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On Maximal Solution to a Degenerate Parabolic Equation Involving a Time Fractional Derivative

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Abstract

In this paper, we consider a degenerate parabolic equation associated with Caputo derivative. Our problem is studied in the unbounded domain and the nonlocal initial condition. Under some suitable conditions of the input data, we show the local existence of the mild solution. Then we show the continuation of the mild solution. We also claim the maximal mild solution. The main analysis in the current paper is based on some estimations of resolvent theory combined with many complex valuations on solutions operators in Banach spaces.

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1 Introduction

Today, fractional calculus refers to the study of so-called derivative operators and fractional operator analysis over real or complex domains and their applications. Over the past decades, fractional calculus began to be used as a powerful tool by many researchers working in several branches of science and engineering. In many fractional models, we notice two operators of interest, namely Caputo derivative (see [1, 2, 5, 7, 8, 10–17]) and Riemann-Liouville operator.

Let us denote $B := I - \Delta$ where I in the identity operator and Δ is the Laplace operator. In this paper, we consider the equation

$$\partial_t^\alpha Bu(x,t) - \Delta u(x,t) = Bf(u(x,t)) \quad \text{in } \mathbb{R}^3 \times (0,\infty), \tag{1.1}$$

equipped with the nonlocal initial condition

$$Bu(t)|_{t=0} = Pu_0, \quad x \in \mathbb{R}^3, \tag{1.2}$$

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where u_0 is the function defined later. The derivative ∂_t^α is the meaning of Caputo sense [1, 8]. When $\alpha = 1$, this above equation (1.1) is called degenerate parabolic equation which is mentioned in [4].

In [3], the authors studied a fractional integrodifferential equations of Sobolev type with nonlocal conditions

$$\partial_t^\alpha (Bu(t)) = Au(t) + Bf\left(t, u(t), \int_0^t \rho(t,s)h(t,s,u(s))ds\right), \ \ 0 < t < T \tag{1.3}$$

and

$$Bu(0) = B\left(u_0 - g(u)\right) \tag{1.4}$$

Here A and B are closed (unbounded) linear operators with domains contained in X. Using the theory of propagation family as well as the theory of the measures of noncompactness and the condensing maps, they obtained the existence result of mild solutions for problem (1.1)-(1.2).

In [6], the authors considered the existence and uniqueness of maxinal mild solutions for non-autonomous integro-differential equations having nonlocal condition. They applied a monotone iterative method with upper and lower solutions in an ordered Banach space X combined with evolution system and measure of noncompactness.

The maximal mild solution is defined in Definition (2.4). To the best of our knowledge, there are not any results on maximal mild solution to problem (1.3)-(1.4). Our paper is the first investigation on maximal mild solution. The difficulty of the paper occurs when we study the solution operators in the space $\mathbb{R}^3$. The principal analysis is to using some estimations of resolvent theory concerning Hankel contour.

This paper is organized as follows. In section 2, we introduce some premilinaries on the resolvent theory, some estimations of the operators. We also obtain the bound of the solution operators. Section 3 provide the property of behaviour of solution operators. The main results are described in section 4.

2 Preliminaries

In this section, we recall some basic settings of resolvent theory. Throughout this paper, we consider the Laplace operator Δ defined on $W_0^{1,p}(\mathbb{R}^3) \cap W^{2,p}(\mathbb{R}^3)$, $1 \le p < \infty$, by $\Delta = \sum_{j=1}^3 \partial^2_{x_j x_j}$.

2.1 Resolvent theory

Since the Laplace operator it is closed linear and densely, by taking $\theta \in (\frac{\pi}{2}, \pi)$ and denoting

$$\Sigma_\theta := \left\{ z \in \mathbb{C} : z \neq 0, |\arg(z)| < \theta \right\},$$

we have Σ_θ is a subset of $\rho(\Delta)$, the resolvent set of Δ. Moreover, it is well-known that there is a constant $C > 0$ such that

$$\left\| (\lambda - \Delta) f \right\|_{W^{2,p}(\mathbb{R}^3)} \le \frac{C}{|\lambda|}, \quad \forall \lambda \in \Sigma_\theta.$$

We say that $\Gamma_{r,\varphi}$ is a Hankel contour if exists $r > 0$ and $\varphi \in (\frac{\pi}{2}, \pi)$ such that $\Gamma = \Gamma_1 \cup \Gamma_2 \cup \Gamma_3$,

$$\begin{aligned}
\Gamma_1 &= \{ l e^{i\varphi} : l \in [r, \infty) \}, \\
\Gamma_2 &= \{ r e^{i\psi} : \psi \in [-\varphi, \varphi] \}, \\
\Gamma_3 &= \{ l e^{-i\varphi} : l \in [r, \infty) \}.
\end{aligned}$$

Along this contour, we define the operators

$$\begin{cases} H_1(\lambda) &= \lambda^{\alpha-1}(\lambda^\alpha B-\Delta)^{-1}B, \quad \lambda\in\Gamma,\\ H_2(\lambda) &= \quad (\lambda^\alpha B-\Delta)^{-1}B, \qquad \lambda\in\Gamma.\end{cases}$$

These can be estimated as the following lemma.

Lemma 2.1. *Given* $1\le p<\infty$. *Then, for all* $\lambda\in\Gamma$,

$$\|H_1(\lambda)\|_{L^p(\mathbb{R}^3)\to L^p(\mathbb{R}^3)}\le \frac{|\lambda^{-1}|(C+|\lambda^\alpha|)}{|\lambda^\alpha+1|},$$

and

$$\|H_2(\lambda)\|_{L^p(\mathbb{R}^3)\to L^p(\mathbb{R}^3)}\le \frac{C\,|\lambda^{-\alpha}|+|\lambda^\alpha|}{|\lambda^\alpha+1|}.$$

Proof. We begin with estimating the operator $H_1(\lambda)$. We firstly note that $\frac{\lambda^\alpha}{\lambda^\alpha+1}\in\rho(\Delta)$ for all $\lambda\in\Gamma_{\epsilon,\varphi}$. Indeed, on the path $\Gamma_{\epsilon,\varphi,1}$ we have $\lambda=re^{i\varphi}$ with $\epsilon\le r<\infty$. A direct calculation yields

$$\begin{aligned}\frac{\lambda^\alpha}{\lambda^\alpha+1}&=\frac{r^\alpha\cos(\alpha\varphi)+r^\alpha i\sin(\alpha\varphi)}{r^\alpha\cos(\alpha\varphi)+1+r^\alpha i\sin(\alpha\varphi)}\\&=\frac{r^{2\alpha}+1+r^\alpha i\sin(\alpha\varphi)}{r^\alpha\cos(\alpha\varphi)+1-r^{2\alpha}\sin^2(\alpha\varphi)}.\end{aligned}$$

Since $\varphi\in(\frac{\pi}{2},\pi)$ and $\alpha\in(0,1)$, it holds $\sin(\alpha\varphi)>0$. Hence, we imply that $\frac{\lambda^\alpha}{\lambda^\alpha+1}\notin(0,\infty)$. On the paths $\Gamma_{\epsilon,\varphi,2}$ and $\lambda\in\Gamma_{\epsilon,\varphi,3}$, by using the analogous way as above, we also have $\frac{\lambda^\alpha}{\lambda^\alpha+1}\notin(0,\infty)$. Therefore, it holds $\frac{\lambda^\alpha}{\lambda^\alpha+1}\in\rho(\Delta)$ for all $\lambda\in\Gamma_{\epsilon,\varphi}$. For $P=I+\Delta$, we have

$$H_1(\lambda)=\frac{\lambda^{\alpha-1}}{\lambda^\alpha+1}\left(\frac{\lambda^\alpha}{\lambda^\alpha+1}-\Delta\right)^{-1}(I-\Delta).$$

This helps to show that

$$\begin{aligned}\frac{\lambda^\alpha+1}{\lambda^{\alpha-1}}H_1(\lambda)&=\left(\frac{\lambda^\alpha}{\lambda^\alpha+1}-\Delta\right)^{-1}(I-\Delta)-(I-\Delta)^{-1}(I-\Delta)+I\\&=\left(\left(\frac{\lambda^\alpha}{\lambda^\alpha+1}-\Delta\right)^{-1}-(I-\Delta)^{-1}\right)(I-\Delta)+I.\end{aligned}$$

This implies that

$$\frac{\lambda^\alpha+1}{\lambda^{\alpha-1}}H_1(\lambda)=\frac{1}{\lambda^\alpha+1}\left(\frac{\lambda^\alpha}{\lambda^\alpha+1}-\Delta\right)^{-1}+I,$$

which deduces the following identity

$$H_1(\lambda)=\frac{\lambda^{\alpha-1}}{(\lambda^\alpha+1)^2}\left(\frac{\lambda^\alpha}{\lambda^\alpha+1}-\Delta\right)^{-1}+\frac{\lambda^{\alpha-1}}{\lambda^\alpha+1}I.$$

Therefore, we infer that

$$\|H_1(\lambda)\|_{B(L^p(\mathbb{R}^3))} \leq \left|\frac{\lambda^{\alpha-1}}{(\lambda^\alpha+1)^2}\frac{C_0}{\lambda^\alpha/(\lambda^\alpha+1)}\right| + \left|\frac{\lambda^{\alpha-1}}{\lambda^\alpha+1}\right|,$$

which directly implies the firstly inequality of this lemma.

