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New Solitary Wave Solutions of the Korteweg-de Vries (KdV) Equation by New Version of the Trial Equation Method

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Abstract

New solitary wave solutions for the Korteweg-de Vries (KdV) equation by a new version of the trial equation method are attained. Proper transformation reduces the Korteweg-de Vries (KdV) equation to a quadratic ordinary differential equation that is fully integrated using the new version trial equation approach. The family of solitary wave solutions of the reduced equation ensures a combined expression for the Korteweg-de Vries (KdV) equation, which contains exact solutions derived in recent years using different integration methods. The analytic solution of the reduced equation permits to find exact solutions for the Korteweg-de Vries (KdV) equation, providing a variety of new solitary wave solutions that have not been reported before.

Keywords: New version of the trial equation method, nonlinear partial differential equations, Korteweg-de Vries (KdV) equation, solitary wave soliton solutions
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1 Introduction

Solitary wave solutions are obtained from the solutions of nonlinear partial differential equations. These solitary wave solutions occur in various scientific fields such as optical fibers, plasma physics, chemical physics, chemical kinematics, fluid mechanics, solid-state physics, and so on, and help us understand many nonlinear phenomena. These nonlinear events are understood by examining the exact solutions of the nonlinear partial differential equations. The examination of these exact solutions and dynamics has been done by many scientists in various fields of science. Some analytical methods have been proposed to find the exact solutions of partial differential equations such as exp-function method, tanh function method, G'/G-expansion

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method, Hirota bilinear method, first integral method, improved G'/G-expansion method, extended trial equation method, multiple extended trial equation method, trial equation method, Jacobi elliptic function method, modified Kudryashov method, extended Kudryashov method and generalized Kudryashov method [1–6, 8–15, 19, 20, 22–37, 39–41]. Recently, a method that allows soliton solutions of the partial differential equations has been proposed by Ma and Fuchssteiner [21]. Their main intention was to find the solution function of a resolvable differential equation in terms of rational polynomial or polynomial functions. This idea was developed in the following years by many scientists. Liu called this approach the trial equation method and applied it to some nonlinear evolution equations [16–18]. Recently this approach was subsequently improved and named as extended trial equation method by Pandir et al [10, 40]. A more general form of this experiment equation method is implemented to the partial derivative equations by Pandir and Gurefe and obtained different new exact solutions. In this study, we offer a new version of the trial equation method and then apply this method to nonlinear Korteweg-de Vries (KdV) equation. A new version of the trial equation method will allow us to find combined solitary wave solutions and combined Jacobi elliptic functions in the solution function. With this developed new method, we tried to attain renewed and dissimilar solutions that are not in the literature. We will research the solutions of the following nonlinear Korteweg-de Vries (KdV) equation

$$w_t + w_{xxx} - 6ww_x = 0. \tag{1.1}$$

The Korteweg-de Vries (KdV) equation is used in many parts of nonlinear mechanical and theoretical physics to describe nonlinear one-dimensional waves. In particular, the mathematical modeling of medium-amplitude shallow water surface waves is based on the Korteweg-de Vries (KdV) equation. This equation has been demonstrated to define the asymptotic development of small but finite amplitude shallow-water waves, hydromagnetic waves in a cold plasma, ion-acoustic waves and acoustic waves in an anharmonic crystal. Here, is a complex function according to the and arguments that ensure the Eq. (1.1) [7, 38]. The Korteweg-de Vries (KdV) equation is a type of nonlinear equation that arises in the theory of hydrodynamic waves. This equation can be fully integrated with the help of the Hamilton system. The article was designed as follows. A new version of the trial equation method is presented in Section 2 for partial differential equations. In the next section, we applied the developed method to the Korteweg-de Vries (KdV) equation and the results are explained in Section 3.

2 Description of the new version of the trial equation method

In this part, the main line of the new version of the trial equation method is offered. With this new developed method, distinct and new results are accomplished by the results acquired with the trial equation method. We suppose general partial differential equation of the type

$$\Psi\left(w,\, w_x,\, w_y, w_z ...,\, w_t, ...,\, w_{xx},\, w_{xy}, w_{xz},\, ..., w_{xt}, ...\right) = 0, \tag{2.1}$$

where w is an unbeknown function and $x, y, z, \ldots, t$ are independent variables. When we apply the following traveling wave transformation to Eq. (2.1)

$$w\left(x, y,\, z, \ldots, t\right) = w\left(\tau\right), \quad \tau = e_1 x + e_2 y + e_3 z + ... + e_m t \tag{2.2}$$

which $e_j (j = 1, 2, 3, ..., m)$ are constants, we degrade Eq. (2.1) to nonlinear differential equation

$$\Omega\left(u, u', u'', u''', ...\right) = 0. \tag{2.3}$$

Let take into account the solution of the Eq.(2.3) in the following form

$$u\left(\tau\right) = A_0 + \sum_{i=1}^{M} A_i P^i\left(\tau\right) + B_i P^{-i}\left(\tau\right), \tag{2.4}$$

where

$$\left(P'\right)^2(\tau) = h_0 + h_1 P(\tau) + h_2 P^2(\tau) + \cdots + h_N P^N(\tau), \tag{2.5}$$

and $A_0, A_i, B_i\ (i = 1, \cdots, M)$ are constants to be fixed coefficients later. By using the solution function (2.4), relevant derivatives in the Eq.(2.3) are obtained

$$\begin{aligned}(u'(\tau))^2 &= P'(\tau)^2\left(\sum_{i=1}^{M} iA_i P^{i-1}(\tau) - iB_i P^{-i-1}(\tau)\right)^2 \\ &= \Lambda(\tau)\left(\sum_{i=1}^{M} iA_i P^{i-1}(\tau) - iB_i P^{-i-1}(\tau)\right)^2\end{aligned} \tag{2.6}$$

