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Blow-up solutions to fractional solid fuel ignition model

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Abstract

In this study, we investigate a time fractional parabolic equation with Caputo-Fabrizio derivative and a pure exponential source term. The model can be seen as a modified version of a solid fuel ignition by considering delay effects. Because of the bad behavior of the source term, the common Banach fixed point theorem is not suitable to be applied. Therefore, we approach it by an iteration method in which we find a super solution to the problem. Then, the monotony of approximating solutions implies the existence of a mild solution. Furthermore, we show that if the given initial data is sufficiently large in terms of norm measure, the corresponding solution will blow up in a finite time. The main techniques of the work are mainly based on calculations related to explicit formulas of solution operator derived from the eigenpair of Helmholtz's equation and Sobolev embeddings between Hilbert scale spaces.

Key words: Caputo-Fabrizio, fractional equation, exponential nonlinearity
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1 Introduction

In the current work, we are concerned with the following Dirichlet boundary problem for a class of time-fractional reaction diffusion equations

$$\begin{cases} {}_{CF}D_t^{\alpha} w(x,t) - \Delta w(x,t) = \gamma \exp(w(x,t)), & (x,t) \in \Omega \times \mathbb{R}^+, \\ w(x,t) = 0, & (x,t) \in \partial\Omega \times \mathbb{R}^+, \\ w(x,t) = w_0(x), & (x,t) \in \Omega \times \{0\}, \end{cases} \tag{1.1}$$

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where, Ω is a bounded domain in $\mathbb{R}^N, N \geqslant 1$ with smooth boundary $\partial\Omega$, $\gamma > 0$ is the Frank-Kamenetski parameter and w_0 is the initial data. The symbol ${}_{CF}D_t^\alpha$ is the Caputo-Fabrizio derivative operator of order $\alpha \in (0,1)$ which is defined as (see [1, 2])

$$ {}_{CF}D_t^\alpha z(t) = \frac{\Upsilon(\alpha)}{1-\alpha} \int_0^t \mathbf{D}_\alpha(t-r) \frac{\partial z(r)}{\partial r} \mathrm{d}r, \quad \text{for } t \geq 0, $$

where $\mathbf{D}_\alpha(r) = \exp\left(-\frac{\alpha}{1-\alpha} r\right)$ and Υ is a normalization satisfying $\Upsilon(0) = \Upsilon(1) = 1$.

Parabolic equations are often associated with classical time-derivative operators. However, in some situations, time-fractional derivatives seem to be more optimal, for example, diffusion with memory. In terms of non-integer order derivatives, there are many kinds of definitions such as Riemann–Liouville derivative, Caputo derivative, Hilfer derivative, etc. These mentioned operators are constructed via some singular kernels, this effectively expresses common memory effects but does not match delay phenomena. In this paper, we consider the time-fractional derivative in the sense of Caputo-Fabrizio which are optimal for the latter case. Diffusion type equations with Caputo-Fabrizio derivative recently attracted much interest from PDEs researchers due. Let us make a short overview of recent beautiful works.

In [1], Tuan and Yong obtained the global of the mild solution for (1.1) for the source functions which are globally Lipschitz functions. For the local Lipschitz term, they got the existence of local mild solutions to the problem. Also in this work, a blow-up alternative was established. In [3], Tuan considered the Cahn-Hilliard equation associated with Caputo-Fabrizio derivative as follows

$$ \begin{cases} {}_{CF}D_t^\alpha w + \Delta^2 w = \Delta(w - w^3), & \text{in} \quad \Omega \times (0,T], \\ w(x,0) = w_0(x), & \text{in} \quad \Omega, \\ w = \Delta w = 0, & \text{in} \quad \partial\Omega. \end{cases} \tag{1.2} $$

He obtained the local existence for a mild solution to problem (1.2). In addition, he showed that the convergence of the mild solution to problem (1.2) when $\alpha \to 1^-$. Recently, the time-fractional integro-differential equation with the Caputo–Fabrizio type derivative has been considered in [4]. We also refer the reader to [2, 5] for interesting works in the spirit of using the mild concept for solutions to parabolic models with the Caputo-Fabrizio derivative

The new contribution of our paper is organized as follows:

- The first contribution of our paper is to show the existence of the mild solution to Problem (1.1) under the bad behavior of the source term $\exp(w)$. Due to this behavior, one cannot derive any Lipschitz type estimate. Therefore, we are unable to use the common Banach fixed point as in [1]. In this situation, we use the method of super solution to show the existence of the mild solution (see [6, Chapter 9]).
- Also based on mentioned behavior nonlinearity, we conjecture that it is very difficult to prove the global existence of solutions. We prove in this paper that when the initial is sufficiently large, all possible mild solutions blow up in a finite time.

