ is the blow-up time.

Remark 2.7. In order to ensure the local existence of the weak solution, in what follows, we always assume that Assumption 2.2 fulfills, and $(u_0, v_0, u_1, v_1) \in \left[H_0^2(\Omega)\right]^2 \times \left[L^2(\Omega)\right]^2$ to ensure the local Hadamard well-posedness result for our model.

3 Stationary problem, potential well setting, invariant sets and vacuum isolating of solutions

3.1 Stationary problem, potential well setting, and a family of potential wells

Stationary solutions of Problem (1)-(4) solve the following nonlinear elliptic problem

$$\begin{cases} \Delta^2 u = f_1(u, v),\ x \in \Omega, \\ \Delta^2 v = f_2(u, v),\ x \in \Omega, \\ u(x) = v(x) = \partial_\nu u(x) = \partial_\nu v(x) = 0,\ x \in \partial\Omega. \end{cases} \tag{21}$$

Problem (21) may be investigated with critical-point theory. For this purpose, we consider the potential energy functional $J : \left[H_0^2(\Omega)\right]^2 \longrightarrow \mathbb{R}$, and Nehari functional $I : \left[H_0^2(\Omega)\right]^2 \longrightarrow \mathbb{R}$ given by

$$J(u, v) := \frac{1}{2}\left(\|\Delta u\|_2^2 + \|\Delta v\|_2^2 \right) - \int_\Omega F(u(x), v(x))\, \mathrm{d}x, \tag{22}$$

and

$$I(u, v) := \left(\|\Delta u\|_2^2 + \|\Delta v\|_2^2 \right) - p \int_\Omega F(u(x), v(x))\, \mathrm{d}x. \tag{23}$$

It is clear that $J, I \in C^1\left(\left[H_0^2(\Omega)\right]^2; \mathbb{R} \right)$, and critical points of J are (weak) solutions of Problem (21). Finally, it follows from (22) and (23) that J and I posses the following identity

$$J(u, v) = \frac{p-2}{2p}\left(\|\Delta u\|_2^2 + \|\Delta v\|_2^2 \right) + \frac{I(u, v)}{p},\ \forall (u, v) \in \left[H_0^2(\Omega)\right]^2. \tag{24}$$

Next, we proceed now to establish the fundamental properties of the potential energy functional J and the Nehari functional I.

Lemma 3.1. *Let $(\phi, \psi) \in \left[H_0^2(\Omega)\right]^2$. Then* *i.* $\lim_{\lambda\downarrow 0} J(\lambda\phi,\lambda\psi)=0$, *and* $\lim_{\lambda\to\infty} J(\lambda\phi,\lambda\psi)=-\infty$;

ii. *On the interval* $(0,\infty)$, *there exists a unique constant* $\lambda_* := \left(\frac{\|\Delta\phi\|_2^2+\|\Delta\psi\|_2^2}{p\int_\Omega F(u(x),v(x))\,dx}\right)^{\frac{1}{p-2}} > 0$ *such that* $\frac{d}{d\lambda}J(\lambda\phi,\lambda\psi)\Big|_{\lambda=\lambda_*}=0$. *Moreover, the function* $\lambda\longmapsto J(\lambda\phi,\lambda\psi)$ *is strictly increasing on* $(0,\lambda_*)$, *strictly decreasing on* (λ_*,∞), *and takes the maximum at* $\lambda=\lambda_*$;

iii. $I(\lambda\phi,\lambda\psi)>0$ *for all* $\lambda\in(0,\lambda_*)$, $I(\lambda\phi,\lambda\psi)<0$ *for all* $\lambda\in(\lambda_*,\infty)$, *and* $I(\lambda_*\phi,\lambda_*\psi)=0$.

Proof of Lemma 3.1. First, by recalling the definition of functional J and (7), we can easily obtain

$$J(\lambda\phi,\lambda\psi)=\frac{\lambda^2}{2}\left(\|\Delta\phi\|_2^2+\|\Delta\psi\|_2^2\right)-\lambda^p\int_\Omega F(\phi(x),\psi(x))\,dx>0,\ \forall\lambda\in(0,\infty).$$

Since $(\phi,\psi)\in\left[H_0^2(\Omega)\right]^2\setminus\{(0,0)\}$, then we have $\int_\Omega F(\psi(x),\psi(x))\,dx>0$. Consequently, $\lim_{\lambda\downarrow 0} J(\lambda\phi,\lambda\psi)=0$, and $\lim_{\lambda\to\infty} J(\lambda\phi,\lambda\psi)=-\infty$.

Next, a direct calculation gives us

$$\frac{d}{d\lambda}J(\lambda\phi,\lambda\psi)=\lambda\left(\|\Delta\phi\|_2^2+\|\Delta\psi\|_2^2\right)-p\lambda^{p-1}\int_\Omega F(\phi(x),\psi(x))\,dx=0\Longleftrightarrow\lambda=\lambda_*.$$

Hence, there exists a unique $\lambda_*=\left(\frac{\|\Delta\phi\|_2^2+\|\Delta\psi\|_2^2}{p\int_\Omega F(u(x),v(x))\,dx}\right)^{\frac{1}{p-2}}>0$ such that $\frac{d}{d\lambda}J(\lambda\phi,\lambda\psi)\Big|_{\lambda=\lambda_*}=0$. As a direct consequence of this fact, we have the function $\lambda\longmapsto J(\lambda\phi,\lambda\psi)$ is strictly increasing on $(0,\lambda_*)$, strictly decreasing on (λ_*,∞), and takes the maximum at $\lambda=\lambda_*$.

Finally, by recalling (7) and (23), for any $(\phi,\psi)\in\left[H_0^2(\Omega)\right]^2$ and $\lambda\in(0,\infty)$, we have

$$I(\lambda\phi,\lambda\psi)=\lambda^2\left(\|\Delta\phi\|_2^2+\|\Delta\psi\|_2^2\right)-p\lambda^p\int_\Omega F(\phi(x),\psi(x))\,dx=\lambda\frac{d}{d\lambda}J(\lambda\phi,\lambda\psi).$$

Accordingly, $I(\lambda\phi,\lambda\psi)>0$ for all $\lambda\in(0,\lambda_*)$, $I(\lambda\phi,\lambda\psi)<0$ for all $\lambda\in(\lambda_*,\infty)$, and $I(\lambda_*\phi,\lambda_*\psi)=0$. Lemma 3.1 is proved completely. □

Lemma 3.2. *Put*

$$S_p:=\sup_{(\phi,\psi)\in\left[H_0^2(\Omega)\right]^2\setminus\{(0,0)\}}\frac{\left(\|\phi\|_p^p+\|\psi\|_p^p\right)^{\frac{1}{p}}}{\sqrt{\|\Delta\phi\|_2^2+\|\Delta\psi\|_2^2}}>0.$$

Let $(\phi,\psi)\in\left[H_0^2(\Omega)\right]^2$, *then the following statements fulfill:*

i. *If* $0<\|\Delta\phi\|_2^2+\|\Delta\psi\|_2^2<\left(\frac{1}{pC_2S_p^p}\right)^{\frac{2}{p-2}}$, *then* $I(\phi,\psi)>0$;

ii. *If* $I(\phi,\psi)<0$, *then* $\|\Delta\phi\|_2^2+\|\Delta\psi\|_2^2>\left(\frac{1}{pC_2S_p^p}\right)^{\frac{2}{p-2}}$;

iii. *If* $I(\phi,\psi)=0$ *and* $(\phi,\psi)\neq(0,0)$, *then* $\|\Delta\phi\|_2^2+\|\Delta\psi\|_2^2\geqslant\left(\frac{1}{pC_2S_p^p}\right)^{\frac{2}{p-2}}$.

Proof of Lemma 3.2. First, by using (10), we have the following estimate

$$\begin{aligned}\int_\Omega F(\phi(x),\psi(x))\,dx &\leqslant C_2\left(\|\phi\|_p^p+\|\psi\|_p^p\right)\\ &\leqslant C_2S_p^p\left(\|\Delta\phi\|_2^2+\|\Delta\psi\|_2^2\right)^{\frac{p}{2}},\ \forall(\phi,\psi)\in\left[H_0^2(\Omega)\right]^2.\end{aligned}$$

Thus, we obtain

$$I(\phi,\psi) \geqslant \left[1 - pC_2S_p^p\left(\|\Delta\phi\|_2^2 + \|\Delta\psi\|_2^2\right)^{\frac{p-2}{2}}\right]\left(\|\Delta\phi\|_2^2 + \|\Delta\psi\|_2^2\right), \forall (\phi,\psi) \in \left[H_0^2(\Omega)\right]^2.$$

Hence,

i. If $0 < \|\Delta\phi\|_2^2 + \|\Delta\psi\|_2^2 < \left(\frac{1}{pC_2S_p^p}\right)^{\frac{2}{p-2}}$, then $I(\phi,\psi) > 0$;

ii. If $I(\phi,\psi) < 0$, then $\|\Delta\phi\|_2^2 + \|\Delta\psi\|_2^2 > \left(\frac{1}{pC_2S_p^p}\right)^{\frac{2}{p-2}}$;

iii. If $I(\phi,\psi) = 0$ and $(\phi,\psi) \neq (0,0)$, then $\|\Delta\phi\|_2^2 + \|\Delta\psi\|_2^2 \geqslant \left(\frac{1}{pC_2S_p^p}\right)^{\frac{2}{p-2}}$.

Lemma 3.2 is proved. □

Next, we define the Nehari manifold

$$\mathscr{N} := \left\{(\phi,\psi) \in \left[H_0^2(\Omega)\right]^2 : I(\phi,\psi) = 0\right\} \setminus \{(0,0)\}. \tag{25}$$

According to Lemma 3.1, it is clear that $\mathscr{N} \neq \emptyset$. We also need to introduce the inside and outside part of Nehari manifold as follows

$$\mathscr{N}_+ := \left\{(\phi,\psi) \in \left[H_0^2(\Omega)\right]^2 : I(\phi,\psi) > 0\right\} \cup \{(0,0)\},$$
$$\mathscr{N}_- = \left\{(\phi,\psi) \in \left[H_0^2(\Omega)\right]^2 : I(\phi,\psi) < 0\right\}.$$

Moreover, as a direct consequence of the identity (24), we have the functional J coercive on the Nehari manifold $\mathscr{N}$. Hence, we can define the depth of potential well or the so-called mountain pass level

$$d := \inf_{(\phi,\psi)\in\mathscr{N}} J(\phi,\psi). \tag{26}$$

Actually, we can show that d is attained by some $(\Phi,\Psi) \in \mathscr{N}$, which is a nontrivial critical point of J on $\left[H_0^2(\Omega)\right]^2 \setminus \{(0,0)\}$. Therefore, there exists a solution to stationary problem associated with Problem (1)-(3).

Lemma 3.3. *The following statements hold:*

i. $d = \inf_{(\phi,\psi)\in\left[H_0^2(\Omega)\right]^2\setminus\{(0,0)\}} \sup_{\lambda\in(0,\infty)} J(\lambda\phi,\lambda\psi)$;

ii. d has a positive lower bound. In particular, we have

$$d \geqslant \frac{p-2}{2p}\left(\frac{1}{pC_2C_*^p}\right)^{\frac{2}{p-2}};$$

iii. There exists an extremal of the variation problem (26). More precisely, there is a couple function $(\Phi,\Psi) \in \mathscr{N}$ such that $J(\Phi,\Psi) = d$. Furthermore, (Φ,Ψ) is a weak solution of Problem (21), and satisfies

$$\frac{\partial J}{\partial\phi}(\Phi,\Psi)w_1 = \frac{\partial J}{\partial\psi}(\Phi,\Psi)w_2 = 0, \forall (w_1,w_2) \in \left[H_0^2(\Omega)\right]^2.$$

Proof of Lemma 3.3. First, with $(\phi,\psi) \in \left[H_0^2(\Omega)\right]^2 \setminus \{(0,0)\}$, it follows from Lemma 3.1 that there exists a unique $\lambda_* > 0$ satisfies

$$\sup_{\lambda\in(0,\infty)} J(\lambda\phi,\lambda\psi) = J(\lambda_*\phi,\lambda_*\psi). \tag{27}$$

By utilizing Lemma 3.1, we deduce that $(\lambda_*\phi,\lambda_*\psi) \in \mathcal{N}$. Combining this with (27) entails

$$\sup_{\lambda\in(0,\infty)} J(\lambda\phi,\lambda\psi) = J(\lambda_*\phi,\lambda_*\psi) \geqslant d.$$

Thus, we have

$$\inf_{(\phi,\psi)\in\left[H_0^2(\Omega)\right]^2\setminus\{(0,0)\}} \sup_{\lambda\in(0,\infty)} J(\lambda\phi,\lambda\psi) \geqslant d. \tag{28}$$

On the other hand, by applying Lemma 3.1, we have $\sup\limits_{\lambda\in(0,\infty)} J(\lambda\phi,\lambda\psi) = J(\phi,\psi)$ for any $(\phi,\psi) \in \mathcal{N}$. Hence, one has

$$\inf_{(\phi,\psi)\in\left[H_0^2(\Omega)\right]^2\setminus\{(0,0)\}} \sup_{\lambda\in(0,\infty)} J(\lambda\phi,\lambda\psi) \leqslant \inf_{(\phi,\psi)\in\mathcal{N}} \sup_{\lambda\in(0,\infty)} J(\lambda\phi,\lambda\psi) = \inf_{(\phi,\psi)\in\mathcal{N}} J(\phi,\psi) = d. \tag{29}$$

Therefore, it follows from (28) and (29) that

$$d = \inf_{(\phi,\psi)\in\left[H_0^2(\Omega)\right]^2\setminus\{(0,0)\}} \sup_{\lambda\in(0,\infty)} J(\lambda\phi,\lambda\psi).$$

Next, by making use Lemma 3.2 and (24), for any $(\phi,\psi) \in \mathcal{N}$, we have

$$J(\phi,\psi) = \frac{p-2}{2p}\left(\|\Delta\phi\|_2^2 + \|\Delta\psi\|_2^2\right) \geqslant \frac{p-2}{2p}\left(\frac{1}{pC_2S_p^p}\right)^{\frac{2}{p-2}}.$$

Thus, as an immediate consequence of this fact, we obtain

$$d = \inf_{(\phi,\psi)\in\mathcal{N}} J(\phi,\psi) \geqslant \frac{p-2}{2p}\left(\frac{1}{pC_2S_p^p}\right)^{\frac{2}{p-2}}.$$