$$\begin{aligned}u''(\tau) = &\frac{\Lambda'(\tau)}{2}\sum_{i=1}^{M} iA_i P^{i-1}(\tau) - iB_i P^{-i-1}(\tau) \\ &+ \Lambda(\tau)\left[i(i-1)A_i P^{i-2}(\tau) + i(i+1)B_i P^{-i-2}(\tau)\right],\end{aligned} \tag{2.7}$$

where $\Lambda(\tau) = \left(P'\right)^2(\tau)$. Here, the P function is attained from the solution of the Eq. (2.5). The number is identified in Eq. (2.4) with the balancing process. The balancing number is a positive integer which is identified by balancing between the highest order derivative terms with the highest power nonlinear terms in Eq. (2.3). In accordance with the determined solution function (2.4), the relevant derivatives are gotten and substituted in Eq. (2.3), and then the polynomial equation is attained related to the function. The coefficients in this polynomial equation are balanced to zero and a system of algebraic equations is created. When the created algebraic equation system is resolved with the help of the Mathematica program, the $e_j (j = 1, 2, 3, ..., m)$ and $A_0, A_i, B_i\ (i = 1, \cdots, M)$ coefficients are attained which are necessary for the solution function. The attained coefficients is written instead of Eq. (2.5) and the function is attained by integrating according to the number . Then, the computed P function is putted in place Eq. (2.4). Thus, the new combined solitary wave solutions and combined Jacobi elliptic functions are reached.

3 Application to nonlinear Korteweg-de Vries (KdV) equation

In this part, a new version of the trial equation method is applied to the nonlinear KdV equation. To implement the new version trial equation method to the Eq. (1.1), let's first consider the traveling wave transformation

$$w(x,t) = w(\tau),\ \tau = k(x - ct), \tag{3.1}$$

where k and c are constants. By using Eq. (3.1), Eq. (1.1) reduces a nonlinear ordinary differential equation

$$-kcw(\tau) - 3kw^2(\tau) + k^3 w''(\tau) = 0 \tag{3.2}$$

The balancing procedure is made between the highest order derivative w'' term and the nonlinear highest degree w^2 term in Eq. (3.2) as follows

$$P'(\xi) = \sqrt{h_0 + h_1 P(\xi) + h_2 P^2(\xi) + \cdots + h_N P^N(\xi)} \cong \pm\sqrt{h_N} P(\xi)^{N/2} + ... \tag{3.3}$$

$$u^3 = A_M^3 P(\xi)^{3M} + ...\ u''(\xi) \cong M(M+1)A_M P(\xi)^{M+2} + ... \tag{3.4}$$

where $N = 4$ By the balance term is found as $M = 2$ from the equivalence of the terms $w'' \approx w^2$. When this attained balance term is substituted in Eq. (2.4), it is attained as the most general case of the new solution function of the Eq. (1.1)

$$w(\tau) = A_0 + A_1 P(\tau) + A_2 P^2(\tau) + \frac{B_1}{P(\tau)} + \frac{B_2}{P^2(\tau)}. \tag{3.5}$$

After calculating the related terms according to the solution function (3.5), it is rewritten in Eq. (3.2) and a polynomial equation relevant to the P function is found. If we take into account the polynomial here as a zero polynomial, a system of algebraic equations is achieved. When this system is resolved with the Mathematica program, A_0, A_1, A_2, $B_1, B_2, h_0, h_1, h_2, h_3, h_4$ and k, c coefficients were obtained. These coefficients are put in Eq. (3.5) and Eq. (2.5). When the coefficients are replaced in Eq. (2.5), P functions are attained by calculating the integral

$$\pm(\tau - \tau_0) = \int \frac{dP}{\sqrt{h_0 + h_1 P(\tau) + h_2 P^2(\tau) + h_3 P^3(\tau) + h_4 P^4(\tau)}}. \tag{3.6}$$

By putting in place these P functions in the Eq. (3.5), the solutions of the Eq. (1.1) in the following situations are attained.

Case 1

$$A_0 = A_0,\ \ A_1 = \frac{2k^2 h_3}{3},\ \ A_2 = \frac{4k^2 h_4}{3},\ \ B_1 = B_2 = 0,\ \ c = \frac{k^2 h_3^3 - 48 A_0 h_3 h_4 + 48 k^2 h_1 h_4^2}{8 h_3 h_4},$$

$$\begin{aligned} h_0 &= \frac{-72 A_0^2 h_3 h_4 - 4k^4 h_1 h_3^2 h_4 + 3k^2 A_0 h_3^3 + 144 k^2 A_0 h_1 h_4^2}{64 k^4 h_3 h_4^2}, \\ h_2 &= \frac{h_3^3 + 8 h_1 h_4^2}{4 h_3 h_4},\ \ h_1 = h_1,\ \ h_3 = h_3,\ \ h_4 = h_4. \end{aligned} \tag{3.7}$$

Considering the attained coefficients, the solution function (3.5) is as follows

$$w_1(\tau_1) = A_0 + A_1 P(\tau_1) + A_2 P^2(\tau_1). \tag{3.8}$$

Additionally, when other obtained coefficients are in its place Eq. (2.5), integral is attained

$$\pm(\tau - \tau_0) = \int \frac{dP}{\sqrt{\frac{h_0}{h_4} + \frac{h_1}{h_4} P(\tau) + \frac{h_2}{h_4} P^2(\tau) + \frac{h_3}{h_4} P^3(\tau) + P^4(\tau)}}, \tag{3.9}$$

where $\alpha_1, \alpha_2, \alpha_3$ and α_4 are the roots of the polynomial equation $P^4 + \frac{h_3}{h_4} P^3 + \frac{h_2}{h_4} P^2 + \frac{h_1}{h_4} P + \frac{h_0}{h_4} = 0$. Integrating Eq. (3.9), we achieve P functions as follows:

$$\pm(\tau - \tau_0) = -\frac{1}{P - \alpha_1}, \tag{3.10}$$

$$\pm(\tau - \tau_0) = \frac{2}{\alpha_1 - \alpha_2} \sqrt{\frac{P - \alpha_2}{P - \alpha_1}},\ \ \alpha_2 > \alpha_1, \tag{3.11}$$