The paper is organized as follows. In section 2, we focus on some preliminary results which are useful to some knowledge of the main results. Section 3 provides the main results of our paper. Theorem 3.1 give the local existence of a mild solution to Problem (1.1).

2 Auxiliary settings and lemmas

First, throughout this work, instead of writing $|A| \leq C|B|$ where $C > 0$, we symbolize $A \lesssim B$. To keep everything short, we use the notation $\mathbb{A} := -\Delta$ and denote by $\sigma(\mathbb{A}), \mathcal{V}$, respectively, the set of eigenvalues of $\mathbb{A}$ and the corresponding orthogonal basis of eigenvectors $L^2(\Omega)$. We notice that all elements of $\sigma(\mathbb{A})$ form a nondecreasing sequence of positive real values. Then, we define by λ_1, v_1 the first element of $\sigma(\mathbb{A}), \mathcal{V}$, respectively, and convention that $(\lambda, v) \in (\sigma(\mathbb{A}), \mathcal{V})$ means $(\lambda, v) \in \sigma(\mathbb{A}) \times \mathcal{V}$ and $\mathbb{A}v = \lambda v$.

We now discuss how mild solutions to Problem (1.1) are defined. Taking L^2-inner product to the first equation of (1.1) with an arbitrary $v \in \mathcal{V}$ we have

$$\int_\Omega \Big({}_{CF}D_t^\alpha w(x,t) + \lambda w(x,t) \Big) v(x) \mathrm{d}x = \gamma \int_\Omega \exp(w(x,t)) v(x) \mathrm{d}x, \tag{2.1}$$

here we note that the Dirichlet condition of w is satisfied. We first consider the fact that

$$LT\left({}_{CF}D_t^\alpha w(t)\right) = \frac{J(\alpha)}{1-\alpha} LT\left(\frac{\mathrm{d}}{\mathrm{d}t} w(t)\right) LT\left(\exp\left(\frac{-\alpha t}{1-\alpha}\right)\right),$$

for an appropriate function. Here $LT(f)$ denotes the Laplace transform of the function f. Then, by applying the Laplace transform to (2.1) one deduces

$$\int_\Omega \left(\frac{\xi LT\left(w(x,\cdot)\right) - w_0(x)}{\xi + \alpha(1-\xi)} + \lambda LT\left(w(x,\cdot)\right) - LT\left(F(w(x,\cdot))\right) \right) v(x) \mathrm{d}x = 0,$$

where $F(w) := \gamma \exp(w)$. Applying the inversion Laplace transform, we immediately obtain

$$\begin{aligned} \int_\Omega w(x) v(x) \mathrm{d}x = {} & \frac{\exp\left(\frac{-\alpha\lambda t}{1+\lambda(1-\alpha)}\right)}{1+\lambda(1-\alpha)} \int_\Omega w_0(x) v(x) \mathrm{d}x \\ & + \int_0^t \int_\Omega \frac{\alpha \exp\left(\frac{-\alpha\lambda(t-\nu)}{1+\lambda(1-\alpha)}\right)}{(1+\lambda(1-\alpha))^2} F(w(x,\nu)) v(x) \mathrm{d}x \mathrm{d}\nu, \end{aligned}$$

for any $(\lambda, v) \in (\sigma(\mathbb{A}), \mathcal{V})$. Based on the above result, we introduce some operators which play an important role in presenting mild solutions to Problem (1.1). Precisely, for any $t \in [0, \infty)$ we define two operator $\mathcal{X}_1(t), \mathcal{X}_2(t)$ from $L^2(\Omega)$ to $\mathbb{R}$ as following