In order to prove the existence of a stationary solution, we will use the direct method. Let $\{(\phi_n,\psi_n)\}_{n=1}^{\infty} \subset \mathcal{N}$ such that $\lim\limits_{n\to\infty} J(\phi_n,\psi_n) = d$. Recalling the identity (24), we deduce that the sequence $\{(\phi_n,\psi_n)\}_{n=1}^{\infty}$ is bounded on $\left[H_0^2(\Omega)\right]^2$. Since the embedding $H_0^2(\Omega) \hookrightarrow L^p(\Omega)$ is compact and $H_0^2(\Omega)$ is a Hilbert space, without loss of generality, we may assume that there exists $(\Phi,\Psi) \in \left[H_0^2(\Omega)\right]^2$ such that

$$\begin{aligned}
&\phi_n \to \Phi \text{ weakly in } H_0^2(\Omega),\ \psi_n \to \Psi \text{ weakly in } H_0^2(\Omega),\\
&\phi_n \to \Phi,\ \psi_n \to \Psi \text{ strongly in } L^p,\\
&\phi_n(x) \to \Phi(x),\ \psi_n(x) \to \Psi(x) \text{ for a.e. } x \in \Omega.
\end{aligned}$$

Thus, by applying Lebesgue Dominate Convergence Theorem, one can deduce that

$$\lim_{n\to\infty} \int_\Omega F(\phi_n(x),\psi_n(x))\,\mathrm{d}x = \int_\Omega F(\Phi(x),\Psi(x))\,\mathrm{d}x.$$

By using the weak lower semicontinuity of the map $H_0^2(\Omega) \ni u \longmapsto \|\Delta u\|_2^2$, it may be conclude that

$$\begin{aligned} J(\Phi,\Psi) &= \frac{1}{2}\left(\|\Delta\Phi\|_2^2 + \|\Delta\Psi\|_2^2\right) - \int_\Omega F(\Phi(x),\Psi(x))\,\mathrm{d}x \\ &\leqslant \liminf_{n\to\infty}\left[\frac{1}{2}\left(\|\Delta\phi_n\|_2^2 + \|\Delta\psi_n\|_2^2\right) - \int_\Omega F(\phi_n(x),\psi_n(x))\,\mathrm{d}x\right] \\ &= \liminf_{n\to\infty} J(\phi_n,\psi_n) = d, \end{aligned}$$

On the other hand, since $\{(\phi_n,\psi_n)\}_{n=1}^{\infty} \subset \mathcal{N}$, we have $(\phi_n,\psi_n) \neq (0,0)$ and $I(\phi_n,\psi_n) = 0$ for all $n \in \mathbb{N}$. By appealing Lemma 3.2, we can easily deduce from (23) that

$$p\int_\Omega F(\phi_n(x),\psi_n(x))\,\mathrm{d}x = \|\Delta\phi_n\|_2^2 + \|\Delta\psi_n\|_2^2 \geqslant \left(\frac{1}{pC_2S_p^p}\right)^{\frac{2}{p-2}},\ \forall n \in \mathbb{N}.$$

This fact permits us to verify that

$$\int_\Omega F(\Phi(x),\Psi(x))\,\mathrm{d}x \geqslant \frac{1}{p}\left(\frac{1}{pC_2S_p^p}\right)^{\frac{2}{p-2}} > 0.$$

And this fact implies $(\Phi,\Psi) \in \left(\left[H_0^2(\Omega)\right]^2\right) \setminus \{(0,0)\}$. With the same argument, we can see that

$$I(\Phi,\Psi) \leqslant \liminf_{n\to\infty} I(\phi_n,\psi_n) = 0.$$

Thus, what is left is to show that $I(\Phi,\Psi) = 0$. Arguing by contradiction, suppose that $I(\Phi,\Psi) < 0$. Thus, by using Lemma 3.1, there exists $\lambda_* < 1$ such that $I(\lambda_*\Phi,\lambda_*\Psi) = 0$. Therefore, we have

$$\begin{aligned} d \leqslant J(\lambda_*\Phi,\lambda_*\Psi) &= \frac{(p-2)\lambda_*^2}{2p}\left(\|\Delta\Phi\|_2^2 + \|\Delta\Psi\|_2^2\right) \\ &< \frac{p-2}{2p}\left(\|\Delta\Phi\|_2^2 + \|\Delta\Psi\|_2^2\right) \\ &\leqslant \liminf_{n\to\infty}\frac{p-2}{2p}\left(\|\Delta\phi_n\|_2^2 + \|\Delta\psi_n\|_2^2\right) \\ &= \liminf_{n\to\infty} J(\phi_n,\psi_n) = d. \end{aligned}$$

This is impossible, so $I(\Phi,\Psi) = 0$, and this fact eventually leads to $(\Phi,\Psi) \in \mathcal{N}$ and $J(\Phi,\Psi) = d$.

In order to conclude this proof, we claim that if $J(\Phi,\Psi) = d$ and $(\Phi,\Psi) \in \mathcal{N}$, then (Φ,Ψ) is a weak solution of Problem (21). By the definition of the depth of potential well d, we have $J(\Phi,\Psi) = \min_{(\phi,\psi)\in\mathcal{N}} J(\phi,\psi)$. Thanks to Lagrange multiplier theorem, there exists a constant $\hbar$ such that

$$\nabla J(\Phi,\Psi)(w_1,w_2) = \hbar\nabla I(\Phi,\Psi)(w_1,w_2),\ \forall (w_1,w_2) \in \left[H_0^2(\Omega)\right]^2. \tag{30}$$

where $\nabla J(\Phi,\Psi),\ \nabla I(\Phi,\Psi) \in \left[H^{-2}(\Omega)\right]^2$ be the Fréchet derivative of J and I at (Φ,Ψ), respectively. By choosing $(w_1,w_2) = (\Phi,\Psi)$ in (30), and noting that $\nabla J(\Phi,\Psi)(\Phi,\Psi) = I(\Phi,\Psi) = 0$. Then we obtain $\hbar\nabla I(\Phi,\Psi)(\Phi,\Psi) = 0$. On the other hand, a direct computation yields

$$\nabla I(\Phi,\Psi)(\Phi,\Psi) = p(2-p)\int_\Omega xF(\Phi(x),\Psi(x))\,\mathrm{d}x < 0.$$

Therefore, we have $\hbar = 0$. Hence, it follows from (30) that

$$\nabla J(\Phi, \Psi)(w_1, w_2) = 0, \forall w_1, w_2 \in H_0^2(\Omega)$$

or

$$\frac{\partial J}{\partial \phi}(\Phi, \Psi) w_1 = \frac{\partial J}{\partial \psi}(\Phi, \Psi) w_2 = 0, \forall w_1, w_2 \in H_0^2(\Omega).$$

The above display enables us to deduce that (Φ, Ψ) is weak solution of Problem (21). This completes the proof of Lemma 3.1. □

Remark 3.4. We introduce the set of all stationary solutions of Problem (1)-(4) by

$$\mathscr{S} = \left\{ (\phi, \psi) \in \left[H_0^2(\Omega)\right]^2 : \frac{\partial J}{\partial \phi}(\phi, \psi) w_1 = \frac{\partial J}{\partial \psi}(\phi, \psi) w_2 = 0, \forall (w_1, w_2) \in \left[H_0^2(\Omega)\right]^2 \right\},$$

and with $\ell \in \{0\} \cup [d, \infty)$, we define the set

$$\mathscr{S}_\ell = \left\{ (\phi, \psi) \in \left[H_0^2(\Omega)\right]^2 : J(\phi, \psi) = \ell \right\}.$$

In order to state as well as prove sharp estimates for the weak solution of Problem (1)-(4), we need to introduce a family of potential wells, the outside set of the corresponding potential well sets, and give some properties of them. Then, the invariant sets, and the vacuum isolating of solutions for Problem (1)-(4) are also discussed, in the next subsection. For this purpose, with $\delta > 0$, we define

$$I_\delta(\phi, \psi) := \delta\left(\|\Delta \phi\|_2^2 + \|\Delta \psi\|_2^2\right) - \int_\Omega F(\phi(x), \psi(x))\, dx, \forall (\phi, \psi) \in \left[H_0^2(\Omega)\right]^2, \tag{31}$$

and

$$r(\delta) := \left(\frac{\delta}{pC_2 S_p^p}\right)^{\frac{2}{p-2}}. \tag{32}$$

The following lemma is given to establish the relations between the sign of I_δ and $\|\phi\|_a^2 + \|\psi\|_b^2$.

Lemma 3.5. *Let $(\phi, \psi) \in \left[H_0^2(\Omega)\right]^2$. Then*

i. If $0 < \|\Delta \phi\|_2^2 + \|\Delta \psi\|_2^2 < r(\delta)$, then $I_\delta(\phi, \psi) > 0$;
ii. If $I_\delta(\phi, \psi) < 0$, then $\|\Delta \phi\|_2^2 + \|\Delta \psi\|_2^2 > r(\delta)$;
iii. If $I_\delta(\phi, \psi) = 0$ and $(\phi, \psi) \neq (0, 0)$, then $\|\Delta \phi\|_2^2 + \|\Delta \psi\|_2^2 \geqslant r(\delta)$;
iv. If $I_\delta(\phi, \psi) = 0$ and $(\phi, \psi) \neq (0, 0)$, then $J(\phi, \psi) > 0$ for $0 < \delta < \frac{p}{2}$, $J(\phi, \psi) = 0$ for $\delta = \frac{p}{2}$ and $J(\phi, \psi) < 0$ for $\delta > \frac{p}{2}$.

Proof of Lemma 3.5. By an argument analogous to that used for the proof of Lemma 3.2, we can see that

$$I_\delta(\phi, \psi) \geqslant \left[\delta - C_2 S_p^p \left(\|\Delta \phi\|_2^2 + \|\Delta \psi\|_2^2\right)^{\frac{p-2}{2}}\right] \left(\|\Delta \phi\|_2^2 + \|\Delta \psi\|_2^2\right), \forall (\phi, \psi) \in \left[H_0^2(\Omega)\right]^2.$$

Therefore, one can easily deduce that

i. If $0 < \|\Delta \phi\|_2^2 + \|\Delta \psi\|_2^2 < r(\delta)$, then $I_\delta(\phi, \psi) > 0$;
ii. If $I_\delta(\phi, \psi) < 0$, then $\|\Delta \phi\|_2^2 + \|\Delta \psi\|_2^2 > r(\delta)$;
iii. If $I_\delta(\phi, \psi) = 0$ and $(\phi, \psi) \neq (0, 0)$, then $\|\Delta \phi\|_2^2 + \|\Delta \psi\|_2^2 \geqslant r(\delta)$.

Furthermore, the representation

$$J(\phi,\psi)=\frac{p-2\delta}{2p}\left(\|\Delta\phi\|_2^2+\|\Delta\psi\|_2^2\right)+\frac{I_\delta(\phi,\psi)}{p},\ \forall(\phi,\psi)\in\left[H_0^2(\Omega)\right]^2,\ \delta\in(0,\infty),\tag{33}$$

allows us to deduce that if $I_\delta(\phi,\psi)=0$ and $(\phi,\psi)\neq(0,0)$, then $J(\phi,\psi)>0$ for $0<\delta<\frac{p}{2}$, $J(\phi,\psi)=0$ for $\delta=\frac{p}{2}$ and $J(\phi,\psi)<0$ for $\delta>\frac{p}{2}$. Lemma 3.5 is proved. □

Now, we define the depth of a family of potential wells, for $\delta\in\left(0,\frac{p}{2}\right)$,

$$d(\delta):=\inf_{(\phi,\psi)\in\mathcal{N}_\delta}J(\phi,\psi),\tag{34}$$

where

$$\mathcal{N}_\delta:=\left\{(\phi,\psi)\in\left[H_0^2(\Omega)\right]^2:I_\delta(\phi,\psi)=0\right\}\setminus\{(0,0)\}.\tag{35}$$

Then, the expression of $d(\delta)$ can be given as follows.

Lemma 3.6. *With $d(\delta)$ defined by (34), the following properties hold:*

i. $d(\delta)\geqslant a(\delta)r(\delta)$ *for* $a(\delta):=\frac{1}{2}\left(1-\frac{2\delta}{p}\right)$ *with* $\delta\in\left(0,\frac{p}{2}\right)$*. In particular,*

$$d\geqslant\frac{p-2}{2p}\left(\frac{1}{pC_2S_p^p}\right)^{\frac{2}{p-2}};$$

ii. $d(\delta)=\frac{\delta^{\frac{2}{p-2}}(p-2\delta)d}{p-2}$ *for* $\delta\in\left(0,\frac{p}{2}\right)$;

iii. $\lim_{\delta\downarrow0}d(\delta)=0$, $\lim_{\delta\uparrow\frac{p}{2}}d(\delta)=0$, *and the function* $\delta\longmapsto d(\delta)$ *is continuous on* $\left(0,\frac{p}{2}\right)$;

iv. The function $\delta\longmapsto d(\delta)$ *is strictly increasing on* $(0,1]$, *strictly decreasing on* $\left[1,\frac{p}{2}\right)$, *and takes the maximum* $d=d(1)$ *at* $\delta=1$.

Proof of Lemma 3.6. For any $(\phi,\psi)\in\mathcal{N}_\delta$, applying Lemma 3.5, we have $\|\Delta\phi\|_2^2+\|\Delta\psi\|_2^2\geqslant r(\delta)$. Thus, by recalling the definitions of $J(\phi,\psi)$ and $I_\delta(\phi,\psi)$, we deduce that

$$J(\phi,\psi)=\frac{p-2\delta}{2p}\left(\|\Delta\phi\|_2^2+\|\Delta\psi\|_2^2\right)+\frac{I_\delta(\phi,\psi)}{p}=a(\delta)\left(\|\Delta\phi\|_2^2+\|\Delta\psi\|_2^2\right)\geqslant a(\delta)r(\delta),$$

which implies $d(\delta)\geqslant a(\delta)r(\delta)$ for $\delta\in\left(0,\frac{p}{2}\right)$.