Our next objective is to estimate the operator $H_2(\lambda)$. It is easily seen that

$$\begin{aligned} H_2(\lambda) &= \big(\lambda^\alpha + (\lambda^\alpha+1)A\big)^{-1}(I+A) \\ &= \frac{1}{\lambda^\alpha+1}\big(\frac{\lambda^\alpha}{\lambda^\alpha+1} - \Delta\big)^{-1}(I-\Delta). \end{aligned}$$

Multiplying the both sides of the above by $\lambda^\alpha+1$,we obtain

$$\begin{aligned} (\lambda^\alpha+1)H_2(\lambda) &= \big(\frac{\lambda^\alpha}{\lambda^\alpha+1} - \Delta\big)^{-1}(I-\Delta) \\ &= \big(\frac{\lambda^\alpha}{\lambda^\alpha+1} - \Delta\big)^{-1}(I-\Delta) - (I-\Delta)^{-1}(I-\Delta) + I. \end{aligned}$$

By an argument similar to the previous one. We get

$$\begin{aligned} (\lambda^\alpha+1)H_2(\lambda) &= \big(\frac{1}{\lambda^\alpha+1}\big)\big(\frac{\lambda^\alpha}{\lambda^\alpha+1} - \Delta\big)^{-1} + I \\ &= \big(\frac{1}{\lambda^\alpha+1}\big)\big(\frac{\lambda^\alpha}{\lambda^\alpha+1} - \Delta\big)^{-1} + I. \end{aligned}$$

Consequently, we find that

$$H_2(\lambda) = \frac{1}{(\lambda^\alpha+1)^2}\big(\frac{\lambda^\alpha}{\lambda^\alpha+1} - \Delta\big)^{-1} + \frac{1}{\lambda^\alpha+1}I.$$

According to the above proof $\frac{\lambda^\alpha}{\lambda^\alpha+1} \in \rho(\Delta)$,it follows that

$$\begin{aligned} \|H_2(\lambda)\|_{B(L^p(\mathbb{R}^3))} &\leq \Big|\frac{1}{(\lambda^\alpha+1)^2}\frac{C_0}{\frac{\lambda^\alpha}{\lambda^\alpha+1}} + \frac{1}{\lambda^\alpha+1}\Big| \\ &= \frac{C_0\,|\lambda^{-\alpha}\,|+1}{|(\lambda^\alpha+1)\,|}. \end{aligned}$$

From been proved, the secondly inequality is hold. □

2.2 Solution formulation

By the above lemma, we are able to define the following families of operators

$$S_j(t) = \frac{1}{2\pi i}\int_{\Gamma_{r,\theta}} e^{\lambda t} H_j(\lambda)d\lambda; \quad j = 1,2. \tag{2.5}$$

Now, taking the Laplace transform into (1.1) yields

$$\widehat{u}(\lambda) = \lambda^{\alpha-1}(\lambda^\alpha B - \Delta)^{-1}Bu_0 + (\lambda^\alpha B - \Delta)^{-1}B\widehat{f}(u(\lambda)), \tag{2.6}$$

where $\widehat{u}(\lambda)$ denotes the Laplace transform of $u(t)$. Thus, we have that

$$\widehat{u}(\lambda) = H_1(\lambda)u_0 + H_2(\lambda)\widehat{f}(u(\lambda)).$$

By means of the inverse Laplace transform, we derive

$$\begin{aligned} u(t) &= \mathcal{L}^{-1}(H_1(.))u_0 + \mathcal{L}^{-1}(H_2(.)\widehat{f}(.)) \\ &= S_1 u_0 + \int_0^t S_2(t-s) f(u(s)) ds. \end{aligned}$$

where

$$S_j(t) = \frac{1}{2\pi i} \int_{\Gamma,\theta} e^{\lambda t} H_j(\lambda) d\lambda; \quad j = 1,2 \tag{2.7}$$

$$\Gamma_{\epsilon,\varphi} \subset \rho(-A) = \rho(\Delta).$$

Definition 2.2. *A funtion $u \in C_\beta([0,T]; L^p(\mathbb{R}^3))$ for some $p \geq 1$, $0 < \beta < 1$ is said to be a mild solution of (1.1)-(1.2) on $[0,T]$ if it satisfies*

$$u(t) = S_1(t)u_0 + \int_0^t S_2(t-s) f(u(s)) ds,$$

for all $t \in [0,T]$.

Definition 2.3. *(Continuation of mild solution) Let u be a mild solution of Problem (1.1)-(1.2) on $[0,T]$. If there exists a time $\widetilde{T} \in (T,\infty)$, and a function $\widetilde{u} \in C_\beta([0,T]; L^p(\mathbb{R}^3)$ such that*

i) $\widetilde{u}|_{[0,T]} = u$,
ii) *$\widetilde{u}$ is a mild solution of Problem (1.1) -(1.2) on $[0,\widetilde{T}]$,*

then $\widetilde{u}$ is called a continuation of u.

Definition 2.4. *(Maximal solution) If a fuction u: $[0,T_*] \longrightarrow L^p(\mathbb{R}^3)$ satisfies that*

i) $u|_{[0,T]}$ is a $C_\beta L^p$-mild solution of Problem (1.1)-(1.2) on $[0,T_*]$,
ii) u has no continuation,

then it is called a maximal mild solution of Problem (1.1)-(1.2).

3 Behavior of solution operators

This section is to construct existence of mild solutions in the case $P = I + \Delta$. For this purpose, we firstly establish boundedness estimates of the solution operators $S_1(t)$, $S_2(t)$ in Subsection 3.1. Secondly, we wil estimates operators.

3.1 Auxiliary lemmas

In this part, we will present some auxiliary lemmas. In lemma 2.1, sharp estimates for the operators H_1, H_2 in the space of bounded linear operators on $L^p(\mathbb{R}^3)$.

Lemma 3.1. *The operators $S_1(t), S_2(t)$ defined in 2.7 have the following properties*
Let $\alpha \in (0,1)$ then there exists constants C such that

$$\|S_1(t)\|_{B(L^p(\mathbb{R}^3))} \leq C(t^\alpha + t^{-\alpha} + 1),$$

and

$$\|S_2(t)\|_{B(L^p(\mathbb{R}^3))} \leq C(t^{2\alpha-1} + t^{\alpha-1}).$$

Proof. We begin by estimating $S_1(t)$.It follows definition operator $S_1(t)$ in (2.7), we have following inequality

$$\|S_1(t)\|_{B(L^p(\mathbb{R}^3))} \leq \frac{1}{2\pi}\sum_{k=1}^{3}\|_{B(L^p(\mathbb{R}^3))}\int_{\Gamma,\theta,k} e^{\lambda t}H_1(\lambda)d\lambda\|$$
$$:= \frac{1}{2\pi}\sum_{k=1}^{3}\|N_k\|_{B(L^p(\mathbb{R}^3))},$$

where we denote

$$N_k = \int_{\Gamma,\theta,k} e^{\lambda t}H_1(\lambda)d\lambda, k = 1,2,3.$$

Hence, to estimate operator $S_1(t)$ we need estimates operator $N_i, i = 1,2,3$.
First, we estimate operator N_1. By using Lemma (2.1) and a trivial verification shows that

$$\|N_1\|_{B(L^p(\mathbb{R}^3))} = \|\int_{\Gamma,\theta,1} e^{\lambda t}H_1(\lambda)d\lambda\|_{B(L^p(\mathbb{R}^3))}$$
$$\leq \int_{\Gamma,\theta,1} |e^{\lambda t}|\frac{|\lambda^{-1}|(C_0 + |\lambda^{\alpha}|)}{|\lambda^{\alpha}+1)|}|d\lambda|.$$