$$\begin{aligned} \mathcal{X}_1(t) w(x) &:= \sum_{(\lambda,v) \in (\sigma(\mathbb{A}),\mathcal{V})} \frac{v(x)}{1+\lambda(1-\alpha)} \exp\left(\frac{-\alpha\lambda t}{1+\lambda(1-\alpha)}\right) \int_\Omega w(x) v(x) \mathrm{d}x, \\ \mathcal{X}_2(t) w(x) &:= \sum_{(\lambda,v) \in (\sigma(\mathbb{A}),\mathcal{V})} \frac{\alpha v(x)}{(1+\lambda(1-\alpha))^2} \exp\left(\frac{-\alpha\lambda t}{1+\lambda(1-\alpha)}\right) \int_\Omega w(x) v(x) \mathrm{d}x. \end{aligned}$$

Having defined these two operators, we can now introduce a precise definition of mild solutions to Problem (1.1) as follows.

Definition 2.1. Let $\alpha \in (0,1)$, $T > 0$ and $w_0 \in L^\infty(\Omega)$. We call $w \in L^\infty(0,T; L^\infty(\Omega))$ a mild solution to Problem (1.1) if it satisfies the following integral equation

$$w(x,t) = \mathcal{X}_1(t) w_0(x) + \mathcal{I}\big(\exp(w)\big)(x,t), \quad t \in [0,T], \tag{2.2}$$

where we define

$$\mathcal{I}(w)(x,t) := \gamma \int_0^t \mathcal{X}_2(t-\nu) w(x,\nu) \mathrm{d}\nu.$$

Definition 2.2. Let $0 < T < \infty$ and $w \in C\left([0,T); L^2(\Omega)\right)$ be the mild solution to Problem (1.1). We call w a finite time blow-up solution if is not smooth enough to satisfy (**??**) or the following result holds

$$\lim_{t \to T^-} \int_\Omega w(x,t) v_1(x) \mathrm{d}x = \infty.$$

Remark 2.1. *Although we need to prove some further differentiable properties to show that the solution w which satisfies (2.2) actually solves (1.1), in the range of mild concept (using Definition 2.1) we do not need to discuss this problem. The advantage of Equality (2.2) is that we do not need to consider the differentiability of w. However, (2.2) is still a nonlinear equation. Thus, one can not find a general form for every solution to (2.2) (with given w). To deal with this trouble, we often visualize w via the so-call Picard iteration $\{w_j\}_{j=\overline{1,\infty}}$, defined as follows*

$$\begin{aligned} w_1(t) &:= \mathcal{X}_1(t)w_0, \\ w_k(t) &:= \mathcal{X}_1(t)w_0 + \mathcal{I}\big(\exp(w_{k-1})\big)(t). \end{aligned}$$

Then, w is considered as the limit of $\{w_j\}_{j=\overline{1,\infty}}$. We will refer to this matter later in Section 3.

We would like to derive some lemmas for obtaining essential estimates for the solution operators $\mathcal{X}_1, \mathcal{X}_2$ and deriving a convergent criterion for $\{w_j\}_{j=\overline{1,\infty}}$.

Lemma 2.3. *Let $w \in L^\infty(\Omega)$ and $\frac{N}{2} < \beta < 2$. Then, the following estimate is satisfied*

$$\Big\|\mathcal{X}_1(t)\exp(w)\Big\|_{L^\infty(\Omega)} \lesssim \exp\left(\|w\|_{L^\infty}\right).$$

Proof. Since $\frac{N}{2} < \beta < 2$, $\mathcal{H}^\beta(\Omega)$ can be embedded in $C(\bar{\Omega})$. Hence, for $w \in L^\infty(\Omega)$ we have

$$\Big\|\exp(w)\Big\|_{L^2(\Omega)} \lesssim \exp\left(\|w\|_{L^\infty(\Omega)}\right). \tag{2.3}$$