Next, let $(\phi,\psi)\in\mathcal{N}$ such that $J(\phi,\psi)=d$. For $\delta>0$, we define $\lambda=\lambda(\delta)$ by

$$I_\delta(\lambda\phi,\lambda\psi)=0\Longleftrightarrow\delta\lambda^2\left(\|\Delta\phi\|_2^2+\|\Delta\psi\|_2^2\right)+p\lambda^p\int_\Omega F(\phi(x),\psi(x))\,\mathrm{d}x=0.$$

Thus, for each $\delta>0$, there exists a unique constant

$$\lambda(\delta)=\left[\frac{\delta\left(\|\Delta\phi\|_2^2+\|\Delta\psi\|_2^2\right)}{p\int_\Omega F(\phi(x),\psi(x))\,\mathrm{d}x}\right]^{\frac{1}{p-2}}=\delta^{\frac{1}{p-2}},$$

such that $I_\delta(\lambda\phi,\lambda\psi)=0$. Hence, according to the definition of $d(\delta)$, we have

$$d(\delta)\leqslant J(\lambda(\delta)\phi,\lambda(\delta)\psi)=\frac{\delta^{\frac{2}{p-2}}}{2}\left(1-\frac{2\delta}{p}\right)\left(\|\Delta\phi\|_2^2+\|\Delta\psi\|_2^2\right)=\frac{\delta^{\frac{2}{p-2}}(p-2\delta)d}{p-2}.$$

On the other hand, let $\delta \in \left(0, \frac{p}{2}\right)$ and $(\phi, \psi) \in \mathscr{N}_\delta$ such that $J(\phi, \psi) = d(\delta)$. By utilizing Lemma 3.1, with $\lambda_* = \delta^{-\frac{1}{p-2}}$, we have $I(\lambda_*\phi, \lambda_*\psi) = 0$. By recalling definition of d, we have

$$d \leqslant J(\lambda_*\phi, \lambda_*\psi) = \delta^{-\frac{2}{p-2}} \frac{p-2}{2p} \left(\|\Delta\phi\|_2^2 + \|\Delta\psi\|_2^2\right) = \delta^{-\frac{2}{p-2}} \frac{p-2}{2p} \frac{2p}{p-2\delta} d(\delta) = \frac{\delta^{-\frac{2}{p-2}}(p-2)}{p-2\delta} d(\delta).$$

This fact implies

$$d(\delta) \geqslant \frac{\delta^{\frac{2}{p-2}}(p-2\delta)\, d}{p-2}.$$

Therefore, $d(\delta) = \frac{\delta^{\frac{2}{p-2}}(p-2\delta)d}{p-2}$, and hence, we immediately obtain the results of the thirst and fourth statements. This completes the proof of Lemma 3.6. □

Remark 3.7. In fact, the existence of an element $(\phi, \psi) \in \mathscr{N}_\delta$ such that $J(\phi, \psi) = d(\delta)$ is not trivial. However, by an argument analogous to that used for the proof of Lemma 3.3, we can easily obtain this result. The proof is left to the reader.

Lemma 3.8. *Let $\delta \in \left(0, \frac{p}{2}\right)$ and $J(\phi, \psi) \leqslant d(\delta)$, the following properties hold:*

i. If $I_\delta(\phi, \psi) > 0$, then $0 < \|\Delta\phi\|_2^2 + \|\Delta\psi\|_2^2 < \frac{d(\delta)}{a(\delta)}$;
ii. If $\|\Delta\phi\|_2^2 + \|\Delta\psi\|_2^2 > \frac{d(\delta)}{a(\delta)}$, then $I_\delta(\phi, \psi) < 0$;
iii. If $I_\delta(\phi, \psi) = 0$, then $\|\Delta\phi\|_2^2 + \|\Delta\psi\|_2^2 \leqslant \frac{d(\delta)}{a(\delta)}$.

Proof of Lemma 3.8. By recalling (33), we have

$$d(\delta) \geqslant J(\phi, \psi) = a(\delta)\left(\|\Delta\phi\|_2^2 + \|\Delta\psi\|_2^2\right) + \frac{I_\delta(\phi, \psi)}{p}, \ \forall (\phi, \psi) \in \left[H_0^2(\Omega)\right]^2.$$

This fact implies

i. If $I_\delta(\phi, \psi) > 0$, then $0 < \|\Delta\phi\|_2^2 + \|\Delta\psi\|_2^2 < \frac{d(\delta)}{a(\delta)}$;
ii. If $\|\Delta\phi\|_2^2 + \|\Delta\psi\|_2^2 > \frac{d(\delta)}{a(\delta)}$, then $I_\delta(\phi, \psi) < 0$;
iii. If $I_\delta(\phi, \psi) = 0$, then $\|\Delta\phi\|_2^2 + \|\Delta\psi\|_2^2 \leqslant \frac{d(\delta)}{a(\delta)}$.

Lemma 3.8 is proved. □

Remark 3.9. Lemmata 3.5 and 3.8 permit us to partition the space $\left[H_0^2(\Omega)\right]^2$ into two parts $I_\delta(\phi, \psi) > 0$ and $I_\delta(\phi, \psi) < 0$ by surface $\mathscr{N}_\delta$. The inside part of $\mathscr{N}_\delta$ is $I_\delta(\phi, \psi) > 0$ and the outside part of $\mathscr{N}_\delta$ is $I_\delta(\phi, \psi) < 0$. A sphere $\|\Delta\phi\|_2^2 + \|\Delta\psi\|_2^2 = r(\delta)$ lies inside of $I_\delta(\phi, \psi) > 0$ and sphere $\|\Delta\phi\|_2^2 + \|\Delta\psi\|_2^2 = \frac{d(\delta)}{a(\delta)}$ lies inside of $I_\delta(\phi, \psi) < 0$.

For the time being, we are in a position to introduce the single potential well $\mathscr{W}$ and a family of potential wells $\mathscr{W}_\delta$ with its outsider $\mathscr{U}$ and $\mathscr{U}_\delta$, respectively. For $\delta \in \left(0, \frac{p}{2}\right)$, define

$$\begin{aligned}
\mathscr{W} &:= \left\{(\phi, \psi) \in \left[H_0^2(\Omega)\right]^2 : J(\phi, \psi) < d, I(\phi, \psi) > 0\right\} \cup \{(0,0)\},\\
\mathscr{W}_\delta &:= \left\{(\phi, \psi) \in \left[H_0^2(\Omega)\right]^2 : J(\phi, \psi) < d(\delta), I_\delta(\phi, \psi) > 0\right\} \cup \{(0,0)\},\\
\overline{\mathscr{W}}_\delta &:= \left\{(\phi, \psi) \in \left[H_0^2(\Omega)\right]^2 : J(\phi, \psi) \leqslant d(\delta), I_\delta(\phi, \psi) \geqslant 0\right\},\\
\mathscr{U} &:= \left\{(\phi, \psi) \in \left[H_0^2(\Omega)\right]^2 : J(\phi, \psi) < d, I(\phi, \psi) < 0\right\},\\
\mathscr{U}_\delta &:= \left\{(\phi, \psi) \in \left[H_0^2(\Omega)\right]^2 : J(\phi, \psi) < d(\delta), I_\delta(\phi, \psi) < 0\right\}.
\end{aligned}$$

Lemma 3.10. *Let $\delta \in \left(0, \frac{p}{2}\right)$. Then, we have the following conclusions*

$$B\left(0, \sqrt{\min\left\{r(\delta), 2d(\delta)\right\}}\right) \subset \mathscr{W}_\delta \subset B\left(0, \sqrt{\frac{d(\delta)}{a(\delta)}}\right), \ \mathscr{U}_\delta \subset \left[H_0^2(\Omega)\right]^2 \setminus B\left[0, \sqrt{r(\delta)}\right],$$

where $B(a, r_0)$ and $B[a, r_0]$ are open and close ball centered at a and with radius r defined by $B(0, r_0) := \left\{(\phi, \psi) \in \left[H_0^2(\Omega)\right]^2 : \|\Delta\phi\|_2^2 + \|\Delta\psi\|_2^2 < r_0^2\right\}$ and $B[0, r_0] := \left\{(\phi, \psi) \in \left[H_0^2(\Omega)\right]^2 : \|\Delta\phi\|_2^2 + \|\Delta\psi\|_2^2 \leqslant r_0^2\right\}$, respectively.

Proof of Lemma 3.10. First, if $\|\Delta\phi\|_2^2 + \|\Delta\psi\|_2^2 < r(\delta)$, then we have $(\phi, \psi) = (0, 0)$ or $I_\delta(\phi, \psi) > 0$. On the other hand, since $J(\phi, \psi) \leqslant \frac{1}{2}\left(\|\Delta\phi\|_2^2 + \|\Delta\psi\|_2^2\right)$ and $\|\Delta\phi\|_2^2 + \|\Delta\psi\|_2^2 < 2d(\delta)$, we also obtain $J(\phi, \psi) < d(\delta)$. Hence, we have

$$B\left(0, \sqrt{\min\left\{r(\delta), 2d(\delta)\right\}}\right) \subset \mathscr{W}_\delta.$$

Next, let $(\phi, \psi) \in \mathscr{W}_\delta$. This fact implies $I_\delta(\phi, \psi) > 0$ or $(\phi, \psi) = (0, 0)$. Hence, one can verify from Lemma 3.8 that $0 < \|\Delta\phi\|_2^2 + \|\Delta\psi\|_2^2 < r(\delta)$ or $(\phi, \psi) = (0, 0)$. That means $\mathscr{W}_\delta \subset B\left(0, \sqrt{\frac{d(\delta)}{a(\delta)}}\right)$. Finally, if $(\phi, \psi) \in \mathscr{U}_\delta$, then $I_\delta(\phi, \psi) < 0$. Hence, it follows from Lemma 3.5 that $\|\Delta\phi\|_2^2 + \|\Delta\psi\|_2^2 > r(\delta)$. Therefore, $\mathscr{U}_\delta \subset \left[H_0^2(\Omega)\right]^2 \setminus B\left[0, \sqrt{r(\delta)}\right]$. Lemma 3.11 is proved completely. □

From the definition of $\mathscr{W}_\delta$, $\mathscr{U}_\delta$ and Lemma 3.6, we can obtain the following result.

Lemma 3.11. *The following properties hold:*

i. If $0 < \delta' < \delta'' \leqslant 1$ then $\mathscr{W}_{\delta'} \subset \mathscr{W}_{\delta''}$;
ii. If $1 \leqslant \delta'' < \delta' < \frac{p}{2}$ then $\mathscr{U}_{\delta'} \subset \mathscr{U}_{\delta''}$.

Lemma 3.12. *Let $0 < J(\phi, \psi) < d$ for some $(\phi, \psi) \in \left[H_0^2(\Omega)\right]^2$, $\delta_1 < \delta_2$ are the two roots of equation $J(\phi, \psi) = d(\delta)$. Then the sign of $I_\delta(\phi, \psi)$ are unchangeable for $\delta_1 < \delta < \delta_2$.*

Proof of Lemma 3.12. If $0 < J(\phi, \psi)$, then $(\phi, \psi) \neq (0, 0)$. Furthermore, if sign of $I_\delta(\phi, \psi)$ are changed for $\delta_1 < \delta < \delta_2$, then there exists a $\bar{\delta} \in (\delta_1, \delta_2)$ such that $I_{\bar{\delta}}(\phi, \psi) = 0$. Thus by the definition of $d(\delta)$ we have $J(\phi, \psi) \geqslant d\left(\bar{\delta}\right)$ which contradicts

$$d(\delta_1) = d(\delta_2) = J(\phi, \psi) \geqslant d\left(\bar{\delta}\right).$$

Lemma 3.12 is proved. □

3.2 Invariant sets and vacuum isolating of solutions

In this section, we discuss the invariance of some sets under the flow of (1)-(4) and vacuum isolating behavior of solutions for Problem (1)-(4). We begin by introducing the following important lemma.

Lemma 3.13. *The Lyapunov function*

$$[0, T_\infty) \ni t \longmapsto E(t) = \frac{1}{2}\left(\|u'(t)\|_2^2 + \|v'(t)\|_2^2\right) + J(u(t), v(t)), \tag{36}$$

is a decreasing function along the trajectories. In particular, we have the following estimate

$$E'(t) = -\|u'(t)\|_{q_1}^{q_1} - \|v'(t)\|_{q_2}^{q_2}, \ \forall t \in [0, T_\infty). \tag{37}$$

Proof of Theorem 3.13. The energy identity (37) can be obtained by testing (1) with u_t, testing (2) with v_t, and adding these two equations. we obtain (37). Lemma 3.13 is proved. □

Next, by using the potential wells above, we can obtain the following invariance for some sets under the flow of (1)-(4) and vacuum isolating behavior for the solutions of Problem (1)-(4).

Theorem 3.14. *Suppose that $0 < e < d$ and $\delta_1 < \delta_2$ be the two roots of equation $d(\delta) = e$. Then:*

i. All solutions of Problem (1)-(4) with $E(0) = e$ belong to $\mathscr{W}_\delta$ for all $\delta \in (\delta_1, \delta_2)$, provided $I(u_0, v_0) > 0$ or $u_0 = v_0 = 0$;
ii. All solutions of Problem (1)-(4) with $E(0) = e$ belong to $\mathscr{U}_\delta$ for all $\delta \in (\delta_1, \delta_2)$, provided $I(u_0, v_0) < 0$.