$$\pm(\tau - \tau_0) = \frac{1}{\alpha_1 - \alpha_2} \ln\left|\frac{P - \alpha_1}{P - \alpha_2}\right|,\ \ \alpha_1 > \alpha_2, \tag{3.12}$$

$$\pm(\tau-\tau_0)=\frac{2}{\sqrt{(\alpha_1-\alpha_2)(\alpha_1-\alpha_3)}}\ln\left|\frac{\sqrt{(P-\alpha_2)(\alpha_1-\alpha_3)}-\sqrt{(P-\alpha_3)(\alpha_1-\alpha_2)}}{\sqrt{(P-\alpha_2)(\alpha_1-\alpha_3)}+\sqrt{(P-\alpha_3)(\alpha_1-\alpha_2)}}\right|,$$
$$\alpha_1>\alpha_2>\alpha_3, \tag{3.13}$$

$$\pm(\tau-\tau_0)=\frac{2F(\varphi,l)}{\sqrt{(\alpha_1-\alpha_3)(\alpha_2-\alpha_4)}},\quad \alpha_1>\alpha_2>\alpha_3>\alpha_4, \tag{3.14}$$

where

$$F(\varphi,l)=\int_0^{\varphi}\frac{d\psi}{\sqrt{1-l^2\sin^2\psi}},\ \varphi=\arcsin\sqrt{\frac{(P-\alpha_1)(\alpha_2-\alpha_4)}{(P-\alpha_2)(\alpha_1-\alpha_4)}},\ l^2=\frac{(\alpha_2-\alpha_3)(\alpha_1-\alpha_4)}{(\alpha_1-\alpha_3)(\alpha_2-\alpha_4)}. \tag{3.15}$$

When P functions are taken from Eq. (3.10)-(3.14) and substituted in Eq. (3.8), we attain

$$w_{1,1}(x,t)=A_0+A_1\left(\alpha_1\pm\frac{1}{\tau_1-\tau_0}\right)+A_2\left(\alpha_1\pm\frac{1}{\tau_1-\tau_0}\right)^2, \tag{3.16}$$

$$w_{1,2}(x,t)=A_0+A_1\left(\alpha_1\pm\frac{4(\alpha_2-\alpha_1)}{4-(\alpha_1-\alpha_2)^2(\tau_1-\tau_0)^2}\right)+A_2\left(\alpha_1\pm\frac{4(\alpha_2-\alpha_1)}{4-(\alpha_1-\alpha_2)^2(\tau_1-\tau_0)^2}\right)^2, \tag{3.17}$$

$$w_{1,3}(x,t)=A_0+A_1\left(\frac{\alpha_2 e^{\pm(\alpha_1-\alpha_2)(\tau_1-\tau_0)}-\alpha_1}{e^{\pm(\alpha_1-\alpha_2)(\tau_1-\tau_0)}-1}\right)+A_2\left(\frac{\alpha_2 e^{\pm(\alpha_1-\alpha_2)(\tau_1-\tau_0)}-\alpha_1}{e^{\pm(\alpha_1-\alpha_2)(\tau_1-\tau_0)}-1}\right)^2, \tag{3.18}$$

$$\begin{aligned}w_{1,4}(x,t)=A_0&+A_1\left(\alpha_1-\frac{2(\alpha_1-\alpha_2)(\alpha_1-\alpha_3)}{2\alpha_1-\alpha_2-\alpha_3+(\alpha_3-\alpha_2)\cosh\left[\sqrt{(\alpha_1-\alpha_2)(\alpha_1-\alpha_3)}(\tau_1-\tau_0)\right]}\right)\\&+A_2\left(\alpha_1-\frac{2(\alpha_1-\alpha_2)(\alpha_1-\alpha_3)}{2\alpha_1-\alpha_2-\alpha_3+(\alpha_3-\alpha_2)\cosh\left[\sqrt{(\alpha_1-\alpha_2)(\alpha_1-\alpha_3)}(\tau_1-\tau_0)\right]}\right)^2,\end{aligned} \tag{3.19}$$

$$\begin{aligned}w_{1,5}(x,t)=A_0&+A_1\left(\alpha_2+\frac{(\alpha_1-\alpha_2)(\alpha_4-\alpha_2)}{\alpha_4-\alpha_2+(\alpha_1-\alpha_4)\,sn^2\left[\frac{\sqrt{(\alpha_1-\alpha_3)(\alpha_2-\alpha_4)}}{2}(\tau_1-\tau_0),\frac{(\alpha_2-\alpha_3)(\alpha_1-\alpha_4)}{(\alpha_1-\alpha_3)(\alpha_2-\alpha_4)}\right]}\right)\\&+A_2\left(\alpha_2+\frac{(\alpha_1-\alpha_2)(\alpha_4-\alpha_2)}{\alpha_4-\alpha_2+(\alpha_1-\alpha_4)\,sn^2\left[\frac{\sqrt{(\alpha_1-\alpha_3)(\alpha_2-\alpha_4)}}{2}(\tau_1-\tau_0),\frac{(\alpha_2-\alpha_3)(\alpha_1-\alpha_4)}{(\alpha_1-\alpha_3)(\alpha_2-\alpha_4)}\right]}\right)^2,\end{aligned} \tag{3.20}$$

where $\tau_1 = k\left(x - \frac{k^2h_3^3 - 48A_0h_3h_4 + 48k^2h_1h_4^2}{8h_3h_4}t\right)$. If we take $A_0 = -A_1\alpha_1$, $A_1 = \frac{A_2}{2}$ and $\tau_0 = 0$, the equations (3.16)-(3.19) are attained respectively rational function solutions

$$w_{1,1}(x,t) = A_2\left[\pm\frac{1}{2k(x-ct)} + \left(\alpha_1 \pm \frac{1}{k(x-ct)}\right)^2\right], \tag{3.21}$$