On the other hand, Parseval's equality implies

$$\begin{aligned} \Big\|\mathcal{X}_1(t)w\Big\|^2_{\mathcal{H}^\beta(\Omega)} &= \sum_{(\lambda,v)\in(\sigma(\mathbb{A}),\mathcal{V})} \frac{\lambda^\beta}{(1+\lambda(1-\alpha))^2}\exp\left(\frac{-2\alpha\lambda t}{1+\lambda(1-\alpha)}\right)\left(\int_\Omega w(x)v(x)\mathrm{d}x\right)^2 \\ &\lesssim \sum_{(\lambda,v)\in(\sigma(\mathbb{A}),\mathcal{V})} \lambda^{\beta-2}\left(\int_\Omega w(x)v(x)\mathrm{d}x\right)^2. \end{aligned}$$

As we mentioned above, since $\sigma(\mathbb{A})$ is a non-decreasing sequence of positive values, the following estimate is satisfied easily

$$\begin{aligned} \Big\|\mathcal{X}_1(t)w\Big\|_{\mathcal{H}^\beta(\Omega)} &\lesssim \lambda_1^{\frac{\beta-2}{2}}\left(\sum_{(\lambda,v)\in(\sigma(\mathbb{A}),\mathcal{V})}\left(\int_\Omega w(x)v(x)\mathrm{d}x\right)^2\right)^{\frac{1}{2}} \\ &\lesssim \|w\|_{L^2(\Omega)}. \end{aligned} \tag{2.4}$$

Combining (2.3) and (2.4) yields

$$\Big\|\mathcal{X}_1(t)\exp(w)\Big\|_{\mathcal{H}^\beta(\Omega)} \lesssim \exp\left(\|w\|_{L^\infty(\Omega)}\right).$$

The embedding between $\mathcal{H}^\beta(\Omega)$ and $C(\bar{\Omega})$ mentioned above helps us to derive the desired result. □

Lemma 2.4. *Let w be a measurable function such that $\|\exp(w)\|_{L^2(\Omega)} < \infty$. The following smoothing effect is satisfied*

$$\Big\|\mathcal{X}_2(t)\mathcal{X}_1(\nu)\exp(w)\Big\|_{\mathcal{H}^\beta(\Omega)} \lesssim \Big\|\exp(w)\Big\|_{L^2(\Omega)}$$

Proof. For $w \in L^2(\Omega)$, by explicit definition of $\mathcal{X}_1$ and $\mathcal{X}_2$, one has

$$\mathcal{X}_2(t)\mathcal{X}_1(\nu)\exp(w(x)) = \sum_{(\lambda,v)\in(\sigma(\mathbb{A}),\mathcal{V})}\sum_{(\lambda',v')\in(\sigma(\mathbb{A}),\mathcal{V})} \frac{\alpha\exp\left[\frac{-\alpha\lambda t}{1+\lambda(1-\alpha)} + \frac{-\alpha\lambda'\nu}{1+\lambda'(1-\alpha)}\right]}{\left[1+\lambda(1-\alpha)\right]^2(1+\sigma_l(1-\alpha))} \times v(x)\int_\Omega\int_\Omega \exp(w(y))v'(y)v(x)v'(x)\mathrm{d}y\mathrm{d}x.$$

Since $\mathcal{V}$ is an orthogonal basis of $L^2(\Omega)$, the above equality is reduced to

$$\mathcal{X}_2(t)\mathcal{X}_1(\nu)\exp(w(x)) = \sum_{(\lambda,v)\in(\sigma(\mathbb{A}),\mathcal{V})} \frac{\alpha\exp\left[\frac{-\alpha\lambda(t+\nu)}{1+\lambda(1-\alpha)}\right]}{\left[1+\lambda(1-\alpha)\right]^3} v(x)\int_\Omega \exp(w(x))v(x)\mathrm{d}x.$$

Thus, we can find the following estimate

$$\left\|\mathcal{X}_2(t)\mathcal{X}_1(\nu)\exp(w)\right\|^2_{\mathcal{H}^\beta(\Omega)} \lesssim \sum_{(\lambda,v)\in(\sigma(\mathbb{A}),\mathcal{V})} \lambda^{\beta-6}\left(\int_\Omega \exp(w(x))v(x)\mathrm{d}x\right)^2.$$