Proof of Theorem 3.14. Let (u, v) be any solution of Problem (1)-(4) with $E(0) = e$ and $I(u_0, v_0) > 0$ or $u_0 = v_0 = 0$. If $u_0 = v_0 = 0$ then $(u_0, v_0) \in \mathscr{W}_\delta$ for all $\delta \in (\delta_1, \delta_2)$. If $I(u_0, v_0) > 0$, then from the definition of $d(\delta)$ and

$$\frac{1}{2}\left(\|u_1\|_2^2 + \|v_1\|_2^2\right) + J(u_0, v_0) = E(0) = d(\delta_1) = d(\delta_2) < d(\delta), \forall \delta \in (\delta_1, \delta_2), \tag{38}$$

it may be concluded that $J(u_0, v_0) < d(\delta)$ for all $\delta \in (\delta_1, \delta_2)$. We will verify that $I_\delta(u_0, v_0) > 0$, for all $\delta \in (\delta_1, \delta_2)$. Indeed, assume that there exists $\bar{\delta} \in (\delta_1, \delta_2)$ such that $I_{\bar{\delta}}(u_0, v_0) = 0$. Thus, by definition of $d(\delta)$, we obtain $J(u_0, v_0) \geqslant d(\bar{\delta}) > d(\delta_1) = d(\delta_2)$, contrary to (38). Therefore, we have $(u_0, v_0) \in \mathscr{W}_\delta$ for all $\delta \in (\delta_1, \delta_2)$. Next we prove that $(u(t), v(t)) \in \mathscr{W}_\delta$ for all $\delta \in (\delta_1, \delta_2)$ and for all $t \in [0, T_\infty)$. If it is false, then it must have a $t_0 \in (0, T_\infty)$ such that $(u(t_0), v(t_0)) \in \partial\mathscr{W}_\delta$ for some $\delta \in (\delta_1, \delta_2)$, i.e. $I_\delta(u(t_0), v(t_0)) = 0$, $(u(t_0), v(t_0)) \neq (0, 0)$ or $J(u(t_0), v(t_0)) = d(\delta)$. From Lemma 3.13, we see that $J(u(t_0), v(t_0)) = d(\delta)$ is impossible. On the other hand, if $I_\delta(u(t_0), v(t_0)) = 0$ and $(u(t_0), v(t_0)) \neq (0, 0)$ then by the definition of $d(\delta)$ we have $J(u(t_0), v(t_0)) \geqslant d(\delta)$ which contradicts with

$$d(\delta) \leqslant J(u(t_0), v(t_0)) \leqslant E(t_0) \leqslant E(0) < d(\delta).$$

Next, we assume that (u, v) be any solution of Problem (1)-(4) with $E(0) = e$ and $I(u_0, v_0) < 0$. First by (38), in the same manner above, we can obtain $I_\delta(u_0, v_0) < 0$ and $J(u_0, v_0) < d(\delta)$ for all $\delta \in (\delta_1, \delta_2)$ i.e. $(u_0, v_0) \in \mathscr{U}_\delta$ for all $\delta \in (\delta_1, \delta_2)$. Next, we prove that $(u(t), v(t)) \in \mathscr{U}_\delta$ for all $\delta \in (\delta_1, \delta_2)$ and for all $t \in [0, T_\infty)$. If it is false, let $t_0 \in (0, T_\infty)$ be the first time such that $(u(t), v(t)) \in \mathscr{U}_\delta$ for all $t \in [0, t_0)$ and $(u(t_0), v(t_0)) \in \partial\mathscr{U}_\delta$ i.e. $I_\delta(u(t_0), v(t_0)) = 0$ or $J(u(t_0), v(t_0)) = d(\delta)$ for some $\delta \in (\delta_1, \delta_2)$. One again, we know that $J(u(t_0), v(t_0)) = d(\delta)$ is impossible. From $(u(t), v(t)) \in \mathscr{U}_\delta$ for all $t \in [0, t_0)$, we obtain $I_\delta(u(t), v(t)) < 0$ for all $t \in (0, t_0)$. This fact implies $\|u(t)\|_a^2 + \|v(t)\|_b^2 > r(\delta)$ for all $t \in (0, t_0)$. Therefore, we obtain $\|u(t_0)\|_a^2 + \|v(t_0)\|_b^2 \geqslant r(\delta) > 0$. Hence, by the definition of $d(\delta)$, we get $J(u(t_0), v(t_0)) \geqslant d(\delta)$ which contradicts with the fact $J(u(t_0), v(t_0)) < d(\delta)$. Theorem 3.14 is proved. □

From Theorem 3.14 and Lemma 3.6, we can obtain the following theorems.

Theorem 3.15. *If in Theorem 3.14 the assumption $E(0) = e$ is replaced by $0 < E(0) \leqslant e$, then the conclusion of Theorem 3.14 also holds.*

Theorem 3.16. *Suppose that $0 < e < d$ and $\delta_1 < \delta_2$ be the two roots of equation $d(\delta) = e$. Then for any $\delta \in (\delta_1, \delta_2)$ both sets $\mathscr{W}_\delta$ and $\mathscr{U}_\delta$ are invariant. Thus, both sets*

$$\mathscr{W}_{\delta_1\delta_2} = \bigcup_{\delta_1 < \delta < \delta_2} \mathscr{W}_\delta, \ \mathscr{U}_{\delta_1\delta_2} = \bigcup_{\delta_1 < \delta < \delta_2} \mathscr{U}_\delta,$$

are invariant under the flow of Problem (1)-(4), provided $0 < E(0) \leqslant e$.

From the above results, we see that if $0 < E(0) \leqslant e < d$, then $I(u_0, v_0) = 0$ and $(u_0, v_0) \neq (0,0)$ is impossible. So the result of Theorem 3.15 shows that for the set of all solutions of Problem (1)-(4) with $0 < E(0) \leqslant e < d$ there exists a vacuum region

$$\mathscr{V}_e = \mathscr{N}_{\delta_1 \delta_2} = \bigcup_{\delta_1 < \delta < \delta_2} \mathscr{N}_\delta = \left\{ (\phi, \psi) \in \left[H_0^2(\Omega) \right]^2 \setminus \{(0,0)\} : I_\delta(\phi, \psi) = 0,\ \delta \in (\delta_1, \delta_2) \right\},$$

such that there is no any solution of Problem (1)-(4) in $\mathscr{V}_e$. And the vacuum region $\mathscr{V}_e$ become bigger and bigger with decreasing of e. As the limit case we obtain

$$\mathscr{V}_0 = \left\{ (\phi, \psi) \in \left[H_0^2(\Omega) \right]^2 \setminus \{(0,0)\} : I_\delta(\phi, \psi) = 0,\ \delta \in \left(0, \frac{p}{2}\right) \right\}.$$

Actually, we have the following theorem.

Theorem 3.17. *Suppose that $e \in (0, d)$ and δ_1, δ_2 are two roots of the equation $d(\delta) = e$ with $\delta_1 < 1 < \delta_2$. Then for all weak solutions of problem (1)-(4) with $0 < E(0) \leqslant e < d$, there exists a vacuum region*

$$\mathscr{V}_e = \mathscr{N}_{\delta_1 \delta_2} = \bigcup_{\delta_1 < \delta < \delta_2} \mathscr{N}_\delta = \left\{ (\phi, \psi) \in \left[H_0^2(\Omega) \right]^2 \setminus \{(0,0)\} : I_\delta(\phi, \psi) = 0,\ \delta \in (\delta_1, \delta_2) \right\},$$

such that no solution of Problem (1)-(4) belongs to $\mathscr{V}_e$. The vacuum region $\mathscr{V}_e$ becomes bigger and bigger when e is decreasing. As the limiting case we obtain

$$\mathscr{V}_0 = \left\{ (\phi, \psi) \in \left[H_0^2(\Omega) \right]^2 \setminus \{(0,0)\} : I_\delta(\phi, \psi) = 0,\ \delta \in \left(0, \frac{p}{2}\right) \right\}.$$

Proof of Theorem 3.17. Let (u, v) be a solution of Problem (1)-(4) with $0 < E(0) \leqslant e < d$. We only need to prove that if $(u(t), v(t)) \neq (0,0)$, then for all $\delta \in (\delta_1, \delta_2)$ we have $I_\delta(u(t), v(t)) \neq 0$ for all $t \in [0, T_\infty)$. Ub fact, we have $I_\delta(u_0, v_0) \neq 0$, otherwise, if $I_\delta(u_0, v_0) = 0$, combine with $(u_0, v_0) \neq (0,0)$, we deduce that $J(u_0, v_0) \geqslant d(\delta) > d(\delta_1) = e$, which is contradictive with $J(u_0, v_0) \leqslant e$. On the other hand, suppose that there exists $t_1 \in (0, T_\infty)$ such that $(u(t_1), v(t_1)) \in \mathscr{V}_e$, which implies that there exists $\delta_0 \in (\delta_1, \delta_2)$ such that $u(t_1) \in \mathscr{N}_{\delta_0}$. It follows from definition of $d(\delta)$ that

$$E(0) \geqslant E(t_1) = J(u(t_1), v(t_1)) \geqslant d(\delta_0) > d(\delta_1) = E(0),$$

which eventually leads to a contradiction. Theorem 3.17 is proved. □

In order to discuss about the invariant of the solutions with nonnegative initial energy, we introduce the following results.

Theorem 3.18. *All nontrivial solutions of Problem (1)-(4) with $E(0) = 0$ belong to*

$$\left[H_0^2(\Omega) \right]^2 \setminus B\left(0, \sqrt{r\left(\frac{p}{2}\right)}\right).$$

Proof of Theorem 3.18. Let (u, v) be any nontrivial solution of Problem (1)-(4) with $E(0) = 0$. From the Lemma 3.13, we deduce that

$$0 = E(0) \geqslant E(t) = \frac{1}{2}\left(\|u'(t)\|_2^2 + \|v'(t)\|_2^2 \right) + J(u(t), v(t)),\ \forall t \in [0, \infty),$$

we get $J(u(t), v(t)) < 0$ for all $t \in [0, T_\infty)$. This fact gives us

$$\frac{1}{2}\left(\|\Delta u(t)\|_2^2 + \|\Delta v(t)\|_2^2\right) < \int_\Omega F(u(x,t), v(x,t))\,dx$$
$$\leqslant C_2 S_p^p \left(\|\Delta u(t)\|_2^2 + \|\Delta v(t)\|_2^2\right)^{\frac{p}{2}}, \forall t \in [0, T_\infty).$$

The previous inequality shows that $\|\Delta u(t)\|_2^2 + \|\Delta v(t)\|_2^2 > \left(\frac{1}{2C_2 S_p^p}\right)^{\frac{2}{p-2}} = r\left(\frac{p}{2}\right)$ for all $t \in [0, T_\infty)$. Theorem 3.18 is proved completely. □

Theorem 3.19. *All solutions of Problem (1)-(4) with $E(0) < 0$ belong to*

$$\left[H_0^2(\Omega)\right]^2 \setminus B(0, \varrho_2),$$

where $\varrho_2 > 0$ be unique solution of the equation $\frac{z^2}{2} - C_2 S_*^p z^p = E(0)$.

Furthermore, we have

$$(u(t), v(t)) \in \mathscr{U}_\delta, \forall \delta \in \left(0, \frac{p}{2}\right), t \in [0, T_\infty).$$

Proof of Theorem 3.19. First, we observe that

$$0 > E(0) \geqslant E(t) \geqslant J(u(t), v(t)) = \frac{1}{2}\left(\|\Delta u(t)\|_2^2 + \|\Delta v(t)\|_2^2\right) - \int_\Omega F(u(x,t), v(x,t))\,dx$$
$$\geqslant \frac{1}{2}\left(\|\Delta u(t)\|_2^2 + \|\Delta v(t)\|_2^2\right) - C_2 S_p^p \left(\|\Delta u(t)\|_2^2 + \|\Delta v(t)\|_2^2\right)^{\frac{p}{2}}$$
$$= \mathcal{H}\left(\sqrt{\|\Delta u(t)\|_2^2 + \|\Delta v(t)\|_2^2}\right), \quad (39)$$

where $\mathcal{H}(z) = \frac{z^2}{2} - C_2 S_p^p z^p$ for $z \in (0, \infty)$. We notice that if $E(0) < 0$ then there exists unique $\varrho_2 > \sqrt{r\left(\frac{p}{2}\right)}$ such that $\mathcal{H}(\varrho_2) = E(0) < 0$. Therefore, from (39), we obtain $\|\Delta u(t)\|_2^2 + \|\Delta v(t)\|_2^2 \geqslant \varrho_2^2$ for all $t \in [0, T_\infty)$. Theorem 3.19 is proved. □

Remark 3.20. If $E(0) < 0$ and $\varrho_2 > 0$ be unique solution of the equation $\frac{z^2}{2} - C_2 S_p^p z^p = E(0)$, then we have

$$C_2 S_p^p \varrho_2^p = \frac{\varrho_2^2}{2} - E(0) \geqslant \sqrt{-2E(0)}\varrho_2.$$

This fact implies

$$\varrho_2 \geqslant \left(\frac{-2E(0)}{C_2^2 S_p^{2p}}\right)^{\frac{1}{2(p-1)}}.$$

So we have $\lim\limits_{E(0) \to -\infty} \varrho_2 = \infty$.

Theorem 3.21. *Assume that $E(0) < d$ and $I(u_0, v_0) < 0$, then for all $t \in [0, T_\infty)$, we have $(u(t), v(t)) \in \mathscr{V}$ and*

$$\|\Delta u(t)\|_2^2 + \|\Delta v(t)\|_2^2 > \frac{2pd}{p-2}, \forall t \in [0, \infty).$$

Proof of Theorem 3.21. With same spirit of Theorem 3.14, we obtain $(u(t), v(t)) \in \mathscr{V}$ for all $t \in [0, T_\infty)$. Furthermore, by recalling the definition of depth of potential well, we have

$$d \leqslant \sup_{\lambda \in (0,\infty)} J(\lambda u(t), \lambda v(t)) = J(\lambda_* u(t), \lambda_* v(t)) < \frac{(p-2)}{2p}\left(\|\Delta u(t)\|_2^2 + \|\Delta v(t)\|_2^2\right).$$

This completes the proof of Theorem 3.21. □

4 Subcritical case initial energy

This section is devoted to the subcritical case initial energy $E(0) < d$. By the combination of modified potential well method with standard continue principle, concavity method and some differential inequalities, we establish some results related to long time behavior of the solutions for Problem (1)-(4).

4.1 Global existence and asymptotic behavior

Theorem 4.1. *Assume that* $0 < E(0) < d$ *and* $I(u_0, v_0) > 0$ *or* $(u_0, v_0) = (0, 0)$ *then the solution of Problem* (1)-(4) *exists globally on* $[0, \infty)$. *Moreover, the solution of Problem* (1)-(4) *satisfies*

$$\|u'(t)\|_2^2 + \|v'(t)\|_2^2 + \|\Delta u(t)\|_2^2 + \|\Delta v(t)\|_2^2 \leqslant \frac{2pE(0)}{p - 2\delta_1}, \forall t \in [0, \infty), \tag{40}$$

where δ_1 *be smallest positive solution of equation* $d(\delta) = E(0)$.