$$w_{1,2}(x,t) = A_2\left[\pm\frac{4(\alpha_2-\alpha_1)}{8-2(\alpha_1-\alpha_2)^2k^2(x-ct)^2} + \left(\alpha_1 \pm \frac{4(\alpha_2-\alpha_1)}{4-(\alpha_1-\alpha_2)^2k^2(x-ct)^2}\right)^2\right], \tag{3.22}$$

travelling wave solution

$$w_{1,3}(x,t) = C_1\coth\left[D_3(x-ct)\right] + C_4\coth^2\left[D_3(x-ct)\right], \tag{3.23}$$

where $C_4 = \pm A_2\alpha_2^2$ and $D_3 = \frac{k(\alpha_1-\alpha_2)}{2}$ soliton solution

$$w_{1,4}(x,t) = \frac{A_1C_2}{E_1+\cosh\left[D_4(x-ct)\right]} + A_2\left(\alpha_1 + \frac{C_2}{E_1+\cosh\left[D_4(x-ct)\right]}\right)^2, \tag{3.24}$$

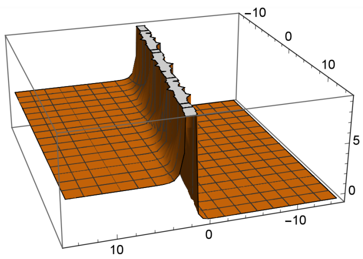
where $C_2 = \frac{-2(\alpha_1-\alpha_2)(\alpha_1-\alpha_3)}{\alpha_3-\alpha_2}$, $E_1 = \frac{2\alpha_1-\alpha_2-\alpha_3}{\alpha_3-\alpha_2}$ and $D_4 = k\sqrt{(\alpha_1-\alpha_2)(\alpha_1-\alpha_3)}$. Also, if we take $A_0 = -A_1\alpha_2$ in Eq. (3.20), a Jacobi elliptic function solution is obtained as

$$w_{1,5}(x,t) = \frac{C_3}{E_2+sn^2(\varphi_3,l_1)} + A_2\left(\alpha_2 + \frac{C_6}{E_2+sn^2(\varphi_3,l_1)}\right)^2, \tag{3.25}$$

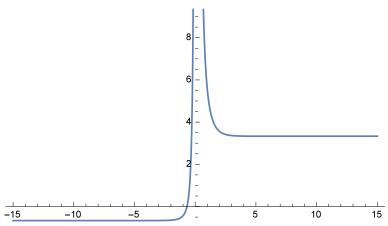
where

$$C_3 = \frac{A_1(\alpha_1-\alpha_2)(\alpha_4-\alpha_2)}{\alpha_1-\alpha_4},\ C_6 = \frac{(\alpha_1-\alpha_2)(\alpha_4-\alpha_2)}{\alpha_1-\alpha_4},\ E_2 = \frac{\alpha_4-\alpha_2}{\alpha_1-\alpha_4}$$

$$\varphi_3 = \frac{k\sqrt{(\alpha_1-\alpha_3)(\alpha_2-\alpha_4)}}{2}(x-ct),\ l_1^2 = \frac{(\alpha_2-\alpha_3)(\alpha_1-\alpha_4)}{(\alpha_1-\alpha_3)(\alpha_2-\alpha_4)}.$$

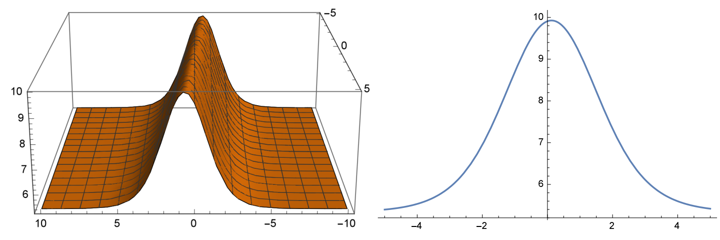


(a)



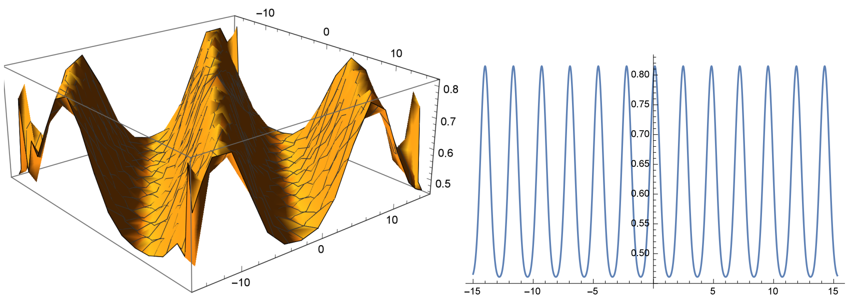
(b)

Figure 1: 3D and 2D graphics representations of the solution $w_{1,3}(x,t)$ with $\alpha_2 = D_3 = 1,\ \alpha_1 = k = C_1 = 2,\ C_4 = A_2 = \frac{4}{3},\ c = \frac{1}{8}$



(a) (b)

Figure 2: 3D and 2D graphics representations of the solution $w_{1,4}(x,t)$ with $\alpha_2 = 1,\ \alpha_1 = k = E_1 = 2,\ C_2 = -1,\ D_4 = \frac{2}{\sqrt{3}},\ c = \frac{1}{8}, A_1 = \frac{2}{3},\ A_2 = \frac{4}{3}$



(a) (b)

Figure 3: 3D and 2D graphics representations of the solution $w_{1,5}(x,t)$ with $\alpha_2 = 1,\ \alpha_1 = k = C_6 = 2,\ C_3 = -1,\ E_2 = \frac{-3}{2},\ c = \frac{1}{8}, \alpha_3 = \frac{5}{3},\ \alpha_4 = 4,\ A_1 = \frac{2}{3}$