Obviously, $\beta < 6$, one thus deduce

$$\left\|\mathcal{X}_2(t)\mathcal{X}_1(\nu)\exp(w)\right\|_{\mathcal{H}^\beta(\Omega)} \lesssim \left\|\exp(w)\right\|_{L^2(\Omega)}.$$

And the proof is completed. □

3 Main results

Theorem 3.1 (Local existence and uniqueness). *Let w_0 be a measurable function such that $\exp(w_0) \in L^2(\Omega)$. Then, there exists a positive constant $T > 0$ such that Problem (1.1) possesses a unique mild solution $w \in L^\infty(0, T; L^\infty(\Omega))$.*

Proof. Let θ be a positive constant and $u(t) := w_1(t) + \theta$. Our aim is to show that u is an (uniform) upper bound of the sequence $\{w_j\}_{j=\overline{1,\infty}}$. We begin by the following observation

$$\exp\left(\mathcal{X}_1(t)w_0\right) \le \mathcal{X}_1(t)\exp(w_0), \tag{3.1}$$

which is easily deduced by Jensen's inequality. Now, we consider the effect of $\mathcal{I}$ on . Evidently, one has

$$\mathcal{I}\big(\exp(u(t))\big) = \gamma\exp(\theta)\int_0^t \mathcal{X}_2(t-\nu)\exp\left(\mathcal{X}_1(\nu)w_0\right)\mathrm{d}\nu.$$

From (3.1), we obtain

$$\mathcal{I}\big(\exp(u(t))\big) \lesssim \gamma\exp(\theta)\int_0^t \mathcal{X}_2(t-\nu)\mathcal{X}_1(\nu)\exp\left(w_0\right)\mathrm{d}\nu$$

Accordingly to the embedding $\mathcal{H}^\beta(\Omega) \hookrightarrow L^\infty(\Omega)$ and Lemma 2.4, the above estimate yields

$$\begin{aligned}\mathcal{I}\big(\exp(u(t))\big) &\lesssim \exp(\theta)\int_0^t \left\|\mathcal{X}_2(t-\nu)\mathcal{X}_1(\nu)\exp\left(w_0\right)\right\|_{\mathcal{H}^\beta(\Omega)}\mathrm{d}\nu \\ &\lesssim T\exp(\theta)\left\|\exp(w_0)\right\|_{L^2(\Omega)}.\end{aligned}$$

Then, if T is small such that the right hand side (RHS) of the above estimate is smaller than or equal to θ, one can deduce

$$w_1(t) + \mathcal{I}\big(\exp(u(t))\big) \le u(t), \tag{3.2}$$

for every $t \in [0, T]$. This estimate plays the most important role in our proof. Indeed, we can evidently see that $w_1(t) < u(t)$ for every $t \in [0, T]$. Suppose that $w_{k-1}(t) < u(t)$ for every $t \in [0, T]$, one can find that

$$\begin{aligned} w_k(t) &= w_1(t) + \mathcal{I}\big(\exp(w_{k-1}(t))\big) \\ &< w_1(t) + \mathcal{I}\big(\exp(u(t))\big), \end{aligned}$$

provided that w_0 is postive in the whole region Ω. Hence, w_k is also uniformly bounded from above by u. By induction, we can easily deduce

$$0 < w_1(t) \le w_2(t) \le \cdots \le w_k(t) \le \cdots \le u(t).$$

Consequently, we have proved $\{w_k\}_{k=\overline{1,\infty}}$ is an increasing sequence which is bounded from above for $(x, t) \in \mathbb{R}^N \times [0, T]$. And thus, $\{w_k\}_{k=\overline{1,\infty}}$ possesses a limit $\widetilde{w}$ which satisfies

$$\begin{aligned} \lim_{k\to\infty} w_k(t) &= \lim_{k\to\infty} \Big(w_1(t) + \mathcal{I}\big(\exp(w_{k-1}(t))\big) \Big) \\ &= w_1(t) + \mathcal{I}\big(\exp(\widetilde{w}(t))\big), \end{aligned}$$

by the dominated convergence theorem. □

Corollary 3.2. Let w_0 satisfy assumitions of Theorem 3.1 and $w \in L^\infty(0, T; L^\infty(\infty))$ be the mild solution of Problem (1.1) associated to w_0. Then, one can derive $w \in C([0, T]; L^\infty(\infty))$.