Proof of Theorem 4.1. In order to prove the existence of global weak solution, it suffices to show that $\|u'(t)\|_2^2 + \|v'(t)\|_2^2 + \|\Delta u(t)\|_2^2 + \|\Delta v(t)\|_2^2$ is bounded uniformly with respect to time variable. Under the hypotheses in Theorem 4.1, accordingly to Theorem 3.14, we have $(u(t), v(t)) \in \mathscr{W}_\delta$ for all $t \in [0, T_\infty)$ and $\delta \in (\delta_1, \delta_2)$ where $\delta_1 < \delta_2$ be the two roots of equation $d(\delta) = E(0)$. Then we have $I_\delta(u(t), v(t)) \geqslant 0$ for all $t \in [0, T_\infty)$ and $\delta \in (\delta_1, \delta_2)$. By letting $\delta \searrow \delta_1$, we obtain $I_{\delta_1}(u(t), v(t)) \geqslant 0$ for all $t \in [0, T_\infty)$. So the following estimate holds on

$$\frac{1}{2}\left(\|u'(t)\|_2^2 + \|v'(t)\|_2^2\right) + \frac{1}{2}\left(1 - \frac{2\delta_1}{p}\right)\left(\|\Delta u(t)\|_2^2 + \|\Delta v(t)\|_2^2\right) \leqslant E(t) \leqslant E(0) < d. \tag{41}$$

Hence, from $\delta_1 < 1$ and (41), we can easily obtain

$$\|u'(t)\|_2^2 + \|v'(t)\|_2^2 + \|\Delta u(t)\|_2^2 + \|\Delta v(t)\|_2^2 \leqslant \frac{2pE(0)}{p - 2\delta_1}, \forall t \in [0, \infty), \tag{42}$$

which eventually implies that $T_\infty = \infty$. This completes the proof of Theorem 4.1. □

Next, we show that the global existence also holds for the case of family potential wells.

Theorem 4.2. *Assume that* $0 < E(0) < d$ *and* $I_{\delta_2}(u_0, v_0) > 0$ *or* $(u_0, v_0) = (0, 0)$ *where* $\delta_1 < \delta_2$ *are two roots of equation* $d(\delta) = E(0)$. *Then, Problem* (1)-(4) *admits a unique global solution* $(u, v) \in C\left([0, \infty); [H_0^2(\Omega)]^2\right)$ *with* $(u_t, v_t) \in C\left([0, \infty); [L^2(\Omega)]^2\right)$, $(u(t), v(t)) \in \mathscr{W}_\delta$ *for all* $\delta_1 < \delta < \delta_2$, $t \in [0, \infty)$.

Proof of Theorem 4.2. From Theorem 4.1 and part 1 of Theorem 3.14, in order to prove Theorem 4.2, it is sufficient to show that $I(u_0, v_0) > 0$ from $I_{\delta_2}(u_0, v_0) > 0$. Indeed, if it is false, then there exists $\overline{\delta} \in [1, \delta_2)$ such that $I_{\overline{\delta}}(u_0, v_0) = 0$. Combining the fact that $(u_0, v_0) \neq (0, 0)$ because of $I_{\delta_2}(u_0, v_0) > 0$, we get

$$J(u_0, v_0) \geqslant d(\overline{\delta}). \tag{43}$$

Furthermore, from definition of the functional E, definition of the functional J, and part 4 of Lemma 3.2, we have

$$\frac{1}{2}\left(\|u_1\|_2^2 + \|v_1\|_2^2\right) + J(u_0, v_0) = E(0) = d(\delta_2) < d(\overline{\delta}).$$

It is a contradiction to (43). Thus, Theorem 4.2 is proved. □

Instead of considering the global existence result that depends on $I(u_0, v_0) > 0$, we study the global existence of Problem (1)-(4) with initial data (u_0, v_0) relying on the $\|\Delta\cdot\|_2^2 + \|\Delta\cdot\|_2^2$ norm.

Theorem 4.3. *Assume that* $0 < E(0) < d$ *and* $\|\Delta u_0\|_2^2 + \|\Delta v_0\|_2^2 < r(\delta_2)$, *where* $\delta_1 < \delta_2$ *are two roots of equation* $d(\delta) = E(0)$. *Then, Problem* (1)-(4) *admits a unique global solution* $(u, v) \in C\left([0,\infty); \left[H_0^2(\Omega)\right]^2\right)$ *with* $(u_t, v_t) \in C\left([0,\infty); \left[L^2(\Omega)\right]^2\right)$ *satisfying*

$$\|\Delta u(t)\|_2^2 + \|\Delta v(t)\|_2^2 \leqslant \frac{E(0)}{a(\delta_1)}, \; \|u'(t)\|_2^2 + \|v'(t)\|_2^2 \leqslant 2E(0), \; \forall t \in [0,\infty). \tag{44}$$

Proof of Theorem 4.3. From $\|\Delta u_0\|_2^2 + \|\Delta v_0\|_2^2 < r(\delta_2)$, we have $0 < \|\Delta u_0\|_2^2 + \|\Delta v_0\|_2^2 < r(\delta_2)$ or $(u_0, v_0) = (0, 0)$. If $0 < \|\Delta u_0\|_2^2 + \|\Delta v_0\|_2^2 < r(\delta_2)$, by applying Lemma 3.5 we get $I_{\delta_2}(u_0, v_0) > 0$. Hence, accordingly to Theorem 4.2, the Problem (1)-(4) has a global solution $(u, v) \in C([0,\infty); V \times V)$ and $(u_t, v_t) \in C^1([0,\infty); L^2 \times L^2)$ and $(u(t), v(t)) \in \mathscr{W}_\delta$ for $\delta_1 < \delta < \delta_2$, $t \in [0,\infty)$. Finally, (44) follows from letting $\delta \searrow \delta_1$ in

$$\frac{1}{2}\left(\|u'(t)\|_2^2 + \|v'(t)\|_2^2\right) + \frac{1}{2}\left(1 - \frac{2\delta}{p}\right)\left(\|\Delta u(t)\|_2^2 + \|\Delta v(t)\|_2^2\right) \leqslant E(t) \leqslant E(0), \; \forall t \in [0,\infty).$$

This completes the proof of Theorem 4.3. □

Theorem 4.4. *Under the same assumptions of Theorem 4.1, we have*

$$u_t \in L^\infty\left(0,\infty; L^2(\Omega)\right) \cap L^{q_1}\left(0,\infty; L^{q_1}(\Omega)\right), \; v_t \in L^\infty\left(0,\infty; L^2(\Omega)\right) \cap L^{q_2}\left(0,\infty; L^{q_2}(\Omega)\right).$$

Proof of Theorem 4.4. Thanks to Theorem 40 and Lemma 3.13, we have

$$\frac{1}{2}\left(\|u'(t)\|_2^2 + \|v'(t)\|_2^2\right) + \int_0^t \|u'(s)\|_{q_1}^{q_1} ds + \int_0^t \|v'(s)\|_{q_2}^{q_2} \leqslant E(0), \; \forall t \in [0,\infty).$$

Letting $t \to \infty$, we obtain our conclusion. Theorem 4.4 is proved. □

Next, we give one useful lemma to estimate the behavior of weak solution. For the proof, we refer the reader to [14].

Lemma 4.5 (The Nakao inequality)**.** *Let* $\varphi : [0,\infty) \longrightarrow [0,\infty)$ *be a bounded function for which there exist constant* $\gamma \geqslant 0$ *such that*

$$\sup_{t\leqslant s\leqslant t+1} \varphi^{1+\gamma}(s) \lesssim \varphi(t) - \varphi(t+1), \; \forall t \in [0,\infty).$$

Then

i. *If* $\gamma = 0$, *then there exist positive constant* $\theta > 0$ *such that* $\varphi(t) \lesssim \exp(-\theta t)$ *for all* $t \in [0,\infty)$.
ii. *If* $\gamma > 0$, *then* $\varphi(t) \lesssim (1+t)^{-\frac{1}{\gamma}}$ *for all* $t \in [0,\infty)$.

The following theorem shows the asymptotic behavior of the global solutions of the Problem (1)-(4).

Theorem 4.6. *Assume that* $E(0) < d$ *and further* $(u_0, v_0) \in \mathscr{W}$. *Thus, we have the following decay estimates:*

i. *If* $q_1 = q_2 = 2$, *then there exists* $\gamma > 0$ *such that*

$$\|u'(t)\|_2^2 + \|v'(t)\|_2^2 + \|\Delta u(t)\|_2^2 + \|\Delta v(t)\|_2^2 \lesssim E(t) \lesssim \exp(-\gamma t), \; \forall t \in [0,\infty).$$

ii. If $q = \max\{q_1, q_2\} \in (2, 2^{**})$*, then*

$$\|u'(t)\|_2^2 + \|v'(t)\|_2^2 + \|\Delta u(t)\|_2^2 + \|\Delta v(t)\|_2^2 \lesssim E(t) \lesssim (1+t)^{-\frac{2}{q-2}}, \forall t \in [0, \infty).$$

Proof of Theorem 4.6. By integrating (38) over $[t, t+1]$, we have

$$E(t) - E(t+1) = \int_t^{t+1} \|u'(s)\|_{q_1}^{q_1} \mathrm{d}s + \int_t^{t+1} \|v'(s)\|_{q_2}^{q_2} \mathrm{d}s =: D(t), \forall t \in [0, \infty). \quad (45)$$

Thus, by using mean value theorem, there exist $t_1 \in \left[t, t+\frac{1}{4}\right]$ and $t_2 \in \left[t+\frac{3}{4}, t+1\right]$ such that

$$\|u'(t_i)\|_{q_1}^{q_1} \lesssim D(t), \|v'(t_i)\|_{q_2}^{q_2} \lesssim D(t), \forall i \in \{1, 2\}. \quad (46)$$

Multiplying the first equation by u, the second equation by v, and integrating it over $\Omega \times (t_1, t_2)$, we obtain

$$\begin{aligned}\int_{t_1}^{t_2} I(u(s), v(s)) \mathrm{d}s = &\int_{t_1}^{t_2} \|u'(s)\|_2^2 \mathrm{d}s + \int_{t_1}^{t_2} \|v'(s)\|_2^2 \mathrm{d}s \\ &+ \langle u'(t_1), u(t_1)\rangle - \langle u'(t_2), u(t_2)\rangle + \langle v'(t_1), v(t_1)\rangle - \langle v'(t_2), v(t_2)\rangle \\ &- \int_{t_1}^{t_2} \left\langle |u'(s)|^{q_1-2} u'(s), u(s)\right\rangle \mathrm{d}s - \int_{t_1}^{t_2} \left\langle |v'(s)|^{q_2-2} v'(s), v(s)\right\rangle \mathrm{d}s.\end{aligned} \quad (47)$$

We estimate the terms on both side of (47) as follows.
Estimate $I_1 = \int_{t_1}^{t_2} \|u'(s)\|_2^2 \mathrm{d}s + \int_{t_1}^{t_2} \|v'(s)\|_2^2 \mathrm{d}s$

By making use Hölder inequality, we deduce that

$$\int_{t_1}^{t_2} \|u'(s)\|_2^2 \mathrm{d}s \lesssim \left(\int_{t_1}^{t_2} \|u'(s)\|_{q_1}^{q_1} \mathrm{d}s\right)^{\frac{2}{q_1}} \lesssim D^{\frac{2}{q_1}}(t),$$

and

$$\int_{t_1}^{t_2} \|v'(s)\|_2^2 \mathrm{d}s \lesssim \left(\int_{t_1}^{t_2} \|v'(s)\|_{q_2}^{q_2} \mathrm{d}s\right)^{\frac{2}{q_2}} \lesssim D^{\frac{2}{q_2}}(t).$$

Therefore, we have

$$I_1 = \int_{t_1}^{t_2} \|u'(s)\|_2^2 \mathrm{d}s + \int_{t_1}^{t_2} \|v'(s)\|_2^2 \mathrm{d}s \lesssim D^{\frac{2}{q_1}}(t) + D^{\frac{2}{q_2}}(t). \quad (48)$$

Estimate $I_2 = \langle u'(t_1), u(t_1)\rangle - \langle u'(t_2), u(t_2)\rangle + \langle v'(t_1), v(t_1)\rangle - \langle v'(t_2), v(t_2)\rangle$

By applying Cauchy–Schwarz inequality, for any $\epsilon > 0$, we have

$$\left|\langle u'(t_i), u(t_i)\rangle\right| \leqslant \|u'(t_i)\|_2 \|u(t_i)\|_2 \lesssim \|u'(t_i)\|_{q_1} \|u(t_i)\|_a \lesssim D^{\frac{1}{q_1}}(t) E^{\frac{1}{2}}(t) \lesssim \epsilon E(t) + C(\epsilon) D^{\frac{2}{q_1}}(t),$$

and

$$\left|\langle v'(t_i), v(t_i)\rangle\right| \lesssim \epsilon E(t) + C(\epsilon) D^{\frac{2}{q_2}}(t),$$

for all $i \in \{1, 2\}$. Therefore, we get

$$I_2 \lesssim \epsilon E(t) + C(\epsilon) D^{\frac{2}{q_1}}(t) + C(\epsilon) D^{\frac{2}{q_2}}(t). \quad (49)$$

Estimate $I_3 = -\int_{t_1}^{t_2} \left\langle |u'(s)|^{q_1-2} u'(s), u(s)\right\rangle \mathrm{d}s - \int_{t_1}^{t_2} \left\langle |v'(s)|^{q_2-2} v'(s), v(s)\right\rangle \mathrm{d}s$

By using Hölder's inequality and Young's inequality, we have

$$\begin{aligned}\int_{t_1}^{t_2} \left\langle \left|u'(s)\right|^{q_1-2} u'(s), u(s) \right\rangle ds &\leqslant \int_{t_1}^{t_2} \|u'(s)\|_{q_1}^{q_1-1} \|u(s)\|_{q_1} ds \\ &\lesssim \epsilon \int_{t_1}^{t_2} \|u(s)\|_{q_1}^{q_1} ds + C(\epsilon) \int_{t_1}^{t_2} \|u'(s)\|_{q_1}^{q_1} ds \\ &\lesssim \epsilon \int_{t_1}^{t_2} E(s) ds + C(\epsilon) D(t),\end{aligned}$$

and

$$\int_{t_1}^{t_2} \left\langle \left|v'(s)\right|^{q_2-2} v'(s), v(s) \right\rangle ds \lesssim \epsilon \int_{t_1}^{t_2} E(s) ds + C(\epsilon) D(t).$$

Therefore, we obtain

$$I_3 \lesssim \epsilon \int_{t_1}^{t_2} E(s) ds + C(\epsilon) D(t). \tag{50}$$

On the other hand, from Theorem 4.1, we have $I_{\delta_1}(u(t), v(t)) \geqslant 0$ for all $t \in [0, \infty)$, then for all $t \in [0, \infty)$, we deduce

$$\begin{aligned}I(u(t), v(t)) &= (1-\delta_1)\left(\|\Delta u(t)\|_2^2 + \|\Delta v(t)\|_2^2\right) + I_{\delta_1}(u(t), v(t)) \\ &\geqslant (1-\delta_1)\left(\|\Delta u(t)\|_2^2 + \|\Delta v(t)\|_2^2\right).\end{aligned}$$

From definition of the functional E, we obtain

$$E(t) \lesssim \left\|u'(t)\right\|_2^2 + \left\|v'(t)\right\|_2^2 + I(u(t), v(t)), \ \forall t \in [0, \infty). \tag{51}$$

From (47)-(51), with $\epsilon > 0$ and small enough, we achieve that

$$\int_{t_1}^{t_2} E(s) ds \lesssim \epsilon E(t) + C(\epsilon) D(t) + C(\epsilon) D^{\frac{2}{q_1}}(t) + C(\epsilon) D^{\frac{2}{q_2}}(t), \ \forall t \in [0, \infty). \tag{52}$$

By using mean value theorem for integral and Lemma 3.13, we deduce that $E(t_2) \lesssim \int_{t_1}^{t_2} E(s) ds$. This fact implies

$$E(t) \lesssim \int_{t_1}^{t_2} E(s) ds + D(t), \ \forall t \in [0, \infty). \tag{53}$$

Combine with (52), (53) leads to

$$E(t) \lesssim D(t) + D^{\frac{2}{q_1}}(t) + D^{\frac{2}{q_2}}(t), \ \forall t \in [0, \infty). \tag{54}$$

Let $q = \max\{q_1, q_2\}$, from (54), we can obtain

$$\max_{t \in [t, t+1]} E^{\frac{q}{2}}(t) = E^{\frac{q}{2}}(t) \lesssim E(t) - E(t+1), \ \forall t \in [0, \infty). \tag{55}$$

By using Lemma 4.5, we can easily obtain decay estimates as in our theorem. This completes the proof of Theorem 4.6. □

4.2 Blow-up in finite time

Our main goal here is to show that with some suitable conditions, the weak solution of Problem (1)-(4) blows up in finite time. We begin by the following lemma.