By the change of variables $\lambda = re^{i\varphi}$,we have

$$\|N_1\|_{B(L^p(\mathbb{R}^3))} \leq \int_{\frac{1}{t}}^{\infty} |e^{tre^{i\varphi}}|\frac{|r^{-1}e^{-i\varphi}|(C_0 + |r^{\alpha}e^{i\varphi\alpha}|)}{|r^{\alpha}e^{i\varphi\alpha}+1|}|e^{i\varphi}|dr.$$

Since

$$|e^{i\varphi}| = 1,\ |e^{tre^{i\varphi}}| = |e^{tr(\cos\varphi+i\sin\varphi)}| = |e^{tr\cos\varphi}||e^{i\sin\varphi}| = |e^{tr\cos\varphi}|$$

and

$$|r^{\alpha}e^{i\varphi\alpha}+1| = \sqrt{((r^{\alpha}\cos\alpha\varphi+1)^2 + (r^{\alpha}\sin\alpha\varphi)^2} \geq r^{\alpha}\sin\alpha\varphi,$$

which impllies that

$$\|N_1\|_{B(L^p(\mathbb{R}^3))} \leq \int_{\frac{1}{t}}^{\infty} e^{tr\cos\varphi}\frac{C_0r^{-\alpha-1}+r^{-1}}{\sin\alpha\varphi}dr,$$

Thank to $e^{-x} \leq C.x^{-\gamma}, \gamma > 0$ we have

$$\|N_1\|_{B(L^p(\mathbb{R}^3))} \leq \frac{C_0}{\sin\alpha\varphi}\int_{\frac{1}{t}}^{\infty} r^{-\alpha-1}C(-tr\cos\varphi)^{-\gamma}dr$$
$$+ \frac{1}{\sin\alpha\varphi}\int_{\frac{1}{t}}^{\infty} Cr^{-1}(-tr\cos\varphi)^{-\gamma}dr,$$

where $\gamma > \alpha$.
By a direct calculation of above integral, we get that

$$\|N_1\|_{B(L^p(\mathbb{R}^3))} \leq \frac{C}{\sin\alpha\varphi(\cos\varphi)^{\gamma}}\Big(\frac{t^{\alpha}}{-\alpha-\gamma} + \frac{1}{-\gamma}\Big) \leq C(t^{\alpha}+1).$$

Next, we estimate operator N_2. By an argument similar to the previous one, we obtain

$$\begin{aligned}\|N_2\|_{B(L^p(\mathbb{R}^3))} &= \| \int_{\Gamma,\theta,2} e^{\lambda t} H_1(\lambda) d\lambda \|_{B(L^p(\mathbb{R}^3))} \\ &\leq \int_{\Gamma,\theta,2} |e^{\lambda t} | \frac{|\lambda^{-1} |(C_0 + |\lambda^\alpha |)}{|\lambda^\alpha + 1) |} |d\lambda |.\end{aligned}$$

By the substitution $\lambda = \epsilon e^{i\psi}$,and let $\epsilon = \frac{1}{t}$, we have

$$\begin{aligned}\|N_2\|_{B(L^p(\mathbb{R}^3))} &\leq \int_{-\varphi}^{\varphi} |e^{t\epsilon e^{i\psi}} | \frac{|\epsilon^{-1} e^{-i\psi} |(C_0 + |\epsilon^\alpha e^{i\psi\alpha} |)}{|\epsilon^\alpha e^{i\psi\alpha} + 1 |} |\epsilon i e^{i\psi} | d\psi \\ &\leq \int_{-\varphi}^{\varphi} e^{\cos\psi} \frac{(C_0 + t^{-\alpha})}{|t^{-\alpha} e^{i\psi\alpha} + 1 |} d\psi.\end{aligned}$$

Using the inequality

$$|t^{-\alpha} e^{i\psi\alpha} + 1 | = \sqrt{(t^{-\alpha}\cos\psi\alpha + 1)^2 + (t^{-\alpha}\sin\psi\alpha)^2} \geq \sin\alpha\psi,$$

we get

$$\begin{aligned}\|N_2\|_{B(L^p(\mathbb{R}^3))} &\leq \int_{-\varphi}^{\varphi} e^{\cos\psi} \frac{(C_0 + t^{-\alpha})}{\sin\alpha\psi} d\psi \\ &\leq (C_0 + t^{-\alpha}) \int_{-\varphi}^{\varphi} \frac{e^{\cos\psi}}{\sin\alpha\psi} d\psi \leq C(C_0 + t^{-\alpha}). \qquad (3.8)\end{aligned}$$

Analysis similar to that in the estimating of operator N_1 shows that

$$\|N_3\|_{B(L^p(\mathbb{R}^3))} \leq C(t^\alpha + 1).$$

Combibing previous estimations, we get that

$$\|S_1(t)\|_{B(L^p(\mathbb{R}^3))} \leq C(t^\alpha + t^{-\alpha} + 1).$$

We proceed now to estimate operator $S_2(t)$. We will estimate operator $S_2(t)$ by using again some techniques of previos step.Applying the triangle inequlity yields that

$$\begin{aligned}\|S_2(t)\|_{B(L^p(\mathbb{R}^3))} &\leq \frac{1}{2\pi} \sum_{k=1}^{3} \|_{B(L^p(\mathbb{R}^3))} \int_{\Gamma,\theta,k} e^{\lambda t} H_2(\lambda) d\lambda \| \\ &= \frac{1}{2\pi} \sum_{i=1}^{3} \|M_i\|_{B(L^p(\mathbb{R}^3))},\end{aligned}$$

where

$$M_i = \int_{\Gamma,\theta,i} e^{\lambda t} H_2(\lambda) d\lambda; i = 1,2,3.$$

Similarly as estimating N_1,one can show that

$$\begin{aligned}\|M_1\|_{B(L^p(\mathbb{R}^3))} &= \| \int_{\Gamma,\theta,1} e^{\lambda t} H_2(\lambda) d\lambda \|_{B(L^p(\mathbb{R}^3))} \\ &\leq \int_{\Gamma,\theta,1} |e^{\lambda t} | \frac{|C_0 \lambda^{-\alpha} | + 1}{|\lambda^\alpha + 1 |} |d\lambda |.\end{aligned}$$

By the substitution $\lambda = re^{i\varphi}$, we have

$$\|M_1\|_{B(L^p(\mathbb{R}^3))} \leq \int_{\frac{1}{t}}^{\infty} |e^{tre^{i\varphi}}| \frac{|C_0 r^{-\alpha} e^{-i\varphi\alpha}| + 1}{|r^\alpha e^{i\varphi\alpha} + 1|} |e^{i\varphi}| dr$$

Note that

$$|e^{tre^{i\varphi}}| = |e^{tr(\cos\varphi + i\sin\varphi)}| = |e^{tr\cos\varphi}| |e^{i\sin\varphi}| = |e^{tr\cos\varphi}|; \ |e^{i\varphi}|_{B(L^p(\mathbb{R}^3))} = 1$$

and

$$|r^\alpha e^{i\varphi\alpha} + 1| = \sqrt{((r^\alpha \cos\alpha\varphi + 1)^2 + (r^\alpha \sin\alpha\varphi)^2} \geq r^\alpha \sin\alpha\varphi$$

which implies that

$$\|M_1\|_{B(L^p(\mathbb{R}^3))} \leq \int_{\frac{1}{t}}^{\infty} e^{tr\cos\varphi} \frac{C_0 r^{-2\alpha} + r^{-\alpha}}{\sin\alpha\varphi} dr.$$