Proof. To make the proof precise we divide it into two parts.
Part 1. We show in this part that $w_1(t)$ is continuous from $[0, T]$ to $L^2(\Omega)$. To this end, for $t \in [0, T]$ and $\varepsilon > 0$ we use Parseval's inequality to find that

$$\begin{aligned} &\Big\| w_1(t+\varepsilon) - w(t) \Big\|^2_{\mathcal{H}^\beta(\Omega)} \\ &\quad = \sum_{(\lambda,v)\in(\sigma(\mathbb{A}),\mathcal{V})} \frac{\lambda^\beta \left| \exp\left(\frac{-\alpha\lambda(t+\varepsilon)}{1+\lambda(1-\alpha)}\right) - \exp\left(\frac{-\alpha\lambda t}{1+\lambda(1-\alpha)}\right) \right|^2}{(1+\lambda(1-\alpha))^2} \left(\int_\Omega w_0(x)v(x)\mathrm{d}x \right)^2 \\ &\quad \lesssim \sum_{(\lambda,v)\in(\sigma(\mathbb{A}),\mathcal{V})} \frac{\lambda^{2+\beta}\varepsilon^2}{(1+\lambda(1-\alpha))^4} \left(\int_\Omega w_0(x)v(x)\mathrm{d}x \right)^2, \end{aligned}$$

here we note that the inequality $|e^{-a} - e^{-b}| \lesssim |a - b|$ holds for any $a, b > 0$. Accordingly, one can find that

$$\Big\| w_1(t+\varepsilon) - w_1(t) \Big\|_{L^\infty(\Omega)} \lesssim \varepsilon \big\| w_0 \big\|_{L^2(\Omega)}. \tag{3.3}$$

Then, the assumption of $w_0 \in L^2(\Omega)$ implies the continuity of w_1.
Part 2. We can prove that $\mathcal{I}(\exp(w(t)))$ is continuous on $[0, T]$ with respect to L^∞−norm for any $w \in L^\infty(0, T; L^\infty(\Omega))$. Indeed, by the triangle inequality for $t \in [0, T]$ and $\varepsilon > 0$ we have

$$\begin{aligned} \Big\| \mathcal{I}(\exp(w(t+\varepsilon))) - \mathcal{I}(\exp(w(t))) \Big\|_{\mathcal{H}^\beta(\Omega)} &\lesssim \int_t^{t+\varepsilon} \Big\| \mathcal{X}_2(t+\varepsilon-\nu)\exp(w(\nu)) \Big\|_{\mathcal{H}^\beta(\Omega)} \mathrm{d}\nu \\ &+ \int_0^t \Big\| \big(\mathcal{X}_2(t+\varepsilon-\nu) - \mathcal{X}_2(t-\nu)\big)\exp(w(\nu)) \Big\|_{\mathcal{H}^\beta(\Omega)} \mathrm{d}\nu. \end{aligned} \tag{3.4}$$

We deal separately with two terms on the RHS of (3.4) as follows. For the first term, one observes that

$$\begin{aligned}\left\|\mathcal{X}_2(t)w(t)\right\|_{\mathcal{H}^\beta(\Omega)} &= \left(\sum_{(\lambda,v)\in(\sigma(\mathbb{A}),\mathcal{V})} \frac{\lambda^\beta \exp\left(\frac{-2\alpha\lambda t}{1+\lambda(1-\alpha)}\right)}{(1+\lambda(1-\alpha))^4}\left(\int_\Omega w(x,t)v(x)\mathrm{d}x\right)^2\right)^{\frac{1}{2}}\\ &\lesssim \left(\sum_{(\lambda,v)\in(\sigma(\mathbb{A}),\mathcal{V})}\left(\int_\Omega w(x,t)v(x)\mathrm{d}x\right)^2\right)^{\frac{1}{2}}\\ &\lesssim ess \sup_{t\in(0,T)} \|w(t)\|_{L^2(\Omega)}, \quad \text{for a.e. } t\in(0,T),\end{aligned}$$

for any $w \in L^\infty(0,T;L^\infty(\Omega))$. Consequently, we find that

$$\left\|\mathcal{X}_2(t)\exp(w(t))\right\|_{\mathcal{H}^\beta(\Omega)} \lesssim ess \sup_{t\in(0,T)} \|\exp(w(t))\|_{L^2(\Omega)} < \infty, \quad \text{for a.e. } t\in(0,T).$$