Lemma 4.7. *Let $\beta \in [2,p]$ then*

$$\|u\|_p^\beta \lesssim \|\Delta u\|_2^2 + \|u\|_p^p, \forall u \in H_0^2(\Omega).$$

Theorem 4.8. *Assume that $E(0) < d$, $(u_0, v_0) \in \mathscr{U}$ and $q = \max\{q_1, q_2\} < p$ then the solution of Problem (1)-(4) blows up in finite time.*

Proof of Theorem 4.8. Our strategy can be explained as follows. First, we suppose that the solution exists for all time domain, and then we reach to a contradiction. For this purpose, we put $H(t) := -E(t)$, and $G(t) := E_* + H(t)$ for all $t \in [0,\infty)$ where $E_* \in (E(0), d)$. By direct calculation and taking (37) into consideration, we obtain

$$G'(t) = -E'(t) = \|u'(t)\|_{q_1}^{q_1} + \|v'(t)\|_{q_2}^{q_2} \geqslant 0, \forall t \in [0,\infty). \tag{56}$$

Therefore, G is increasing function and $G(t) \geqslant G(0) = E_* - E(0) > 0$ for all $t \in [0,\infty)$. Moreover, in this case, accordingly to Theorem 3.21, the function G satisfies

$$\begin{aligned} 0 < G(t) &= E_* - \frac{1}{2}\left(\|u'(t)\|_2^2 + \|v'(t)\|_2^2\right) - \frac{1}{2}\left(\|\Delta u(t)\|_2^2 + \|\Delta v(t)\|_2^2\right) \\ &\quad + \int_\Omega F(u(x,t), v(x,t))\,\mathrm{d}x \\ &\leqslant E_* - \frac{pd}{p-2} + C_2\left(\|u(t)\|_p^p + \|v(t)\|_p^p\right) \\ &\leqslant C_2\left(\|u(t)\|_p^p + \|v(t)\|_p^p\right), \forall t \in [0,\infty). \end{aligned} \tag{57}$$

We consider the function M defined by

$$M(t) := G^{1-\sigma}(t) + \epsilon\left(\langle u'(t), u(t)\rangle + \langle v'(t), v(t)\rangle\right), \forall t \in [0,\infty), \tag{58}$$

for $\epsilon > 0$ small enough, to be determined later, and

$$0 < \sigma < \min\left\{\frac{p-2}{2p}, \frac{p-q_1}{p(q_1-1)}, \frac{p-q_2}{p(q_2-1)}\right\}. \tag{59}$$

Our goal is to show that M satisfies Riccati differential inequality

$$M'(t) \gtrsim M^\alpha(t), \forall t \in [0,\infty).$$

for some constants $\alpha > 1$. This fact, of course, leads to a blow-up in finite time.

First, we note that $\lim_{\epsilon_1 \downarrow \frac{2}{p}}\left(\frac{1}{\epsilon_1} - 1\right) = \frac{p-2}{2} > \frac{(p-2)E_*}{2d}$, thus there exists $\epsilon_1 \in \left(\frac{2}{p}, 1\right)$ such that $\frac{1}{\epsilon_1} - 1 > \frac{(p-2)E_*}{2d}$. Next, we know that $\lim_{\epsilon_2 \downarrow 0}\left(\frac{1}{\epsilon_1} - 1 - \frac{\epsilon_2}{\epsilon_1}\right) = \frac{1}{\epsilon_1} - 1 > \frac{(p-2)E_*}{2d}$. Therefore, there exists $\epsilon_2 \in (0,1)$ such that

$$\frac{1}{\epsilon_1} - 1 - \frac{\epsilon_2}{\epsilon_1} > \frac{(p-2)E_*}{2d} \Longleftrightarrow (1-\epsilon_1-\epsilon_2)\frac{2d}{p-2} > \epsilon_1 E_*.$$

This fact actually implies

$$
\begin{aligned}
(1-\epsilon_1-\epsilon_2)\int_\Omega F\left(u\left(x,t\right),v\left(x,t\right)\right)\mathrm{d}x &> \frac{(1-\epsilon_1-\epsilon_2)\left(\|\Delta u\left(t\right)\|_2^2+\|\Delta v\left(t\right)\|_2^2\right)}{p} \\
&> \frac{(1-\epsilon_1-\epsilon_2)\,2d}{p-2} > \epsilon_1 E_*.
\end{aligned}
$$

By taking a derivative of (58) and using (1) and (2), we obtain

$$
\begin{aligned}
M'\left(t\right) &= (1-\sigma)\,G^{-\sigma}\left(t\right)G'\left(t\right)+\epsilon\left(\|u'\left(t\right)\|_2^2+\|v'\left(t\right)\|_2^2\right)+\epsilon\left(\langle u''\left(t\right),u\left(t\right)\rangle+\langle v''\left(t\right),v\left(t\right)\rangle\right) \\
&= (1-\sigma)\,G^{-\sigma}\left(t\right)G'\left(t\right)+\epsilon\left(\|u'\left(t\right)\|_2^2+\|v'\left(t\right)\|_2^2\right)-\epsilon\left(\|\Delta u\left(t\right)\|_2^2+\|\Delta v\left(t\right)\|_2^2\right) \\
&\quad+\epsilon p\int_\Omega F\left(u\left(x,t\right),v\left(x,t\right)\right)\mathrm{d}x-\epsilon\left\langle\left|u'\left(t\right)\right|^{q_1-2}u'\left(t\right),u\left(t\right)\right\rangle-\epsilon\left\langle\left|v'\left(t\right)\right|^{q_2-2}v'\left(t\right),v\left(t\right)\right\rangle \\
&= (1-\sigma)\,G^{-\sigma}\left(t\right)G'\left(t\right)+\epsilon\left(\|u'\left(t\right)\|_2^2+\|v'\left(t\right)\|_2^2\right)-\epsilon\left(\|\Delta u\left(t\right)\|_2^2+\|\Delta v\left(t\right)\|_2^2\right) \\
&\quad+\epsilon\epsilon_1\left[pG\left(t\right)+\frac{p}{2}\left(\|u'\left(t\right)\|_2^2+\|v'\left(t\right)\|_2^2\right)+\frac{p}{2}\left(\|\Delta u\left(t\right)\|_2^2+\|\Delta v\left(t\right)\|_2^2\right)-pE_*\right] \\
&\quad+\epsilon\epsilon_2 p\int_\Omega F\left(u\left(x,t\right),v\left(x,t\right)\right)\mathrm{d}x+\epsilon\left(1-\epsilon_1-\epsilon_2\right)p\int_\Omega F\left(u\left(x,t\right),v\left(x,t\right)\right)\mathrm{d}x \\
&\quad-\epsilon\left\langle\left|u'\left(t\right)\right|^{q_1-2}u'\left(t\right),u\left(t\right)\right\rangle-\epsilon\left\langle\left|v'\left(t\right)\right|^{q_2-2}v'\left(t\right),v\left(t\right)\right\rangle \\
&= (1-\sigma)\,G^{-\sigma}\left(t\right)G'\left(t\right)+\epsilon\left(\frac{p\epsilon_1}{2}+1\right)\left(\|u'\left(t\right)\|_2^2+\|v'\left(t\right)\|_2^2\right) \\
&\quad+\epsilon\left(\frac{p\epsilon_1}{2}-1\right)\left(\|\Delta u\left(t\right)\|_2^2+\|\Delta v\left(t\right)\|_2^2\right) \\
&\quad+\epsilon\epsilon_1 pG\left(t\right)+\epsilon\epsilon_2 p\int_\Omega F\left(u\left(x,t\right),v\left(x,t\right)\right)\mathrm{d}x \\
&\quad-\epsilon\left\langle\left|u'\left(t\right)\right|^{q_1-2}u'\left(t\right),u\left(t\right)\right\rangle-\epsilon\left\langle\left|v'\left(t\right)\right|^{q_2-2}v'\left(t\right),v\left(t\right)\right\rangle \\
&\quad+\epsilon p\left[(1-\epsilon_1-\epsilon_2)\int_\Omega F\left(u\left(x,t\right),v\left(x,t\right)\right)\mathrm{d}x-\epsilon_1 E_*\right] \\
&\geqslant (1-\sigma)\,G^{-\sigma}\left(t\right)G'\left(t\right)+\epsilon\left(\frac{p\epsilon_1}{2}+1\right)\left(\|u'\left(t\right)\|_2^2+\|v'\left(t\right)\|_2^2\right) \qquad (60) \\
&\quad+\epsilon\left(\frac{p\epsilon_1}{2}-1\right)\left(\|u\left(t\right)\|_a^2+\|v\left(t\right)\|_b^2\right) \\
&\quad+\epsilon\epsilon_1 pG\left(t\right)+\epsilon\epsilon_2 p\int_\Omega F\left(u\left(x,t\right),v\left(x,t\right)\right)\mathrm{d}x \\
&\quad-\epsilon\left\langle\left|u'\left(t\right)\right|^{q_1-2}u'\left(t\right),u\left(t\right)\right\rangle-\epsilon\left\langle\left|v'\left(t\right)\right|^{q_2-2}v'\left(t\right),v\left(t\right)\right\rangle. \qquad (61)
\end{aligned}
$$

By using Young's inequality, we have following estimates

$$
\left\langle\left|u'\left(t\right)\right|^{q_1-2}u'\left(t\right),u\left(t\right)\right\rangle \leqslant \frac{\epsilon_3^{q_1}}{q_1}\|u\left(t\right)\|_{q_1}^{q_1}+\frac{\epsilon_3^{-q_1'}}{q_1'}\|u'\left(t\right)\|_{q_1}^{q_1} \leqslant \frac{\epsilon_3^{q_1}}{q_1}\|u\left(t\right)\|_{q_1}^{q_1}+\frac{\epsilon_3^{-q_1'}}{q_1'}G'\left(t\right), \quad (62)
$$

and

$$
\left\langle\left|v'\left(t\right)\right|^{q_2-2}v'\left(t\right),v\left(t\right)\right\rangle \leqslant \frac{\epsilon_4^{q_2}}{q_2}\|v\left(t\right)\|_{q_2}^{q_2}+\frac{\epsilon_4^{-q_2'}}{q_2'}\|v'\left(t\right)\|_{q_2}^{q_2} \leqslant \frac{\epsilon_4^{q_2}}{q_2}\|v\left(t\right)\|_{q_2}^{q_2}+\frac{\epsilon_4^{-q_2'}}{q_2'}G'\left(t\right), \quad (63)
$$

for any $\epsilon_3 > 0$, $\epsilon_4 > 0$, where q_i' is the conjugate exponent of q_i. Furthermore, from (61)-(63), with $\epsilon_3^{-q_1'} = K_1 G^{-\sigma}(t)$, $\epsilon_4^{-q_2'} = K_2 G^{-\sigma}(t)$ where $K_1 > 0$, $K_2 > 0$ are specified later, we obtain

$$\begin{aligned} M'(t) \geqslant & \left[1 - \sigma - \epsilon\left(\frac{K_1}{q_1'} + \frac{K_2}{q_2'}\right)\right] G^{-\sigma}(t)\, G'(t) + \epsilon\left(\frac{p\epsilon_1}{2} + 1\right)\left(\|u'(t)\|_2^2 + \|v'(t)\|_2^2\right) \\ & + \epsilon\left(\frac{p\epsilon_1}{2} - 1\right)\left(\|\Delta u(t)\|_2^2 + \|\Delta v(t)\|_2^2\right) + \epsilon\epsilon_1 p G(t) + \epsilon\epsilon_2 p \int_\Omega F(u(x,t), v(x,t))\,\mathrm{d}x \\ & - \frac{\epsilon}{q_1 K_1^{q_1-1}} G^{\sigma(q_1-1)}(t)\, \|u(t)\|_{q_1}^{q_1} - \frac{\epsilon}{q_2 K_2^{q_2-2}} G^{\sigma(q_2-1)}(t)\, \|v(t)\|_{q_2}^{q_2}. \end{aligned} \tag{64}$$

By using (11) and Theorem 3.21, we have following estimate

$$\|u(t)\|_p^p + \|v(t)\|_p^p \lesssim G(t) + \|u'(t)\|_2^2 + \|v'(t)\|_2^2 + \|\Delta u(t)\|_2^2 + \|\Delta v(t)\|_2^2,\ \forall t \in [0, \infty). \tag{65}$$