Note that $e^{-x} \leq C_0 x^{-\gamma}, \gamma > 0$ we have

$$\begin{aligned} M_1 \leq & \frac{C}{\sin\alpha\varphi} \int_{\frac{1}{t}}^{\infty} r^{-2\alpha} C_0 (-tr\cos\varphi)^{-\gamma} dr \\ & + \frac{1}{\sin\alpha\varphi} \int_{\frac{1}{t}}^{\infty} r^{-\alpha} (-tr\cos\varphi)^{-\gamma} dr. \end{aligned}$$

where $\gamma > \alpha$. By a direct calculation of integral terms, we get that

$$\begin{aligned} \|M_1\|_{B(L^p(\mathbb{R}^3))} \leq & \frac{C}{\sin\alpha\varphi(\cos\varphi)^\gamma} \Big[\frac{t^{2\alpha-1}}{-2\alpha - \gamma + 1} + \frac{t^{\alpha-1}}{-\alpha - \gamma + 1} \Big] \\ \leq & C(t^{2\alpha-1} + t^{\alpha-1}). \end{aligned}$$

On other hand, by the same away as (3.8), the following estimate also holds

$$\|M_2\|_{B(L^p(\mathbb{R}^3))} \leq C(C_0 t^{2\alpha-1} + t^{\alpha-1})$$

By an argument analogous to the previous one. We also get

$$\|M_3\|_{B(L^p(\mathbb{R}^3))} \leq C(t^{2\alpha-1} + t^{\alpha-1}).$$

Summarily, $\|S_2(t)\|_{B(L^p(\mathbb{R}^3))} \leq C(t^{2\alpha-1} + t^{\alpha-1})$. □

Lemma 3.2. *Let $\alpha \in (0,1)$.Consider the operators $S_1(t), S_2(t)$ defined in 2.7. Then,exists C such that*

$$\|\partial_t S_1(t)\|_{B(L^p(\mathbb{R}^3))} \leq C(t^{\alpha-1} + t^{-1} + t^{-\alpha-1}),$$

and

$$\|\partial_t S_2(t)\|_{B(L^p(\mathbb{R}^3))} \leq C(t^{2\alpha-2} + t^{\alpha-2}).$$

Proof. Applying the partial derivative formula, we have

$$\partial_t S_j(t) = \frac{1}{2\pi i} \int_{\Gamma,\theta} \lambda e^{\lambda t} H_j(\lambda) d\lambda; j = 1, 2$$

Continuing to use the method as in the lemma 3.1, we have

$$\|\partial_t S_1(t)\|_{B(L^p(\mathbb{R}^3))} \leq \frac{1}{2\pi}\sum_{k=1}^{3}\left\|\int_{\Gamma,\theta,k}\lambda e^{\lambda t}H_1(\lambda)d\lambda\right\|_{B(L^p(\mathbb{R}^3))} = \frac{1}{2\pi}\sum_{k=1}^{3}\|N_k^{'}\|_{B(L^p(\mathbb{R}^3))},$$

where

$$N_k^{'} = \int_{\Gamma,\theta,k}\lambda e^{\lambda t}H_1(\lambda)d\lambda; k = 1,2,3.$$

Now, we estimate the operator $N_1^{'}$:

$$\begin{aligned}\|N_1^{'}\|_{B(L^p(\mathbb{R}^3))} &= \|\int_{\Gamma,\theta,1}\lambda e^{\lambda t}H_1(\lambda)d\lambda\|_{B(L^p(\mathbb{R}^3))}\\ &\leq \int_{\Gamma,\theta,1}|\lambda e^{\lambda t}|\frac{|\lambda^{-1}|(C_0+|\lambda^{\alpha}|)}{|\lambda^{\alpha}+1)|}|d\lambda|.\end{aligned}$$

By the substitution $\lambda = re^{i\varphi}$, we have

$$\|N_1^{'}\|_{B(L^p(\mathbb{R}^3))} \leq \int_{\frac{1}{t}}^{\infty}|r.e^{i\varphi}e^{tre^{i\varphi}}|\frac{|r^{-1}e^{-i\varphi}|(C_0+|r^{\alpha}e^{i\varphi\alpha}|)}{|r^{\alpha}e^{i\varphi\alpha}+1|}|e^{i\varphi}|dr$$

By similar approach as above, we get

$$\begin{aligned}\|N_1^{'}\|_{B(L^p(\mathbb{R}^3))} &\leq \frac{C_0}{\sin\alpha\varphi}\int_{\frac{1}{t}}^{\infty}r^{-\alpha}C(-tr\cos\varphi)^{-\gamma}dr + \frac{1}{\sin\alpha\varphi}\int_{\frac{1}{t}}^{\infty}C(-tr\cos\varphi)^{-\gamma}dr\\ &\leq \frac{C}{\sin\alpha\varphi(\cos\varphi)^{\gamma}}[\frac{t^{\alpha-1}}{-\alpha-\gamma+1}+\frac{t^{-1}}{-\gamma+1}] \leq C(t^{\alpha-1}+t^{-1}).\end{aligned}$$

By employing the same techniques as above estimates, one can get the following chain

$$\begin{aligned}\|N_2^{'}\|_{B(L^p(\mathbb{R}^3))} &= \left\|\int_{\Gamma,\theta,2}\lambda e^{\lambda t}H_1(\lambda)d\lambda\right\|\\ &\leq \int_{\Gamma,\theta,2}|\lambda e^{\lambda t}|\frac{|\lambda^{-1}|(C_0+|\lambda^{\alpha}|)}{|\lambda^{\alpha}+1)|}|d\lambda|.\end{aligned}$$

By the substitution $\lambda = \epsilon e^{i\psi}$and let $\epsilon = \frac{1}{t}$, we have

$$\begin{aligned}\|N_2^{'}\|_{B(L^p(\mathbb{R}^3))} &\leq \int_{-\varphi}^{\varphi}|\epsilon e^{i\psi}e^{t\epsilon e^{i\psi}}|\frac{|\epsilon^{-1}e^{-i\psi}|(C_0+|\epsilon^{\alpha}e^{i\psi\alpha}|)}{|\epsilon^{\alpha}e^{i\psi\alpha}+1|}|\epsilon i e^{i\psi}|d\psi\\ &\leq \int_{-\varphi}^{\varphi}t^{-1}e^{\cos\psi}\frac{(C_0+t^{-\alpha})}{|t^{-\alpha}e^{i\psi\alpha}+1|}d\psi.\end{aligned}$$

Using the inequality $|t^{-\alpha}e^{i\psi\alpha}+1| = \sqrt{(t^{-\alpha}\cos\psi\alpha+1)^2+(t^{-\alpha}\sin\psi\alpha)^2} \geq \sin\alpha\psi$, we get

$$\begin{aligned}\|N_1^{'}\|_{B(L^p(\mathbb{R}^3))} &\leq \int_{-\varphi}^{\varphi}t^{-1}e^{\cos\psi}\frac{(C_0+t^{-\alpha})}{\sin\alpha\psi}d\psi \leq (C_0t^{-1}+t^{-\alpha-1})\int_{-\varphi}^{\varphi}\frac{e^{\cos\psi}}{\sin\alpha\psi}d\psi\\ &\leq C(C_0t^{-1}+t^{-\alpha-1}).\end{aligned}$$

Our next objective is to evaluate $N_3^{'}$, atrivial verification shows that

$$\|N_3^{'}\|_{B(L^p(\mathbb{R}^3))} = \| \int_{\Gamma,\theta,3} \lambda e^{\lambda t} H_1(\lambda) d\lambda \|$$
$$\leq \int_{\Gamma,\theta,3} |\lambda e^{\lambda t} | \frac{|\lambda^{-1} |(C_0 + |\lambda^{\alpha} |)}{|\lambda^{\alpha} + 1) |} |d\lambda |.$$

By the substitution $\lambda = re^{-i\varphi}$, we have

$$\|N_3^{'}\|_{B(L^p(\mathbb{R}^3))} \leq \int_{\frac{1}{t}}^{\infty} |re^{-i\varphi}.e^{tre^{-i\varphi}} | \frac{|r^{-1}e^{i\varphi} |(C_0 + |r^{\alpha} e^{-i\varphi\alpha} |)}{|r^{\alpha} e^{-i\varphi\alpha} + 1 |} |e^{-i\varphi} |dr.$$