Case 2:

$$A_0 = \frac{131\left(\sqrt{24492985}-2585\right)k^2h_3^2 - 5k^2h_3^2\sqrt{262\left(36077\sqrt{24492985}-246508345\right)}}{3294912h_4},$$

$$A_1 = \frac{2k^2h_3}{3},\ \ A_2 = \frac{4k^2h_4}{3},\ \ B_1 = -\frac{55k^2h_3^3\left(\sqrt{24492985}-3109\right)}{1647456h_4^2},$$

$$B_2 = \frac{55k^2h_3^4\left(4157\sqrt{24492985}-20337665\right)}{3453067776h_4^3},\ \ c = \frac{5k^2h_3^3\sqrt{36077\sqrt{24492985}-246568345}}{2096\sqrt{262}h_4},$$

$$h_0 = h_0,\ h_1 = h_1,\ h_2 = h_2,\ h_3 = h_3,\ h_4 = h_4. \tag{3.26}$$

By the obtained coefficients the solution function (3.5) is

$$w_2(\tau_2) = A_0 + A_1 P(\tau_2) + A_2 P^2(\tau_2) + \frac{B_1}{P(\tau_2)} + \frac{B_2}{P^2(\tau_2)}. \tag{3.27}$$

Also, when other attained coefficients are replaced in Eq. (2.5), integral is attained as follows

$$\pm(\tau-\tau_0) = \int \frac{dP}{\sqrt{\frac{h_0}{h_4} + \frac{h_1}{h_4}P(\tau) + \frac{h_2}{h_4}P^2(\tau) + \frac{h_3}{h_4}P^3(\tau) + P^4(\tau)}}, \tag{3.28}$$

where $\alpha_1, \alpha_2, \alpha_3$ and α_4 are the roots of the polynomial equation $P^4 + \frac{h_3}{h_4}P^3 + \frac{h_2}{h_4}P^2 + \frac{h_1}{h_4}P + \frac{h_0}{h_4} = 0$. When Eq. (3.28) is integrated, the expressions like in Eq. (3.10)-(3.14) are attained. When P functions are withdrawn from here and replaced in Eq. (3.27), we obtain

$$P^4 + \frac{h_3}{h_4}P^3 + \frac{h_2}{h_4}P^2 + \frac{h_1}{h_4}P + \frac{h_0}{h_4} = 0. \tag{3.29}$$

$$\begin{aligned} w_{2,2}(x,t) = {} & A_0 + A_1\left(\alpha_1 \pm \frac{4(\alpha_2-\alpha_1)}{4-(\alpha_1-\alpha_2)^2(\tau_2-\tau_0)^2}\right) + A_2\left(\alpha_1 \pm \frac{4(\alpha_2-\alpha_1)}{4-(\alpha_1-\alpha_2)^2(\tau_2-\tau_0)^2}\right)^2 \\ & + \frac{B_1}{\alpha_1 \pm \frac{4(\alpha_2-\alpha_1)}{4-(\alpha_1-\alpha_2)^2(\tau_2-\tau_0)^2}} + \frac{B_2}{\left(\alpha_1 \pm \frac{4(\alpha_2-\alpha_1)}{4-(\alpha_1-\alpha_2)^2(\tau_2-\tau_0)^2}\right)^2}, \end{aligned} \tag{3.30}$$

$$\begin{aligned} w_{2,3}(x,t) = {} & A_0 + A_1\left(\frac{\alpha_2 e^{\pm(\alpha_1-\alpha_2)(\tau_2-\tau_0)} - \alpha_1}{e^{\pm(\alpha_1-\alpha_2)(\tau_2-\tau_0)} - 1}\right) + A_2\left(\frac{\alpha_2 e^{\pm(\alpha_1-\alpha_2)(\tau_2-\tau_0)} - \alpha_1}{e^{\pm(\alpha_1-\alpha_2)(\tau_2-\tau_0)} - 1}\right)^2 \\ & + \frac{B_1}{\frac{\alpha_2 e^{\pm(\alpha_1-\alpha_2)(\tau_2-\tau_0)} - \alpha_1}{e^{\pm(\alpha_1-\alpha_2)(\tau_2-\tau_0)} - 1}} + \frac{B_2}{\left(\frac{\alpha_2 e^{\pm(\alpha_1-\alpha_2)(\tau_2-\tau_0)} - \alpha_1}{e^{\pm(\alpha_1-\alpha_2)(\tau_2-\tau_0)} - 1}\right)^2}, \end{aligned} \tag{3.31}$$

$$w_{2,4}(x,t) = A_0 + A_1\left(\alpha_1 - \frac{2(\alpha_1-\alpha_2)(\alpha_1-\alpha_3)}{2\alpha_1-\alpha_2-\alpha_3+(\alpha_3-\alpha_2)\cosh\left[\sqrt{(\alpha_1-\alpha_2)(\alpha_1-\alpha_3)}(\tau_2-\tau_0)\right]}\right)$$
$$+ A_2\left(\alpha_1 - \frac{2(\alpha_1-\alpha_2)(\alpha_1-\alpha_3)}{2\alpha_1-\alpha_2-\alpha_3+(\alpha_3-\alpha_2)\cosh\left[\sqrt{(\alpha_1-\alpha_2)(\alpha_1-\alpha_3)}(\tau_2-\tau_0)\right]}\right)^2$$
$$+ \frac{B_1\left(2\alpha_1-\alpha_2-\alpha_3+(\alpha_3-\alpha_2)\cosh\left[\sqrt{(\alpha_1-\alpha_2)(\alpha_1-\alpha_3)}(\tau_2-\tau_0)\right]\right)}{\alpha_1\alpha_2+\alpha_1\alpha_3-2\alpha_2\alpha_3+\alpha_1(\alpha_3-\alpha_2)\cosh\left[\sqrt{(\alpha_1-\alpha_2)(\alpha_1-\alpha_3)}(\tau_2-\tau_0)\right]}$$
$$+ B_2\left(\frac{2\alpha_1-\alpha_2-\alpha_3+(\alpha_3-\alpha_2)\cosh\left[\sqrt{(\alpha_1-\alpha_2)(\alpha_1-\alpha_3)}(\tau_2-\tau_0)\right]}{\alpha_1\alpha_2+\alpha_1\alpha_3-2\alpha_2\alpha_3+\alpha_1(\alpha_3-\alpha_2)\cosh\left[\sqrt{(\alpha_1-\alpha_2)(\alpha_1-\alpha_3)}(\tau_2-\tau_0)\right]}\right)^2, \tag{3.32}$$