Therefore, by making use (57) and (65), we get

$$\begin{aligned} G^{\sigma(q_1-1)}(t)\, \|u(t)\|_{q_1}^{q_1} & \lesssim \left(\|u(t)\|_p^p + \|v(t)\|_p^p\right)^{\sigma(q_1-1)} \|u(t)\|_{q_1}^{q_1} \\ & \lesssim \|u(t)\|_p^{\sigma p(q_1-1)+q_1} + \|v(t)\|_p^{\sigma p(q_1-1)+q_1} \\ & \lesssim \|\Delta u(t)\|_2^2 + \|\Delta v(t)\|_2^2 + \|u(t)\|_p^p + \|v(t)\|_p^p \\ & \lesssim G(t) + \|u'(t)\|_2^2 + \|v'(t)\|_2^2 + \|\Delta u(t)\|_2^2 + \|\Delta v(t)\|_2^2, \end{aligned} \tag{66}$$

provided by Lemma 4.7 with $\beta = \sigma p(q_1 - 1) + q_1 < p$. With the same spirit, we also obtain

$$G^{\sigma(q_2-1)}(t)\, \|v(t)\|_{q_2}^{q_2} \lesssim G(t) + \|u'(t)\|_2^2 + \|v'(t)\|_2^2 + \|\Delta u(t)\|_2^2 + \|\Delta v(t)\|_2^2. \tag{67}$$

From (64), (66) and (67), there exists a constant $C_* > 0$ such that

$$\begin{aligned} M'(t) \geqslant & \left[1 - \sigma - \epsilon\left(\frac{K_1}{q_1'} + \frac{K_2}{q_2'}\right)\right] G^{-\sigma}(t)\, G'(t) \\ & + \epsilon\left[\frac{p\epsilon_1}{2} + 1 - C_*\left(\frac{1}{K_1^{q_1-1}} + \frac{1}{K_2^{q_2-1}}\right)\right]\left(\|u'(t)\|_2^2 + \|v'(t)\|_2^2\right) \\ & + \epsilon\left[\frac{p\epsilon_1}{2} - 1 - C_*\left(\frac{1}{K_1^{q_1-1}} + \frac{1}{K_2^{q_2-1}}\right)\right]\left(\|\Delta u(t)\|_2^2 + \|\Delta v(t)\|_2^2\right) \\ & + \epsilon\left[\epsilon_1 p - C_*\left(\frac{1}{K_1^{q_1-q}} + \frac{1}{K_2^{q_2-2}}\right)\right] G(t) + p\epsilon\epsilon_1 \int_\Omega F(u(x,t), v(x,t))\,\mathrm{d}x. \end{aligned} \tag{68}$$

At this point, for large values of K_1 and K_2 such that

$$\frac{p\epsilon_1}{2} - 1 - C_*\left(\frac{1}{K_1^{q_1-1}} + \frac{1}{K_2^{q_2-1}}\right) > 0,$$

we choose $\epsilon > 0$ such that

$$1 - \sigma - \epsilon\left(\frac{K_1}{q_1'} + \frac{K_2}{q_2'}\right) > 0,\ M(0) = G^{1-\sigma}(0) + \epsilon\left(\langle u_0, u_1\rangle + \langle v_0, v_1\rangle\right) > 0,$$

then (68) gives us

$$\begin{aligned} M'(t) \gtrsim & G(t) + \|u'(t)\|_2^2 + \|v'(t)\|_2^2 + \|\Delta u(t)\|_2^2 + \|\Delta v(t)\|_2^2 \\ & + \int_\Omega F(u(x,t), v(x,t))\,\mathrm{d}x,\ \forall t \in [0, \infty). \end{aligned} \tag{69}$$

On the other hand, applying Hölder's inequality, we obtain

$$\begin{aligned}\left(\left\langle u'(t),u(t)\right\rangle+\left\langle v'(t),v(t)\right\rangle\right)^{\frac{1}{1-\sigma}} &\lesssim \left\|u'(t)\right\|_2^{\frac{1}{1-\sigma}}\left\|u(t)\right\|_2^{\frac{1}{1-\sigma}}+\left\|v'(t)\right\|_2^{\frac{1}{1-\sigma}}\left\|v(t)\right\|_2^{\frac{1}{1-\sigma}}\\ &\lesssim \left\|u'(t)\right\|_2^{\frac{1}{1-\sigma}}\left\|u(t)\right\|_p^{\frac{1}{1-\sigma}}+\left\|v'(t)\right\|_2^{\frac{1}{1-\sigma}}\left\|v(t)\right\|_p^{\frac{1}{1-\sigma}}\\ &\lesssim \left\|u'(t)\right\|_2^{2}+\left\|u(t)\right\|_p^{\frac{2}{1-2\sigma}}+\left\|v'(t)\right\|_2^{2}+\left\|v(t)\right\|_p^{\frac{2}{1-2\sigma}}\\ &\lesssim \left\|u'(t)\right\|_2^{2}+\left\|v'(t)\right\|_2^{2}+\left\|\Delta u(t)\right\|_2^{2}+\left\|\Delta v(t)\right\|_2^{2}\\ &+\left\|u(t)\right\|_p^{p}+\left\|v(t)\right\|_p^{p}\\ &\lesssim \left\|u'(t)\right\|_2^{2}+\left\|v'(t)\right\|_2^{2}+\left\|\Delta u(t)\right\|_2^{2}+\left\|\Delta v(t)\right\|_2^{2}\\ &+\int_\Omega F(u(x,t),v(x,t))\,\mathrm{d}x.\end{aligned}$$

Therefore, we have

$$\begin{aligned}M^{\frac{1}{1-\sigma}}(t) &= \left[G^{1-\sigma}(t)+\epsilon\left(\left\langle u'(t),u(t)\right\rangle+\left\langle v'(t),v(t)\right\rangle\right)\right]^{\frac{1}{1-\sigma}}\\ &\lesssim G(t)+\left(\left\langle u'(t),u(t)\right\rangle+\left\langle v'(t),v(t)\right\rangle\right)^{\frac{1}{1-\sigma}}\\ &\lesssim G(t)+\left\|u'(t)\right\|_2^{2}+\left\|v'(t)\right\|_2^{2}+\left\|\Delta u(t)\right\|_2^{2}+\left\|\Delta v(t)\right\|_2^{2}+\int_\Omega F(u(x,t),v(x,t))\,\mathrm{d}x.\end{aligned} \tag{70}$$

From (69) and (70), we obtain

$$M'(t)\gtrsim M^{\frac{1}{1-\sigma}}(t),\ \forall t\in[0,\infty).$$

And this estimate implies that the solution blows up in a finite time. Theorem 4.8 is proved. □

Theorem 4.9. *Assume that $E(0)<d$, $(u_0,v_0)\in\mathscr{U}$ and $q_1=q_2=2$ then the weak solution of Problem (1)-(4) blows up in finite time. Furthermore, we have the following estimate*

$$T_\infty\leqslant\frac{2}{p-2}\,\frac{\zeta+\sqrt{\zeta^2+2(d-E(0))\left(\|u_0\|_2^2+\|v_0\|_2^2\right)}}{(d-E(0))}, \tag{71}$$

where

$$\zeta:=\frac{2}{p-2}\left(\|u_0\|_2^2+\|v_0\|_2^2\right)-\langle u_0,u_1\rangle-\langle v_0,v_1\rangle. \tag{72}$$

Proof of Theorem 4.9. By last statement in Theorem 2.6, it is enough to prove that no global solution in $[0,\infty)$ can exists. Then, we will assume, by contradiction, that weak solutions exist in the whole interval $[0,\infty)$. The main tool in proving the blow-up result is the concavity method where the basis idea of the method is to construct a positive defined functional M of the solution by the energy inequality and show that $M^{-\alpha}$ is concave function of time variable. For this purpose, with $T_0>0$, $\beta>0$, and $\tau>0$ specified later, we define the auxiliary functional $M:[0,T_0]\longrightarrow\mathbb{R}$ by

$$\begin{aligned}M(t)=&\ \|u(t)\|_2^2+\|v(t)\|_2^2+\int_0^t\|u(s)\|_2^2\,\mathrm{d}s+\int_0^t\|v(s)\|_2^2\,\mathrm{d}s\\ &+(T_0-t)\left(\|u_0\|_2^2+\|v_0\|_2^2\right)+\beta(t+\tau)^2.\end{aligned} \tag{73}$$

By direct computation, we achieve that

$$
\begin{aligned}
M'(t) &= 2\left\langle u'(t), u(t)\right\rangle + 2\left\langle v'(t), v(t)\right\rangle + \left(\|u(t)\|_2^2 - \|u_0\|_2^2\right) \\
&\quad + \left(\|v(t)\|_2^2 - \|v_0\|_2^2\right) + 2\beta(t+\tau) \\
&= 2\left\langle u'(t), u(t)\right\rangle + 2\left\langle v'(t), v(t)\right\rangle + 2\int_0^t \left\langle u'(s), u(s)\right\rangle \mathrm{d}s \\
&\quad + 2\int_0^t \left\langle v'(s), v(s)\right\rangle \mathrm{d}s + 2\beta(t+\tau),
\end{aligned}
\tag{74}
$$

and

$$
\begin{aligned}
M''(t) &= 2\left\langle u''(t), u(t)\right\rangle + 2\left\|u'(t)\right\|_2^2 + 2\left\langle v''(t), v(t)\right\rangle + 2\left\|v'(t)\right\|_2^2 \\
&\quad + 2\left\langle u'(t), u(t)\right\rangle + 2\left\langle v'(t), v(t)\right\rangle + 2\beta \\
&= 2\left\|u'(t)\right\|_2^2 + 2\left\|v'(t)\right\|_2^2 - 2\left(\|\Delta u(t)\|_2^2 + \|\Delta v(t)\|_2^2\right) + 2p\int_\Omega F(u(x,t), v(x,t))\,\mathrm{d}x + 2\beta.
\end{aligned}
\tag{75}
$$

From (73) and (74), we have $M(t) \geqslant \beta\tau^2 > 0$ for all $t \in [0, T_0]$, and $M'(0) = 2\langle u_0, u_1\rangle + 2\langle v_0, v_1\rangle + 2\beta\tau > 0$ for $\beta\tau$ large enough.

From (74), thanks to Cauchy-Schwartz inequality, we obtain

$$
\begin{aligned}
&\frac{\left(M'(t)\right)^2}{4} \\
&= \left[\left\langle u'(t), u(t)\right\rangle + \left\langle v'(t), v(t)\right\rangle + \int_0^t \left\langle u'(s), u(s)\right\rangle \mathrm{d}s + \int_0^t \left\langle v'(s), v(s)\right\rangle \mathrm{d}s + \beta(t+\tau)\right]^2 \\
&\leqslant \left(\|u(t)\|_2^2 + \|v(t)\|_2^2 + \int_0^t \|u(s)\|_2^2 \mathrm{d}s + \int_0^t \|v(s)\|_2^2 \mathrm{d}s + \beta(t+\tau)^2\right) \\
&\quad \times \left(\left\|u'(t)\right\|_2^2 + \left\|v'(t)\right\|_2^2 + \int_0^t \left\|u'(s)\right\|_2^2 \mathrm{d}s + \int_0^t \left\|v'(s)\right\|_2^2 \mathrm{d}s + \beta\right) \\
&\leqslant \left(\left\|u'(t)\right\|_2^2 + \left\|v'(t)\right\|_2^2 + \int_0^t \left\|u'(s)\right\|_2^2 \mathrm{d}s + \int_0^t \left\|v'(s)\right\|_2^2 \mathrm{d}s + \beta\right) M(t).
\end{aligned}
\tag{76}
$$

From (73)-(76), we obtain

$$
M''(t)\,M(t) - \frac{(p+2)\left(M'(t)\right)^2}{4} \geqslant M(t)\,D(t),\ \forall t \in [0, T_0],
\tag{77}
$$

where

$$
\begin{aligned}
D(t) = &-2p\left[\frac{1}{2}\left(\left\|u'(t)\right\|_2^2 + \left\|v'(t)\right\|_2^2\right) - \int_\Omega F(u(x,t), v(x,t))\,\mathrm{d}x\right] \\
&- 2\left(\|\Delta u(t)\|_2^2 + \|\Delta v(t)\|_2^2\right) \\
&- (p+2)\left(\int_0^t \left\|u'(s)\right\|_2^2 \mathrm{d}s + \int_0^t \left\|v'(s)\right\|_2^2 \mathrm{d}s\right) - p\beta.
\end{aligned}
\tag{78}
$$

From definition of the functional E and Theorem 3.21, we achieve that

$$\begin{aligned} D(t) = & -2p\left[E(0) - \int_0^t \|u'(s)\|_2^2 ds - \int_0^t \|v'(s)\|_2^2 ds - \frac{1}{2}\left(\|u(t)\|_a^2 + \|v(t)\|_b^2\right)\right] \\ & -2\left(\|u(t)\|_a^2 + \|v(t)\|_b^2\right) - (p+2)\left(\int_0^t \|u'(s)\|_2^2 ds + \int_0^t \|v'(s)\|_2^2 ds\right) - p\beta \\ = & \ (p-2)\left(\|\Delta u(t)\|_2^2 + \|\Delta v(t)\|_2^2\right) - 2pE(0) - p\beta \\ & + (p-2)\left(\int_0^t \|u'(s)\|_2^2 ds + \int_0^t \|v'(s)\|_2^2 ds\right) \\ > & \ 2p\left[d - E(0)\right] - p\beta. \end{aligned} \tag{79}$$

Choose $\beta \in (0, 2(d - E(0))]$. From (77)-(79), we obtain

$$M(t) \geqslant M(0)\left[1 - \frac{(p-2)}{4}\frac{M'(0)\,t}{M(0)}\right]^{-\frac{4}{p-2}}, \forall t \in [0, T_0]. \tag{80}$$

Choose $\tau \in (\tau_*, \infty)$ where

$$\tau_* = \begin{cases} 0 \text{ if } \zeta = \dfrac{2}{p-2}\left(\|u_0\|_2^2 + \|v_0\|_2^2\right) - \langle u_0, u_1\rangle - \langle v_0, v_1\rangle \leqslant 0, \\ \dfrac{\zeta}{\beta} \text{ if } \zeta > 0, \end{cases} \tag{81}$$

and $T_0 \in \left[\frac{2}{p-2}\frac{\beta\tau^2 + \|u_0\|_2^2 + \|v_0\|_2^2}{\beta\tau - \zeta}, \infty\right)$, we get

$$T_* = \frac{4M(0)}{(p-2)M'(0)} = \frac{2\left[\|u_0\|_2^2 + \|v_0\|_2^2 + T_0\left(\|u_0\|_2^2 + \|v_0\|_2^2\right) + \beta\tau^2\right]}{(p-2)\left(\beta\tau + \langle u_0, u_1\rangle + \langle v_0, v_1\rangle\right)} \leqslant T_0.$$

From (80), we get $\lim_{t\nearrow T_*} M(t) = \infty$. This is a contradiction with the fact that the solution is global and it shows that the solution blows up at finite time.