Similar to estimate of $N_1^{'}$, it may be concluded that

$$\|N_3^{'}\|_{B(L^p(\mathbb{R}^3))} \leq C(t^{\alpha-1} + t^{-1}).$$

Consequently,

$$\|\partial_t S_1(t)\|_{B(L^p(\mathbb{R}^3))} \leq C(t^{\alpha-1} + t^{-1} + t^{-\alpha-1}).$$

We proceed now to estimate the $\partial_t S_2(t)$. Continue to apply the formula for calculating the derivative

$$\partial_t S_2(t) = \frac{1}{2\pi i} \int_{\Gamma,\theta} \lambda e^{\lambda t} H_2(\lambda) d\lambda$$

$$\|\partial_t S_2(t)\| \leq \frac{1}{2\pi} \sum_{k=1}^{3} \| \int_{\Gamma,\theta,k} \lambda e^{\lambda t} H_2(\lambda) d\lambda \| = \frac{1}{2\pi} \sum_{k=1}^{3} M_k^{'}$$

where

$$M_k^{'} = \int_{\Gamma,\theta,k} \lambda e^{\lambda t} H_2(\lambda) d\lambda; k = 1, 2, 3$$

Estimate $M_1^{'}$:

$$\|M_1^{'}\|_{B(L^p(\mathbb{R}^3))} = \| \int_{\Gamma,\theta,1} \lambda e^{\lambda t} H_2(\lambda) d\lambda \|_{B(L^p(\mathbb{R}^3))} \leq \int_{\Gamma,\theta,1} |\lambda e^{\lambda t} | \frac{|C_0 \lambda^{-\alpha} | + 1}{|\lambda^{\alpha} + 1 |} |d\lambda |.$$

By the substitution $\lambda = re^{i\varphi}$, We have

$$\|M_1^{'}\|_{B(L^p(\mathbb{R}^3))} \leq \int_{\frac{1}{t}}^{\infty} |re^{i\varphi} e^{tre^{i\varphi}} | \frac{|C_0 r^{-\alpha} e^{-i\varphi\alpha} | + 1}{|r^{\alpha} e^{i\varphi\alpha} + 1 |} |e^{i\varphi} |dr$$

Since

$$|e^{tre^{i\varphi}} | = |e^{tr(\cos\varphi + i\sin\varphi)} | = |e^{tr\cos\varphi} | |e^{i\sin\varphi} | = |e^{tr\cos\varphi} |; |e^{i\varphi} | = 1$$

and

$$|r^{\alpha} e^{i\varphi\alpha} + 1 | = \sqrt{((r^{\alpha}\cos\alpha\varphi + 1)^2 + (r^{\alpha}\sin\alpha\varphi)^2} \geq r^{\alpha}\sin\alpha\varphi,$$

this implies immediately that

$$\|M_1^{'}\|_{B(L^p(\mathbb{R}^3))} \leq \int_{\frac{1}{t}}^{\infty} e^{tr\cos\varphi} \frac{C_0 r^{-2\alpha+1} + r^{-\alpha+1}}{\sin\alpha\varphi} dr$$

Thank to $e^{-x} \leq C_0 x^{-\gamma}, \gamma > 0$ we have

$$\|M_1^{'}\|_{B(L^p(\mathbb{R}^3))} \leq \frac{C}{\sin\alpha\varphi}\int_{\frac{1}{t}}^{\infty} r^{-2\alpha+1}C_0(-tr\cos\varphi)^{-\gamma}dr + \frac{1}{\sin\alpha\varphi}\int_{\frac{1}{t}}^{\infty} r^{-\alpha+1}(-tr\cos\varphi)^{-\gamma}dr$$

where $\gamma > \alpha$
Direct calculation of Integral we get

$$\begin{aligned}\|M_1^{'}\|_{B(L^p(\mathbb{R}^3))} &\leq \frac{C}{\sin\alpha\varphi(\cos\varphi)^{\gamma}}\frac{t^{2\alpha-2}}{-2\alpha-\gamma+2} + \frac{C}{\sin\alpha\varphi(\cos\varphi)^{\gamma}}\frac{t^{\alpha-2}}{-\alpha-\gamma+2}\\ &\leq C(t^{2\alpha-2}+t^{\alpha-2})\end{aligned}$$

Estimate $M_2^{'}$:

$$\begin{aligned}\|M_2^{'}\|_{B(L^p(\mathbb{R}^3))} &= \|\int_{\Gamma,\theta,2}\lambda e^{\lambda t}H_2(\lambda)d\lambda\|\\ &\leq \int_{\Gamma,\theta,2}|\lambda e^{\lambda t}|\frac{|C_0\lambda^{-\alpha}|+1}{|\lambda^{\alpha}+1|}|d\lambda|\end{aligned}$$

By the substitution $\lambda = \epsilon e^{i\psi}$, and ,and let $\epsilon = \frac{1}{t}$, we have

$$\|M_2^{'}\|_{B(L^p(\mathbb{R}^3))} \leq \int_{-\varphi}^{\varphi}|\epsilon e^{i\psi}e^{t\epsilon e^{i\psi}}|\frac{|C_0\epsilon^{-\alpha}e^{-i\psi\alpha}|+1}{|\epsilon^{\alpha}e^{i\psi\alpha}+1|}|\epsilon i e^{i\psi}|d\psi$$

Using techniqe semilar above we have

$$\begin{aligned}\|M_2^{'}\|_{B(L^p(\mathbb{R}^3))} &\leq \int_{-\varphi}^{\varphi}e^{\cos\psi}\frac{(C_0t^{2\alpha-2}+t^{-2})}{\sin\alpha\psi}d\psi \leq (C_0t^{2\alpha-2}+t^{\alpha-2})\int_{-\varphi}^{\varphi}\frac{e^{\cos\psi}}{\sin\alpha\psi}d\psi\\ &\leq C(C_0t^{2\alpha-2}+t^{\alpha-2})\end{aligned}$$

Estimate $M_3^{'}$:

$$\|M_3^{'}\|_{B(L^p(\mathbb{R}^3))} = \|\int_{\Gamma,\theta,3}\lambda e^{\lambda t}H_2(\lambda)d\lambda\| \leq \int_{\Gamma,\theta,3}|\lambda e^{\lambda t}|\frac{|C_0\lambda^{-\alpha}|+1}{|\lambda^{\alpha}+1)|}|d\lambda|$$

By the substitution $\lambda = re^{-i\varphi}$, We have

$$\|M_3^{'}\|_{B(L^p(\mathbb{R}^3))} \leq \int_{\frac{1}{t}}^{\infty}|re^{-i\varphi}e^{tre^{-i\varphi}}|\frac{|C_0r^{-\alpha}e^{i\varphi\alpha}|+1}{|r^{\alpha}e^{-i\varphi\alpha}+1|}|e^{-i\varphi}|dr$$

In the same manner we can see that

$$\|M_3^{'}\|_{B(L^p(\mathbb{R}^3))} \leq C(t^{2\alpha-2}+t^{\alpha-2})$$

From some above observations, we get that

$$\|\partial_t S_2(t)\|_{B(L^p(\mathbb{R}^3))} \leq C(t^{2\alpha-2}+t^{\alpha-2}).$$

□

4 Main result on maximal solution

In this section, we present our main results, which are the existence of locally mild solutions. We consider the following assumptions on the nonlinearities f.

The function $f : \mathbb{R} \longrightarrow \mathbb{R}$ is continuous and there exist constants $L > 0$, $p \geq 1$ such that

- (H1) $|f(x)| \leq L|x|$ for all $x \in \mathbb{R}$.
- (H2) $|f(x) - f(y)| \leq L|x - y|$ for all $x, y \in \mathbb{R}$.