$$w_{2,5}(x,t) = A_0 + A_1\left(\alpha_2 + \frac{(\alpha_1-\alpha_2)(\alpha_4-\alpha_2)}{\alpha_4-\alpha_2+(\alpha_1-\alpha_4)sn^2\left[\frac{\sqrt{(\alpha_1-\alpha_3)(\alpha_2-\alpha_4)}}{2}(\tau_2-\tau_0), \frac{(\alpha_2-\alpha_3)(\alpha_1-\alpha_4)}{(\alpha_1-\alpha_3)(\alpha_2-\alpha_4)}\right]}\right)$$
$$+ A_2\left(\alpha_2 + \frac{(\alpha_1-\alpha_2)(\alpha_4-\alpha_2)}{\alpha_4-\alpha_2+(\alpha_1-\alpha_4)sn^2\left[\frac{\sqrt{(\alpha_1-\alpha_3)(\alpha_2-\alpha_4)}}{2}(\tau_2-\tau_0), \frac{(\alpha_2-\alpha_3)(\alpha_1-\alpha_4)}{(\alpha_1-\alpha_3)(\alpha_2-\alpha_4)}\right]}\right)^2$$
$$+ \frac{B_1}{\alpha_2 + \frac{(\alpha_1-\alpha_2)(\alpha_4-\alpha_2)}{\alpha_4-\alpha_2+(\alpha_1-\alpha_4)sn^2\left[\frac{\sqrt{(\alpha_1-\alpha_3)(\alpha_2-\alpha_4)}}{2}(\tau_2-\tau_0), \frac{(\alpha_2-\alpha_3)(\alpha_1-\alpha_4)}{(\alpha_1-\alpha_3)(\alpha_2-\alpha_4)}\right]}}$$
$$+ \frac{B_2}{\left(\alpha_2 + \frac{(\alpha_1-\alpha_2)(\alpha_4-\alpha_2)}{\alpha_4-\alpha_2+(\alpha_1-\alpha_4)sn^2\left[\frac{\sqrt{(\alpha_1-\alpha_3)(\alpha_2-\alpha_4)}}{2}(\tau_2-\tau_0), \frac{(\alpha_2-\alpha_3)(\alpha_1-\alpha_4)}{(\alpha_1-\alpha_3)(\alpha_2-\alpha_4)}\right]}\right)^2}, \tag{3.33}$$

where $\tau_2 = k\left(x - \frac{5k^2h_3^3\sqrt{36077\sqrt{24492985}-246568345}}{2096\sqrt{262}h_4}t\right)$. If we take $A_0 = -A_1\alpha_1$ and $\tau_0 = 0$ the equations (3.29) (3.30) and (3.31) are obtained, respectively rational function solutions

$$w_{2,1}(x,t) = \alpha_1 \pm \frac{1}{k(x-ct)} + A_2\left(\alpha_1 \pm \frac{1}{k(x-ct)}\right)^2 + \frac{B_1}{\alpha_1 \pm \frac{1}{k(x-ct)}} + \frac{B_2}{\left(\alpha_1 \pm \frac{1}{k(x-ct)}\right)^2}, \tag{3.34}$$

$$w_{2,2}(x,t) = \pm\frac{4(\alpha_2-\alpha_1)}{4-(\alpha_1-\alpha_2)^2k^2(x-ct)^2} + A_2\left(\alpha_1 \pm \frac{4(\alpha_2-\alpha_1)}{4-(\alpha_1-\alpha_2)^2k^2(x-ct)^2}\right)^2$$
$$+ \frac{B_1}{\alpha_1 \pm \frac{4(\alpha_2-\alpha_1)}{4-(\alpha_1-\alpha_2)^2k^2(x-ct)^2}} + \frac{B_2}{\left(\alpha_1 \pm \frac{4(\alpha_2-\alpha_1)}{4-(\alpha_1-\alpha_2)^2k^2(x-ct)^2}\right)^2}, \tag{3.35}$$

combined soliton solution

$$w_{2,3}(x,t) = C_1 \coth\left[D_3(x-ct)\right] + C_4 \coth^2\left[D_3(x-ct)\right] + C_7 \tanh\left[D_3(x-ct)\right] + C_8 \tanh^2\left[D_3(x-ct)\right], \tag{3.36}$$

where $C_7 = \frac{B_1}{\alpha_2}, C_8 = \frac{B_2}{\alpha_2^2}$ and combined soliton solution

$$w_{2,4}(x,t) = \frac{A_1 C_2}{E_1 + \cosh\left[D_4(x-ct)\right]} + A_2\left(\alpha_1 + \frac{C_2}{E_1 + \cosh\left[D_4(x-ct)\right]}\right)^2 + B_3\left(\frac{E_1 + \cosh\left[D_4(x-ct)\right]}{E_3 + \cosh\left[D_4(x-ct)\right]}\right) + B_4\left(\frac{E_1 + \cosh\left[D_4(x-ct)\right]}{E_3 + \cosh\left[D_4(x-ct)\right]}\right)^2, \tag{3.37}$$

where $B_3 = \alpha_1 B_1,\ B_4 = \alpha_1 B_2,\ E_3 = \frac{\alpha_1\alpha_2 + \alpha_1\alpha_3 - 2\alpha_2\alpha_3}{\alpha_1(\alpha_3 - \alpha_2)}$. If $A_0 = -A_1\alpha_2$ is taken in Eq. (3.33), a combined Jacobi elliptic function solution is found as

$$w_{2,5}(x,t) = \frac{C_3}{E_2 + sn^2(\varphi_3, l_1)} + A_2\left(\alpha_2 + \frac{C_6}{E_2 + sn^2(\varphi_3, l_1)}\right)^2 + B_4 \frac{E_2 + sn^2(\varphi_3, l_1)}{E_4 + sn^2(\varphi_3, l_1)} + B_5\left(\frac{E_2 + sn^2(\varphi_3, l_1)}{E_4 + sn^2(\varphi_3, l_1)}\right)^2, \tag{3.38}$$

where $B_4 = \alpha_2 B_1, \quad B_5 = \alpha_2^2 B_2, \quad E_4 = \frac{\alpha_1(\alpha_4 - \alpha_2)}{\alpha_2(\alpha_1 - \alpha_4)}$.