To derive the upper bound for T_∞, we know that

$$T_\infty \leqslant \frac{2}{p-2}\frac{\beta\tau^2 + \|u_0\|_2^2 + \|v_0\|_2^2}{\beta\tau - \zeta} = \frac{2}{p-2} f(\beta, \tau), \forall (\beta, \tau) \in (0, 2(d - E(0))] \times (\tau_*, \infty).$$

We have

$$f_\tau(\beta, \tau) = \frac{\beta\left(\beta\tau^2 - 2\zeta\tau - \|u_0\|_2^2 - \|v_0\|_2^2\right)}{(\beta\tau - \zeta)^2} = 0 \Longleftrightarrow \tau = \frac{\zeta \pm \sqrt{\zeta^2 + \beta\left(\|u_0\|_2^2 + \|v_0\|_2^2\right)}}{\beta}.$$

Therefore

$$f(\beta, \tau) \geqslant f\left(\beta, \frac{\zeta + \sqrt{\zeta^2 + \beta\left(\|u_0\|_2^2 + \|v_0\|_2^2\right)}}{\beta}\right) = 2\frac{\zeta + \sqrt{\zeta^2 + \beta\left(\|u_0\|_2^2 + \|v_0\|_2^2\right)}}{\beta} = h(\beta),$$

for all $(\beta, \tau) \in (0, 2(d - E(0))] \times (\tau_*, \infty)$. Moreover, we also have

$$h'(\beta) = -\frac{\left(\zeta + \sqrt{\zeta^2 + \beta\left(\|u_0\|_2^2 + \|v_0\|_2^2\right)}\right)}{\beta^2\sqrt{\zeta^2 + \beta\left(\|u_0\|_2^2 + \|v_0\|_2^2\right)}} \leqslant 0, \forall \beta \in (0, 2(d - E(0))],$$

Therefore, we achieve

$$T_\infty \leqslant \frac{2}{p-2} \frac{\zeta + \sqrt{\zeta^2 + 2\left(d - E\left(0\right)\right)\left(\|u_0\|_2^2 + \|v_0\|_2^2\right)}}{\left(d - E\left(0\right)\right)}.$$

Theorem 4.9 is proved. □

5 Critical case initial energy

In this section, we shall extend all the results obtained for $E\left(0\right) < d$ to $E\left(0\right) = d$. Although their results are similar, the proofs need necessary modification. Based on this, we do not give the complete proofs, but just prove these results by modifying the corresponding proofs in the case of $E\left(0\right) < d$.

First, we can easily to obtain the following lemma.

Lemma 5.1. *Assume that (u, v) is a solution to Problem (1)-(4). These three statements are logically equivalent*

i. $\left(u\left(t\right), v\left(t\right)\right) = \left(u_0, v_0\right)$ *for all* $t \in [0, \infty)$;
ii. $\left(u_1, v_1\right) = (0, 0)$ *and* $\left(u_0, v_0\right) \in \mathscr{S}$, *where* $\mathscr{S}$ *is the set of all stationary solutions of Problem* (4)-(4);
iii. $\int_0^t \|u'\left(s\right)\|_{q_1}^{q_1} \mathrm{d}s = \int_0^t \|v'\left(s\right)\|_{q_2}^{q_2} \mathrm{d}s = 0$ *for all* $t \in (0, \infty)$.

Lemma 5.2. *Assume that* $E\left(0\right) = d$. *Then* $I\left(u_0, v_0\right) = 0$ *and* $u_1 = v_1 = 0$ *if and only if* $\left(u\left(t\right), v\left(t\right)\right) = \left(u_0, v_0\right)$ *for all* $t \in (0, \infty)$.

Proof of Lemma 5.2. It is clear that if $\left(u\left(t\right), v\left(t\right)\right) = \left(u_0, v_0\right)$ for all $t \in (0, \infty)$, then $\left(u_1, v_1\right) = (0, 0)$ and $\left(u_0, v_0\right) \in \mathscr{S}$. Furthermore, we know that if $\left(u_0, v_0\right) \in \mathscr{S}$ then $I\left(u_0, v_0\right) = 0$. So next, we assume that $I\left(u_0, v_0\right) = 0$ and $u_1 = v_1 = 0$ and prove that $\left(u\left(t\right), v\left(t\right)\right) = \left(u_0, v_0\right)$ for all $t \in (0, \infty)$. From the equality $d = E\left(0\right) = J\left(u_0, v_0\right)$, we get $\left(u_0, v_0\right) \in \mathscr{S}_d$ and this fact implies $\left(u\left(t\right), v\left(t\right)\right) = \left(u_0, v_0\right)$ for all $t \in (0, \infty)$. Lemma 5.2 is proved. □

Theorem 5.3. *Assume that* $E\left(0\right) = d$. *If* $\left(u_0, v_0\right) \in \mathscr{W}$ *then the weak solution of Problem (1)-(4) is global and limit of total energy of Problem (1)-(4) is zero at infinity.*

Proof of Theorem 5.3. First, we prove that there exists $t_* \in (0, T_\infty)$ such that

$$\int_0^{t_*} \|u'\left(s\right)\|_{q_1}^{q_1} \mathrm{d}s + \int_0^{t_*} \|v'\left(s\right)\|_{q_2}^{q_2} \mathrm{d}s > 0.$$

Suppose $\int_0^t \|u'\left(s\right)\|_{q_1}^{q_1} \mathrm{d}s + \int_0^t \|v'\left(s\right)\|_{q_2}^{q_2} \mathrm{d}s = 0$ for all $t \in (0, T_\infty)$. By Lemmas 5.1 and 5.2, we have $I\left(u_0, v_0\right) = 0$ and $u_1 = v_1 = 0$. From $\left(u_0, v_0\right) \in \mathscr{W}$, we obtain $u_0 = v_0 = 0$. Therefore, $d = E\left(0\right) = 0$ which contradicts with $d > 0$. This fact implies there exists $t_* \in (0, T_\infty)$ such that $E\left(t_*\right) < d$ and $\left(u\left(t_*\right), v\left(t_*\right)\right) \in \mathscr{W}$. By Theorem 4.1, we know that the weak solution is global. Theorem 5.3 is proved. □

With the same spirit, we also have the following result.

Theorem 5.4. *Assume that* $E\left(0\right) = d$. *If* $\left(u_0, v_0\right) \in \mathscr{U}$ *and* $q = \max\left\{q_1, q_2\right\} < p$ *then the weak solution of Problem (1)-(4) is blows up in finite time.*

6 High energy initial data

In this section, we investigate the conditions to ensure the existence of finite time blow-up solutions to Problem (1)-(4) with $E(0) > d$ with $q_1 = q_2 = 2$. We begin by introducing the following lemma.

Lemma 6.1. *Let $\delta \geqslant 0$, $T > 0$ and let h be a Lipschitzian function over $[0, T)$. Assume that $h(0) \geqslant 0$ and $h'(t) + \delta h(t) > 0$ for a.e. $t \in (0, T)$. Then $h(t) > 0$ for all $t \in (0, T)$.*

We may now prove the weak antidissipativity of the flow whenever $(u(t), v(t)) \in \mathcal{N}_-$.

Lemma 6.2. *Assume that $(u_0, v_0) \in \mathcal{N}_-$, $(u_1, v_1) \in \left[L^2(\Omega)\right]^2$ are such that*

$$\langle u_0, u_1\rangle + \langle v_0, v_1\rangle \geqslant 0. \tag{82}$$

Then the map $t \longmapsto \|u(t)\|_2^2 + \|v(t)\|_2^2$ is strictly increasing as long as $(u(t), v(t)) \in \mathcal{N}_-$.

Proof of Lemma 6.2. Let $H(t) = \|u(t)\|_2^2 + \|v(t)\|_2^2$, and $G(t) = H'(t) = 2\langle u'(t), u(t)\rangle + 2\langle v'(t), v(t)\rangle$. By direct calculation, we obtain

$$\begin{aligned} G'(t) &= 2\left\langle u''(t), u(t)\right\rangle + 2\left\langle v''(t), v(t)\right\rangle + 2\left\|u'(t)\right\|_0^2 + 2\left\|v'(t)\right\|_0^2 \\ &= 2\left\|u'(t)\right\|_2^2 + 2\left\|v'(t)\right\|_2^2 - I(u(t), v(t)) - \frac{1}{2}G(t). \end{aligned}$$

This fact implies $G'(t) + \frac{1}{2}G(t) > 0$ as long as $(u(t), v(t)) \in \mathcal{N}_-$. Combine with Lemma 6.1 and definition of function H, we have $H(t)$ is strictly increasing. Lemma 6.2 is proved. □

We now prove the invariance set of $\mathcal{N}_-$ with some suitable conditions.

Lemma 6.3. *Suppose that $(u_0, v_0, u_1, v_1) \in \left[H_0^2(\Omega)\right]^2 \times \left[L^2(\Omega)\right]^2$. Assume that the initial data satisfy*

$$2\langle u_0, u_1\rangle + 2\langle v_0, v_1\rangle - \|u_0\|_2^2 - \|v_0\|_2^2 > \frac{2p}{(p-2)\aleph_*^2}E(0), \tag{83}$$

where $\aleph_* := \inf\limits_{u \in H_0^2(\Omega)\setminus\{0\}} \frac{\|\Delta u\|_2}{\|u\|_2} > 0$. Then the solution (u, v) of the Problem (1)-(4) with $E(0) > 0$ belong to $\mathcal{N}_-$, provided by $(u_0, v_0) \in \mathcal{N}_-$.

Proof of Lemma 6.3. Arguing by contradiction, by the continuity of I in t, we suppose that there exists a $t_* \in (0, T_\infty)$ such that $I(u(t), v(t)) < 0$ for all $t \in [0, t_*)$ and $I(u(t_*), v(t_*)) = 0$. By the Cauchy-Schwarz inequality, we have

$$2\langle u_0, u_1\rangle + 2\langle v_0, v_1\rangle \leqslant \|u_0\|_2^2 + \|v_0\|_2^2 + \|u_1\|_2^2 + \|v_1\|_2^2.$$

Therefore, by Lemma 6.2, for any $t \in [0, T_\infty)$, we have

$$\begin{aligned} H(t) = \|u(t)\|_2^2 + \|v(t)\|_2^2 > \|u_0\|_2^2 + \|v_0\|_2^2 &\geqslant 2\langle u_0, u_1\rangle + 2\langle v_0, v_1\rangle - \|u_1\|_2^2 - \|v_1\|_2^2 \\ &> \frac{2p}{(p-2)\aleph_*^2}E(0). \end{aligned} \tag{84}$$

Moreover, from the continuity of solution with respect to t, we have

$$H(t_*) > \frac{2p}{(p-2)\aleph_*^2}E(0).$$

From Lemma 3.13, we obtain

$$E(0) \geqslant E(t_*) = \frac{1}{2}\left(\|u'(t_*)\|_2^2 + \|v'(t_*)\|_2^2\right) + \frac{p-2}{2p}\left(\|\Delta u(t_*)\|_2^2 + \|\Delta v(t_*)\|_2^2\right) + \frac{I(u(t_*), v(t_*))}{p} \geqslant \frac{(p-2)\aleph_*^2}{2p}\left(\|u(t_*)\|_2^2 + \|v(t_*)\|_2^2\right) > E(0). \tag{85}$$

that is contraction. Therefore, $(u(t), v(t)) \in \mathcal{N}_-$ for all $t \in [0, T_\infty)$. This completes the proof of Lemma 6.3. □

We now present the main blow-up theorem for the weak solution with arbitrary positive initial energy.

Theorem 6.4. *Suppose that* $(u_0, v_0, u_1, v_1) \in \left[H_0^2(\Omega)\right]^2 \times \left[L^2(\Omega)\right]^2$. *Assume that the initial data satisfy*

$$2\langle u_0, u_1\rangle + 2\langle v_0, v_1\rangle - \|u_1\|_2^2 - \|v_1\|_2^2 > \frac{2p}{(p-2)\aleph_*^2}E(0). \tag{86}$$

Then the solution (u, v) *of the Problem (1)-(4) blows up in finite time provided by* $(u_0, v_0) \in \mathcal{N}_-$ *and* $E(0) > 0$.

Further, we have the following estimate

$$T_\infty \leqslant \frac{4\left[\zeta + \sqrt{\zeta^2 + \beta_*\left(\|u_0\|_2^2 + \|v_0\|_2^2\right)}\right]}{(p-2)\beta_*},$$

where

$$\beta_* := \frac{(p-2)\aleph_*^2}{p}\left[2\langle u_0, u_1\rangle + 2\langle v_0, v_1\rangle - \|u_1\|_2^2 - \|v_1\|_2^2 - \frac{2p}{(p-2)\aleph_*^2}E(0)\right] > 0,$$

and

$$\zeta := \frac{2}{p-2}\left(\|u_0\|_2^2 + \|v_0\|_2^2\right) - \langle u_0, u_1\rangle - \langle v_0, v_1\rangle.$$

Proof of Theorem 6.4. We use the same method as in Theorem 4.9. From Theorem 4.9, we just need to prove that $D(t) \geqslant 0$. From (79), we have

$$\begin{aligned} D(t) &\geqslant (p-2)\left(\|\Delta u(t)\|_2^2 + \|\Delta v(t)\|_2^2\right) - 2pE(0) - p\beta \\ &\geqslant (p-2)\aleph_*^2\left(\|u(t)\|_2^2 + \|v(t)\|_2^2\right) - 2pE(0) - p\beta \\ &\geqslant (p-2)\aleph_*^2\left[2\langle u_0, u_1\rangle + 2\langle v_0, v_1\rangle - \|u_1\|_2^2 - \|v_1\|_2^2 - \frac{2p}{(p-2)\aleph_*^2}E(0)\right] - p\beta \\ &= p(\beta_* - \beta). \end{aligned}$$

Choose $\beta \in (0, \beta_*]$, we have $D(t) \geqslant 0$. With same calculation as in Theorem 4.9, we can easily obtain the rest results in our theorem. Theorem 6.4 is proved. □

7 Declarations

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