Lemma 4.1. *Let $1 < p < \infty$ and $u_0 \in L^p(\mathbb{R}^3)$. Then, $\mathcal{S}_1(\cdot)u_0 \in C_\beta([0,T]; L^p(\mathbb{R}^3))$ with $\alpha < \beta$. Moreover, for $t \in (0,T)$, $h > 0$ such that $t + h \in [0,T]$,*

$$\|S_1(t+h)u_0 - S_1(t)u_0\|_{L^p(\mathbb{R}^3)} \leq Ch^\epsilon t^{-\alpha-\epsilon}\|u_0\|_{L^p(\mathbb{R}^3)}. \tag{4.9}$$

Proof. Let $t \in (0,T)$ and $h > 0$ such that $t + h \in [0,T]$. Then, by using fundamental theorem of Calculus we have $S_1(t+h)u_0 - S_1(t)u_0 = \int_t^{t+h} \partial_\tau S_1(\tau)u_0 d\tau$. This integral, in view of Lemma 3.2, can be estimated as follows

$$\begin{aligned}
\left\| \int_t^{t+h} \partial_\tau S_1(\tau)u_0 d\tau \right\|_{L^p(\mathbb{R}^3)} &\leq \|u_0\|_{L^p(\mathbb{R}^3)} \int_t^{t+h} \|\partial_\tau S_1(\tau)\|_{L^p(\mathbb{R}^3)\to L^p(\mathbb{R}^3)} d\tau \\
&\leq M\|u_0\|_{L^p(\mathbb{R}^3)} \int_t^{t+h} (t^{\alpha-1} + t^{-1} + t^{-\alpha-1}) d\tau \\
&\leq M(T^{2\alpha} + T^\alpha + 1)\|u_0\|_{L^p(\mathbb{R}^3)} \int_t^{t+h} t^{-\alpha-1} d\tau,
\end{aligned}$$

Let us choose a constant $\epsilon > 0$ such that $\epsilon < \alpha$. By simple computations, there hold

$$\int_t^{t+h} t^{-\alpha-1} d\tau = \frac{(t+h)^\alpha - t^\alpha}{\alpha t^\alpha (t+h)^\alpha} \leq \frac{h^\alpha}{\alpha t^\alpha h^{\alpha-\epsilon} t^\epsilon} = \frac{1}{\alpha} h^\epsilon t^{-\alpha-\epsilon},$$

where $(t+h)^\alpha \geq h^{\alpha-\epsilon}t^\epsilon$. This deduces $\mathcal{S}_1(\cdot)u_0 \in C_\beta([0,T]; L^p(\mathbb{R}^3))$ as well as (4.9). ☐

Lemma 4.2. *Assume that f satisfies the assumptions (H1)-(H2), and $v \in C_\beta([0,T]; L^p(\mathbb{R}^3))$ for some $1 < p < \infty$ and $0 < \beta < \alpha$.*

a) For $t \in [0,T)$, $h > 0$ such that $t + h \in [0,T]$, and $0 < \epsilon < \alpha \wedge (1-\alpha)$,

$$\begin{aligned}
&\left\| \int_0^t (S_2(t+h-s) - S_2(t-s)) f(v(s)) ds \right\|_{L^p(\mathbb{R}^3)} \\
&\qquad \leq \frac{ML(T^\alpha + 1)B(\alpha - \epsilon, 1 - \beta)}{1 - \alpha} t^{\alpha-\beta-\epsilon} h^\epsilon \|v\|_{C_\beta([0,T];L^p(\mathbb{R}^3))}.
\end{aligned}$$

b) For $t \in [0,T)$, $h > 0$ such that $t + h \in [0,T]$,

$$\begin{aligned}
&\left\| \int_t^{t+h} S_2(t+h-s) f(v(s)) ds \right\|_{L^p(\mathbb{R}^3)} \\
&\qquad \leq ML(T^\alpha + 1)B(\alpha, 1-\beta)h^{\alpha-\beta}\|v\|_{C_\beta([0,T_0];L^p(\mathbb{R}^3))}.
\end{aligned}$$

c) If $v \in C_\beta([0,T]; L^p(\mathbb{R}^3))$ then $\left(t \mapsto \int_0^t S_2(t-s) f(v(s)) ds\right) \in C_\beta([0,T]; L^p(\mathbb{R}^3))$.

Proof. We firstly prove Part a. Let $t \in [0,T)$ and $h > 0$ such that $t+h \in [0,T]$. By the Lipschitz continuity of f, we have

$$\begin{aligned}
&\left\| \int_0^t \big(S_2(t+h-s) - S_2(t-s)\big) f(v(s))ds \right\|_{L^p(\mathbb{R}^3)} \\
&\qquad \le \int_0^t \| [S_2(t+h-s) - S_2(t-s)] f(v(s))ds \|_{L^p(\mathbb{R}^3)} \\
&\qquad \le L \left(\int_0^t \left(\int_{t-s}^{t+h-s} \|\partial_\tau S_2(\tau)\|_{B(L^p(\mathbb{R}^3))} d\tau \right) s^{-\beta} ds \right) \|v\|_{C_\beta([0,T];L^p(\mathbb{R}^3))} \\
&\qquad \le ML \left(\int_0^t \int_{t-s}^{t+h-s} (\tau^{2\alpha-2} + \tau^{\alpha-2}) s^{-\beta} d\tau ds \right) \|v\|_{C_\beta([0,T];L^p(\mathbb{R}^3))} \\
&\qquad \le ML(T^\alpha + 1) \left(\int_0^t \int_{t-s}^{t+h-s} \tau^{\alpha-2} s^{-\beta} d\tau ds \right) \|v\|_{C_\beta([0,T];L^p(\mathbb{R}^3))},
\end{aligned}$$

where we have used Lemma 3.2. The latter double integral can be estimated by fundamental computations. With $0 < \epsilon < \alpha \wedge (1-\alpha)$, one has

$$(t+h-s)^{1-\alpha} = (t+h-s)^{1-\epsilon-\alpha}(t+h-s)^{\epsilon} \ge h^{1-\epsilon-\alpha}(t-s)^{\epsilon}.$$

Consequenty, we derive that

$$\begin{aligned}
&\int_0^t \int_{t-s}^{t+h-s} \tau^{\alpha-2} s^{-\beta} d\tau ds \\
&\qquad = \frac{1}{1-\alpha} \int_0^t \frac{(t+h-s)^{1-\alpha} - (t-s)^{1-\alpha}}{(t-s)^{1-\alpha}(t+h-s)^{1-\alpha}} s^{-\beta} ds \\
&\qquad \le \frac{h^\epsilon}{1-\alpha} \int_0^t (t-s)^{\alpha-\epsilon-1} s^{-\beta} ds \\
&\qquad = \frac{h^\epsilon}{1-\alpha} B(\alpha-\epsilon, 1-\beta) t^{\alpha-\beta-\epsilon},
\end{aligned}$$

which claims Part a. Next, we will prove Part b. According to Lemma 3.1 and the Lipschitz continuity of f, we have

$$\begin{aligned}
&\left\| \int_t^{t+h} S_2(t+h-s) f(v(s))ds \right\|_{L^p(\mathbb{R}^3)} \\
&\qquad \le ML \left(\int_t^{t+h} ((t+h-s)^{2\alpha-1} + (t+h-s)^{\alpha-1}) s^{-\beta} ds \right) \|v\|_{C_\beta([0,T];L^p(\mathbb{R}^3))} \\
&\qquad \le ML(T^\alpha + 1) \left(\int_t^{t+h} (t+h-s)^{\alpha-1} (s-t)^{-\beta} ds \right) \|v\|_{C_\beta([0,T];L^p(\mathbb{R}^3))}, \qquad (4.10)
\end{aligned}$$

where we note that $s^{-\beta} \le (s-t)^{-\beta}$. Part b is immediately obtained since the latter integral is equal to $h^{\alpha-\beta} B(\alpha, 1-\beta)$. Thanks to Parts a,b, the function $t \mapsto \int_0^t S_2(t-s) f(v(s))ds$ belongs to the space $C([0,T]; L^p(\mathbb{R}^3))$. Additionally, by the same techniques as above, one can see that

$$\left\| \int_0^t S_2(t-s) f(v(s))ds \right\|_{L^p(\mathbb{R}^3)} \le MLT^\alpha B(\alpha, 1-\beta) \|v\|_{C_\beta([0,T];L^p(\mathbb{R}^3))} t^{-\beta}, \qquad (4.11)$$

for $t \in (0,T]$, which implies Part c and completes the proof. □

Theorem 4.3. *Let $u_0 \in L^p(\mathbb{R}^3)$, and f be defined by the assumptions (H1)-(H2). Then Problem (1.1)-(1.2) has only a locally mild solution in $[0,T]$ with $0 < \beta < min\{2\alpha, 1\}$.*