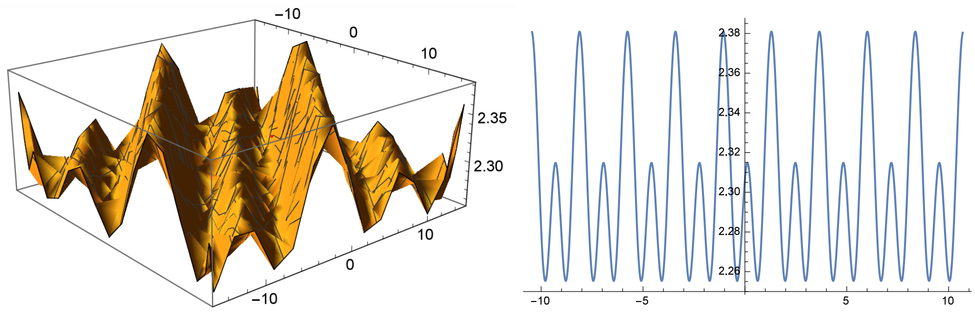


Figure 4: 3D and 2D graphics representations of the solution $w_{2,5}(x,t)$ with $\alpha_2 = B_1 = B_2 = 1,\ \alpha_4 = 4, A_2 = \frac{4}{3}, C_3 = -1,\ E_2 = \frac{-3}{2},\ E_4 = -3, c = \frac{1}{8}, \alpha_1 = k = C_6 = B_4 = B_5 = 2$

When all obtained exact solutions for the Korteweg-de Vries (KdV) equation are analyzed, new solitary wave solutions are not located in the literature. And it can be said that they are new solitary wave solutions attained by the new version trial equation method. According to the appropriate coefficients two and three-dimensional graphics of the attained solution functions are shown in Figure (1)-(4).

4 Conclusion

In this study, the new version of the trial equation method has been successfully implemented to get the new solitary wave solutions of the Korteweg-de Vries (KdV) equation. This method makes it possible to find new closed-form optical solutions. These solitary wave solutions include dark soliton, bright soliton, singular soliton, Jacobi elliptic function solutions and combined solitary wave solutions. The examination of the specifications of the new solitary wave solutions found in this paper is assumed to play a major role in the nonlinear physical phenomena modeled by the Korteweg-de Vries (KdV) equation.

5 Declarations

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The author(s) declare that they have no competing interests.

Ethical Approval

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Authors' Contributions

All authors contributed equally. All the authors read and approved the final manuscript.

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References

[1] M. A. Abdou, *Further improved f-expansion and new exact solutions for nonlinear evolution equations*, Nonlinear Dyn. **52** (2008), 277–288.

[2] M. A. Akbar, N. H. M. Ali, and S. T. Mohyud-Din, *The modified alternative (g'/g)-expansion method to nonlinear evolution equation: application to the (1+1)dimensional drinfel'd-sokolov-wilson equation*, SpringerPlus **327** (2013), 2–16.

[3] A. Bekir and O. Unsal, *Analytic treatment of nonlinear evolution equations using first integral method*, Pramana-J. Phys. **79** (2012), no. 1, 3–17.

[4] S. T. Demiray, Y. Pandir, and H. Bulut, *New solitary wave solutions of maccari system*, Ocean Eng. **103** (2015), 153–159.

[5] S. T. Demiray, Y. Pandir, and H. Bulut, *New soliton solutions for sasa-satsuma equation*, Waves Random Complex Media **25** (2015), no. 3, 417–418.

[6] M. M. El-Borai, H. M. El-Owaidy, H. M. Ahmed, A. H. Arnous, S. Moshokoa, A. Biswas, and M. Belic, *Topological and singular soliton solution to kundu-eckhaus equation with extended kudryashov's method*, Optik **128** (2017), 57–62.

[7] C. S. Gardner, J. M. Greene, M. D. Kruskal, and R. M. Muira, *Method for solving the korteweg-devries equation*, Phys. Rev. Lett. **19** (1967), 1095–1097.

[8] C. A. Gómez, A. Jhangeer, H. Rezazadeh, R. A. Talarposhti, and A. Bekir, *Closed form solutions of the perturbed gerdjikov-ivanov equation with variable coefficients*, East Asian J. Appl. M **11** (2021), no. 1, 207–218.

[9] S. Guo and Y. Zhou, *The extended (g'/g)-expansion method and its applications to the whitham-broer-kaup-like equations and coupled hirota-satsuma kdv equations*, Appl. Math. Comput. **215** (2010), 3214–3221.

[10] Y. Gurefe, E. Misirli, A. Sonmezoglu, and M. Ekici, *Extended trial equation method to generalized nonlinear partial differential equations*, Appl. Math. Comput. **219** (2013), no. 10, 5253–5260.

[11] Y. Gurefe, A. Sonmezoglu, and E. Misirli, *Application of trial equation method to the nonlinear partial differential equations arising in mathematical physics*, Pramana-J. Phys. **77** (2011), no. 6, 1023–1029.

[12] Y. Gurefe, A. Sonmezoglu, and E. Misirli, *Application of an irrational trial equation method to high dimensional nonlinear evolution equations*, J. Adv. Math. Stud. **5** (2012), no. 1, 41–47.