Based on this result, the first term on the RHS of (3.4) is treated as follows

$$\int_t^{t+\varepsilon}\left\|\mathcal{X}_2(t+\varepsilon-\nu)\exp(w(\nu))\right\|_{\mathcal{H}^\beta(\Omega)}\mathrm{d}\nu \lesssim \epsilon ess \sup_{t\in(0,T)} \|\exp(w(t))\|_{L^2(\Omega)}. \tag{3.5}$$

Before considering the other term on the RHS of (3.4), we see that

$$\left|\exp\left(\frac{-\alpha\lambda(t+\varepsilon)}{1+\lambda(1-\alpha)}\right) - \exp\left(\frac{-\alpha\lambda t}{1+\lambda(1-\alpha)}\right)\right| \lesssim \frac{\alpha\lambda\varepsilon}{1+\lambda(1-\alpha)},$$

for any $\lambda > 0$. Based on this estimate, we derive

$$\left\|(\mathcal{X}_2(t+\varepsilon)-\mathcal{X}_2(t))w\right\|^2_{\mathcal{H}^\beta(\Omega)} \lesssim \sum_{(\lambda,v)\in(\sigma(\mathbb{A}),\mathcal{V})} \frac{\lambda^{\beta+2}\varepsilon^2}{(1+\lambda(1-\alpha))^6}\left(\int_\Omega w(x,t)v(x)\mathrm{d}x\right)^2, \quad t>0.$$

Therefore, we obtain the following estimate for the second term on the RHS of (3.4)

$$\int_0^t\left\|(\mathcal{X}_2(t+\varepsilon-\nu)-\mathcal{X}_2(t-\nu))\exp(w(\nu))\right\|_{\mathcal{H}^\beta(\Omega)}\mathrm{d}\nu \lesssim \varepsilon\int_0^t \|\exp(w(\nu))\|_{L^2(\Omega)}\mathrm{d}\nu.$$

Since $w \in L^\infty(0,T;L^\infty(\Omega))$ we find that the integral term on the RHS of the above estimate is finite and deduce

$$\int_0^t\left\|(\mathcal{X}_2(t+\varepsilon-\nu)-\mathcal{X}_2(t-\nu))\exp(w(\nu))\right\|_{\mathcal{H}^\beta(\Omega)} \xrightarrow{\varepsilon\to 0} 0. \tag{3.6}$$

Then, combining (3.4), (3.5) and (3.6) yields

$$\left\|\mathcal{I}(\exp(w(t+\varepsilon)))-\mathcal{I}(\exp(w(t)))\right\|_{\mathcal{H}^\beta(\Omega)} \xrightarrow{\varepsilon\to 0} 0,$$

for any $t\in[0,T]$. This implies the main goal of Part 2. The proof is thus completed. □

Theorem 3.3 (Blow-up solution)**.** *Let (λ_1, v_1) be the first pair of the Helmholtz's equation such that*

$$\int_\Omega v(x)\mathrm{d}x = 1.$$

Suppose that w_0 is given in Theorem 3.1 and satisfies further that

$$\int_\Omega w_0(x)v_1(x)\mathrm{d}x > \frac{\lambda_1}{1+\lambda_1(1-\alpha)}.$$

Then, there exists a constant $T_{\max} < \infty$ such that the mild solution $w \in C([0,T_{\max});L^\infty(\Omega))$ to Problem (1.1) arising from w_0 is a blow-up solution.