Proof. This proof will be based on the contraction mapping theorem. Firstly, it is a simple matter to choose T_0 small enough such that

$$C'L\big(T_0^{2\alpha+\beta}B(2\alpha, 1-\beta) + T_0^{\alpha+\beta}B(\alpha, 1-\beta)\big) < \frac{1}{2}, \tag{4.12}$$

where $B(\cdot,\cdot)$ is Beta function, and C' is given in Lemma 3.1. Let us choose R such that

$$R > 2C\big(T_0^{\alpha+\beta} + T_0^{\beta-\alpha} + T_0^{\beta}\big)\|u_0\|_{L^2(\mathbb{R}^3)}, \tag{4.13}$$

and denote by B_R the ball in $C_\beta([0,T_0]; L^p(\mathbb{R}^3))$ of radius R centered at the origin. Then, B_R is closed convex, nonempty. Let us consider the mapping $\Phi : B_R \mapsto B_R$ defined by

$$\Phi v(t) = S_1(t)u_0 + \int_0^t S_2(t-s)f(v(s))ds,$$

for all $v \in B_R$. We need to prove that Φ has a unique fixed point, which is a mild solution of Problem (1.1). The proof will be divided into two steps as follows.

Step 1: Proving Φ well-defined on B_R. By using Lemma 4.1 and 4.2 ,we have function $t \mapsto \Phi v(t)$ is continuous on $[0,T]$. Hence, it is necessary to prove that Φ maps B_R onto itself. For $v \in B_R$, we have

$$\begin{aligned}\|\Phi(v)\|_{C_\beta([0,T];L^p(\mathbb{R}^3)} \le &\|S_1(t)u_0\|_{C_\beta([0,T];L^p(\mathbb{R}^3)} \\ &+ \left\| \int_0^t S_2(t-s)f(v(s))ds \right\|_{C_\beta([0,T];L^p(\mathbb{R}^3))}.\end{aligned}$$

It is plain that the term $S_1(t)u_0$ can be estimated by Lemma 3.1, which follows that

$$t^\beta\|S_1(t)u_0\|_{L^p(\mathbb{R}^3)} \le C\big(T_0^{\beta+\alpha} + T_0^{\beta-\alpha} + T_0^{\beta}\big)\|u_0\|_{L^p(\mathbb{R}^3)}, \quad \forall t \in [0,T_0].$$

On the other hand, by using Lemma 3.1 and Lipschitz property of f ,we derive that the following estimate.

$$\begin{aligned}&\left\| \int_0^t S_2(t-s)f(v(s))ds \right\|_{L^p(\mathbb{R}^3)} \\ &\quad \le C' \int_0^t \big((t-s)^{2\alpha-1} + (t-s)^{\alpha-1}\big)\|f(v(s))\|_{L^p(\mathbb{R}^3)}ds \\ &\quad \le C'L\|v\|_{C_\beta([0,T];L^p(\mathbb{R}^3))} \int_0^t \big((t-s)^{2\alpha-1} + (t-s)^{\alpha-1}\big)s^{-\beta}ds \\ &\quad \le C'L\|v\|_{C_\beta([0,T];L^p(\mathbb{R}^3))}\big(T_0^{2\alpha}B(2\alpha,1-\beta) + T_0^{\alpha}B(\alpha,1-\beta)\big).\end{aligned} \tag{4.14}$$

Consequently, we obtain that

$$\begin{aligned}&\left\| \int_0^t S_2(t-s)f(v(s))ds \right\|_{C_\beta([0,T];L^p(\mathbb{R}^3))} \\ &\quad \le C'L\|v\|_{C_\beta([0,T];L^p(\mathbb{R}^3))}\big(T_0^{2\alpha+\beta}B(2\alpha,1-\beta) + T_0^{\alpha+\beta}B(\alpha,1-\beta)\big).\end{aligned} \tag{4.15}$$

By the definitions of T_0 and R above, we imply

$$\|\Phi(v)\|_{C_\beta([0,T];L^p(\mathbb{R}^3))} \le R/2 + R/2 = R,$$

namely, we can assert that $\Phi(v) \in B_R$.

Step 2. Proving that Φ is a contraction mapping. Let us arbitrarily take $v_1, v_2 \in B_R$. Then

$$\begin{aligned}
&\|\Phi v_1(t) - \Phi v_2(t)\|_{L^p(\mathbb{R}^3)} \\
&\quad \le \int_0^t \big\|S_2(t-s)\big(f(v_1(s)) - f(v_2(s))\big)\big\|_{L^p(\mathbb{R}^3)} ds \\
&\quad \le C \int_0^t \big((t-s)^{2\alpha-1} + (t-s)^{\alpha-1}\big)\big\|f(v_1(s)) - f(v_2(s))\big\|_{L^p(\mathbb{R}^3)} ds.
\end{aligned}$$

Basing on the fact that

$$\big\|f(v_1(s)) - f(v_2(s))\big\|_{L^p(\mathbb{R}^3)} \le Ls^{-\beta}\|v_1 - v_2\|_{C_\beta([0,T];L^p(\mathbb{R}^3))},$$

we implies

$$\begin{aligned}
&\|\Phi v_1(t) - \Phi v_2(t)\|_{L^p(\mathbb{R}^3)} \\
&\quad \le CL \int_0^t \big((t-s)^{2\alpha-1} + (t-s)^{\alpha-1}\big)\|v_1(s) - v_2(s)\|_{L^p(\mathbb{R}^3)} ds \\
&\quad \le CL\|v_1 - v_2\|_{C_\beta([0,T];L^p(\mathbb{R}^3))} \int_0^t \big((t-s)^{2\alpha-1} + (t-s)^{\alpha-1}\big)s^{-\beta} ds.
\end{aligned}$$

Multiplying the both sides of the above by t^β, we obtain

$$\begin{aligned}
&t^\beta\|\Phi v_1(t) - \Phi v_2(t)\|_{L^p(\mathbb{R}^3)} \\
&\quad \le CL\|v_1 - v_2\|_{C_\beta([0,T];L^p(\mathbb{R}^3))} t^\beta \int_0^t \big((t-s)^{2\alpha-1} + (t-s)^{\alpha-1}\big)s^{-\beta} ds \\
&\quad = CL\|v_1 - v_2\|_{C_\beta([0,T];L^p(\mathbb{R}^3))} t^\beta \big(t^{2\alpha-\beta}B(2\alpha, 1-\beta) + t^{\alpha-\beta}B(\alpha, 1-\beta)\big) \\
&\quad \le CL\|v_1 - v_2\|_{C_\beta([0,T];L^p(\mathbb{R}^3))} \big(T^{2\alpha}B(2\alpha, 1-\beta) + T^{\alpha}B(\alpha, 1-\beta)\big).
\end{aligned}$$

It follows from (4.12) that

$$t^\beta\|\Phi v_1(t) - \Phi v_2(t)\|_{L^p(\mathbb{R}^3)} \le \frac{1}{2}\|v_1 - v_2\|_{C_\beta([0,T];L^p(\mathbb{R}^3))}, \quad \forall t \in [0, T_0].$$

So, by taking supremum with respect to t, we obtain

$$\|\Phi v_1 - \Phi v_2\|_{C_\beta([0,T];L^p(\mathbb{R}^3))} \le \frac{1}{2}\|v_1 - v_2\|_{C_\beta([0,T];L^p(\mathbb{R}^3))},$$

i.e., we can conclude that $\mathcal{T}$ is a contraction mapping. □

In the next section, we will research existence of the maximal mild solution to our problem. Explicitly, we will study Problem (1.1) - (1.2). We will show that Problem (1.1) - (1.2) has a globally mild solution $u \in C_\beta([0,T]; L^p(\mathbb{R}^3))$, or there exists a time $T_{\max} < \infty$ such that Problem (1.1) - (1.2) has a maximal mild solution $u \in C_\beta([0,T]; L^p(\mathbb{R}^3))$ with blowup at the finite time $T_{\max}$.

Before stating the main theorem we first define continuation of mild solution.