[13] J. H. He and X. H. Wu, *Exp-function method for nonlinear wave equations*, Chaos, Soliton Fract. **30** (2006), 700–708.

[14] J. Hietarinta, *Hirota's bilinear method and its generalization*, Int. J. Mod. Phys. A **12** (1997), no. 1, 43–51.

[15] X. J. Laia, J. F. Zhang, and S. H. Meia, *Application of the weierstrass elliptic expansion method to the long-wave and short-wave resonance interaction system*, Z. Naturforsch. A **63** (2008), 273–279.

[16] C. S. Liu, *Trial equation method and its applications to nonlinear evolution equations*, Phys. Sinica **54** (2005), no. 6, 2505–2509.

[17] C. S. Liu, *Trial equation to solve the exact solutions for two kinds of kdv equations with variable coeffients*, Acta Phys. Sinica **54** (2005), 4506–4510.

[18] C. S. Liu, *A new trial equation method and its applications*, Commun. Theor. Phys. **45** (2006), no. 3, 395–397.

[19] C. S. Liu, *Trial equation method for nonlinear evolution equations with rank inhomogeneous: mathematical discussions and applications*, Commun. Theor. Phys. **45** (2006), no. 2, 219–223.

[20] C. S. Liu, *Applications of complete discrimination system for polynomial for classifications of traveling wave solutions to nonlinear differential equations*, Comput. Phys. Commun. **181** (2010), no. 2, 317–324.

[21] W. X. Ma and B. Fuchssteiner, *Explicit and exact solutions to a kolmogrov-petrovskipiskunov equation*, Int. J. Nonlinear Mech. **31** (1996), no. 3, 329–338.

[22] W. X. Ma, Y. Zhang, and Y. Tang, *Symbolic computation of lump solutions to a combined equation involving three types of nonlinear terms*, East Asian J. Appl. M **10** (2020), no. 4, 732–745.

[23] W. Malfliet and W. Hereman, *The tanh method: I exact solutions of nonlinear evolution and wave equations*, Phys. Scripta **54** (1996), 563–568.

[24] Y. Pandir, *Symmetric fibonacci function solutions of some nonlinear partial differential equations*, Appl. Math. Inf. Sci. **8** (2014), 2237–2241.

[25] Y. Pandir, *A new type of the generalized f-expansion method and its application to sine-gordon equation*, Celal Bayar Univ. J. Sci. **13** (2017), no. 3, 647–650.

[26] Y. Pandir, S. T. Demiray, and H. Bulut, *A new approach for some nldes with variable coefficients*, Optik **127** (2016), 11183–11190.

[27] Y. Pandir and A. Ekin, *Dynamics of combined soliton solutions of unstable nonlinear schrodinger equation with new version of the trial equation method*, Chinese J. Physics **67** (2020), 534–543.

[28] Y. Pandir, Y. Gurefe, U. Kadak, and E. Misirli, *Classifications of exact solutions for some nonlinear partial differential equations with generalized evolution*, Abstr. Appl. Anal. **2012** (2012), 1–16.

[29] Y. Pandir, Y. Gurefe, and E. Misirli, *Classification of exact solutions to the generalized kadomtsev-petviashvili equation*, Phys. Scr. **87** (2013), no. 2, 1–12.

[30] Y. Pandir, Y. Gurefe, and E. Misirli, *A multiple extended trial equation method for the fractional sharma-tasso-olver equation*, AIP Conf. Proc., vol. 1558, 2013, p. 1927.

[31] Y. Pandir, A. Sonmezoglu, H. H. Duzgun, and N. Turhan, *Exact solutions of nonlinear schrödinger's equation by using generalized kudryashov method*, AIP Conf. Proc., vol. 1648, 2015, p. 370004.

[32] Y. Pandir and N. Turhan, *A new version of the generalized f-expansion method and its applications*, AIP Conf. Proc., vol. 1798, 2017, p. 020122.

[33] L. K. Ravi, S. S. Ray, and S. Sahoo, *New exact solutions of coupled boussinesqburgers equations by exp-function method*, J. Ocean Eng. Sci **2** (2017), 34–46.

[34] M. Shakeel and S. T. Mohyud-Din, *New* (g'/g)*-expansion method and its application to the zakharov-kuznetsov-benjamin-bona-mahony (KK-BBM) equation*, J. Assoc. Arab Univ. Basic Appl. Sci. **18** (2015), no. 1, 66–81.

[35] S. Shikuo, F. Zuntao, L. Shida, and Z. Qiang, *Jacobi elliptic function expansion method and periodic wave solutions of nonlinear wave equations*, Phys. Lett. A **289** (2001), 69–74.

[36] Y. A. Tandogan, Y. Pandir, and Y. Gurefe, *Solutions of the nonlinear differential equations by use of modified kudryashov method*, Turkish J. Math. Comput. Sci. **1** (2013), 54–60.

[37] K. Yang and J. Liu, *The extended f-expansion method and exact solutions of nonlinear pdes*, Chaos, Solitons Fract. **22** (2014), 111–121.

[38] V. E. Zakharov and L. D. Faddeev, *Korteweg-de vries equation: A completely integrable hamiltonian system*, Funct. Anal. Appl. **5** (1971), 280–287.

[39] H. A. Zedan, *New approach for tanh and extended-tanh methods with applications on hirota-satsuma equations*, Comput. Appl. Math. **28** (2009), no. 1, 1–14.

[40] J. Zhang, F. Jiang, and X. Zhao, *An improved* (g'/g)*-expansion method for solving nonlinear evolution equations*, Int. J. Comput. Math. **87** (2010), no. 8, 1716–1725.

[41] Z. Y. Zhang, *Jacobi elliptic function expansion method for the modified korteweg-de vries-zakharov-kuznetsov and the*

hirota equations, Rom. J. Phys. **60** (2015), no. 9-10, 1384–1394.