Lemma 4.4. *(Unique continuation) Suppose f be defined by the assumptions (H1)-(H2), $u_0 \in L^p(\mathbb{R}^3)$, and $0 < \beta < min\{2\alpha, 1\}$.Let u be a mild solution of (1.1) - (1.2. Then,it possesses a unique continuation.*

Proof. We will prove this lemma by making uses of usual fixed point arguments also. Thank to theorem 4.3, we recall that, u(t) is the unique mild solution of (1.1)(1.2) and

$$u(t) = S_1(t)u_0 + \int_0^t S_2(t-s)f(u(s))ds, \quad \forall t \in [0,T]. \tag{4.16}$$

Let $\mathcal{M} = \sup\{\|u(t)\|_{L^p} : 0 \le t \le T\}$ and put $\mathcal{N} = \mathcal{M} + \|u(T)\|_{L^p}$. We take a real number $0 < \rho < \mathcal{M}$. Then, exists $\xi > 0$ sufficiently small such that following conditions hold for all $t \in [T, T+\xi]$

$$\begin{cases} t^\beta \|u_0\|_{L^p(\mathbb{R}^3)} C_1\left((t-T)^\alpha + (t-T)^\delta T^{-(\alpha+\delta)} + \frac{t-T}{T}\right) \le \dfrac{\rho}{6}, \\ t^\beta C_2\left(t^{2\alpha-\beta} - T^{2\alpha-\beta} + (t-T)^{1-\alpha}.T^{\alpha-\epsilon-\beta}\right) \le \dfrac{\rho}{6\mathcal{N}}, \\ t^\beta C_3\left((t-T)^{2\alpha-\beta} + (t-T)^{\alpha-\beta}\right) \le \dfrac{\rho}{6\mathcal{N}}, \end{cases} \tag{4.17}$$

where C_1 following from lemma 4.1 such that

$$\left\|(\mathcal{S}_1(t+h) - \mathcal{S}_1(t))u_0\right\|_{L^p(\mathbb{R}^3)} \le C\left(h^\alpha + h^\delta t^{-(\alpha+\delta)} + \frac{h}{t}\right)\|u_0\|_{L^p(\mathbb{R}^3)}. \tag{4.18}$$

and C_2, C_3 following from lema 4.2) such that

$$\begin{aligned} &\left\| \int_0^T (S_2(t-s) - S_2(T-s))f(u(s))ds \right\|_{L^p(\mathbb{R}^3)} \\ &\qquad \le C_2\left(t^{2\alpha-\beta} - T^{2\alpha-\beta} + (t-T)^{1-\alpha}.T^{\alpha-\epsilon-\beta}\right)\|u\|_{L^p(\mathbb{R}^3)} \end{aligned}$$

and the following bound

$$\left\| \int_T^t S_2(t-s)f(\widetilde{u}(t))ds \right\|_{L^p(\mathbb{R}^3)} \le C_3\left((t-T)^{2\alpha-\beta} + (t-T)^{\alpha-\beta}\right)\|\widetilde{u}\|_{L^p(\mathbb{R}^3)}.$$

With the parameters ρ, ξ as above, we now define the following space

$$\mathbb{Z}_p^{\rho,\xi,T} := \left\{ v \in C_\beta([0,T+\xi]; L^p(\mathbb{R}^3)) \,\middle|\, v|_{[0,T]} = u \right.$$

and

$$\left. \|v(t) - u(T)\|_{L^p(\mathbb{R}^3)} \le \rho t^{-\beta}, \forall t \in [T, T+\xi] \right\},$$

and the mapping $\widetilde{\Phi} : \mathbb{Z}_p^{\rho,\xi,T} \to \mathbb{Z}_p^{\rho,\xi,T}$ as follows

$$\widetilde{\Phi}\widetilde{u}(t) := S_1(t)u_0 + \int_0^t S_2(t-s)f(\widetilde{u}(s))ds \tag{4.19}$$

for all $t \in [0, T+\xi]$. Let us note that $\widetilde{\Phi}\widetilde{u} = \Phi u = u$.By Lemma 4.2,

$$\widetilde{\Phi}\widetilde{u} \in C_\beta([0,T+\xi]; L^p(\mathbb{R}^3)).$$

Hence, in order to show that the mapping $\widetilde{\Phi}$ is well-defined, we need to prove the below inequality

$$\|\widetilde{\Phi}\widetilde{u}(t) - u(T)\|_{L^p(\mathbb{R}^3)} \le \rho.t^{-\beta}, \quad t \in [T, T+\xi]. \tag{4.20}$$

For $t \in [T, T+\xi]$, we have

$$\begin{aligned}
&\|\widetilde{\Phi}\widetilde{u}(t) - u(T)\|_{L^p\left(\mathbb{R}^3\right)} \\
&\leq \|(S_1(t) - S_1(T))u_0)\|_{L^p\left(\mathbb{R}^3\right)} + \| \int_0^t S_2(t-s) f(\widetilde{u}(s))ds \\
&\quad - \int_0^T S_2(T-s) f(\widetilde{u}(s))ds\|_{L^p\left(\mathbb{R}^3\right)} \\
&\leq \|(S_1(t) - S_1(T))u_0)\|_{L^p\left(\mathbb{R}^3\right)} \\
&\quad + \| \int_0^T \big(S_2(t-s) - S_2(T-s)\big) f(u)(s))ds\|_{L^p\left(\mathbb{R}^3\right)} \\
&\quad + \| \int_T^t S_2(t-s) f(\widetilde{u}(s))ds \Big\|_{L^p\left(\mathbb{R}^3\right)} \\
&\leq \|u_0\|_{L^p(\mathbb{R}^3)} C_1 \big((t-T)^{\alpha} + (t-T)^{\delta} T^{-(\alpha+\delta)} + \frac{t-T}{T}\big) \\
&\quad + \big(C_2 \Big(t^{2\alpha-\beta} - T^{2\alpha-\beta} + (t-T)^{1-\alpha}.T^{\alpha-\epsilon-\beta}\Big) \|u\|_{L^p(\mathbb{R}^3)} \\
&\quad + C_3 \Big((t-T)^{2\alpha-\beta} + (t-T)^{\alpha-\beta}\Big) \|\widetilde{u}\|_{L^p(\mathbb{R}^3)} \leq t^{\beta}.\rho.
\end{aligned}$$

From the above it follows that, we can assert that

$$|\widetilde{\Phi}\widetilde{u}(t) - u(T)| \leq \rho.t^{-\beta}, \quad t \in [T, T+\xi].$$

Summarily,we conclude that the mapping $\widetilde{\Phi}$ well- defined on $\mathbb{Z}_p^{\rho,\xi,T}$. Finally, by an argument analogous to that used for the proof of Theorem (4.3)we can show that $\widetilde{\Phi}$ is a contraction mapping.Thus,$\widetilde{\Phi}$ has a unique fixed point in $\mathbb{Z}_p^{\rho,\xi,T}$, which will be denoted by $\widetilde{u_*}$.Then,$\widetilde{u_*}|_{[0,T]} = u$ and

$$\widetilde{u_*} = S_1(t)u_0 + \int_0^t S_2(t-s) f(\widetilde{u_*}(s))ds, t \in [0, T+\xi].$$

Hence, $\widetilde{u_*}$ is the unique continuation of u. We finish the proof. □

Let us now proceed to establish the existence of a globally mild solution or a maximal mild solution with blowup at the finite time $T_{\max}$ in the following lemma.

Theorem 4.5. *(Maximal solution).Assume that F satisfies the assumption (H1)-(H2). Then,*

i) Problem (1.1) - (1.2 has a globally $C_\beta L^p$ mild solution $u \in C_\beta([0,T]; L^p(\mathbb{R}^3))$ or

ii) There exists $T_{max} < \infty$ such that problem (1.1) - (1.2has a maximal local $C_\beta L^p$-mild solution $u \in C_\beta([0,T]; L^p(\mathbb{R}^3))$ with

$$\lim_{t \to T_{\max}^-} \sup \|u(t)\|_{C_\beta([0,T]; L^p\left(\mathbb{R}^3\right))} = \infty.$$

Proof. The proof is similar to [9] and we omit it here. □

5 Declarations

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Ethical Approval

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Authors' Contributions

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