Proof. Presume that w is the mild solution to Problem (1.1). We begin by deducing the following estimate

$$\begin{aligned}\int_\Omega \exp(w(x,t))v_1(x)\mathrm{d}x &\geq \int_\Omega w^2(x,t)v_1(x)\mathrm{d}x\\ &\geq \left(\int_\Omega w(x,t)v_1(x)\mathrm{d}x\right)^2,\end{aligned} \tag{3.7}$$

here we applied a simple inequality and Hölder's inequality with the note that

$$\int_\Omega v_1(x)\mathrm{d}x = 1.$$

Next, we recall the equation (2.1) with respect to (λ_1, v_1), as follows

$$\int_\Omega \left({}_{CF}D_t^\alpha w(x,t) + \lambda_1 w(x,t)\right)v_1(x)\mathrm{d}x = \gamma\int_\Omega \exp(w(x,t))v_1(x)\mathrm{d}x.$$

And, based on the above result and (3.7), we can find that

$$\begin{aligned}\int_\Omega w(x,t)v_1(x)\mathrm{d}x \geq\ & \frac{\exp\left(\frac{-\alpha\lambda_1 t}{1+\lambda_1(1-\alpha)}\right)}{1+\lambda_1(1-\alpha)}\int_\Omega w_0(x)v_1(x)\mathrm{d}x\\ &+\int_0^t \frac{\alpha\exp\left(\frac{-\alpha\lambda(t-\nu)}{1+\lambda(1-\alpha)}\right)}{(1+\lambda(1-\alpha))^2}\left(\int_\Omega w(x,t)v_1(x)\mathrm{d}x\right)^2\mathrm{d}x\mathrm{d}\nu,\end{aligned}$$

provided v_1 is nonnegative on Ω. This estimate is equivalent to the following result

$$\Im(t) \geq \langle w_0(\cdot)\rangle + \frac{\alpha}{(1+\lambda(1-\alpha))^3}\int_0^t \exp\left(\frac{-\alpha\lambda_1\nu}{1+\lambda(1-\alpha)}\right)\Im^2(\nu)\mathrm{d}\nu,$$

here we denote

$$\langle w(\cdot,t)\rangle := \int_\Omega w(x,t)v_1(x)\mathrm{d}x.$$

and

$$\Im(t) := (1+\lambda_1(1-\alpha))\exp\left(\frac{\alpha\lambda_1 t}{1+\lambda_1(1-\alpha)}\right)\langle w(\cdot,t)\rangle.$$

Then, suppose that $w \in C^1\left([0,T];L^2(\Omega)\right)$ we can find that

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{-1}{\Im(t)}\right) \geq \frac{\alpha}{(1+\lambda_1(1-\alpha))^3}\exp\left(\frac{-\alpha\lambda_1 t}{1+\lambda_1(1-\alpha)}\right),$$

by the fundamental theorem. As a consequence, one obtains

$$\Im(t) \geq \frac{\Im(0)\lambda_1(1+\lambda_1(1-\alpha))^2}{\lambda_1(1+\lambda_1(1-\alpha))^2 - \Im(0)\left(1-\exp\left(\frac{-\alpha\lambda_1 t}{1+\lambda_1(1-\alpha)}\right)\right)}. \tag{3.8}$$

Suppose that $\langle w_0 \rangle > \lambda_1(1+\lambda_1(1-\alpha))^{-1}$, we can find

$$T_{\max} = -\frac{1+\lambda_1(1-\alpha)}{\alpha\lambda_1} \log\left(\frac{\Im(0) - \lambda_1(1+\lambda_1(1-\alpha))^2}{\Im(0)}\right).$$

As a consequence of (3.8), one can deduce

$$\lim_{t\to T_{\max}^-} \langle w(\cdot,t)\rangle = \lim_{t\to T_{\max}^-} \frac{1}{1+\lambda_1(1-\alpha)} \exp\left(\frac{-\alpha\lambda_1 t}{1+\lambda_1(1-\alpha)}\right)\Im(t) = +\infty.$$

This result together with Parseval's equality implies

$$\lim_{t\to T_{\max}^-} \|w(\cdot,t)\|_{L^2(\Omega)} = +\infty.$$

In conclusion, if $\langle w_0 \rangle > \lambda_1(1+\lambda_1(1-\alpha))^{-1}$ we can deduce that either the solution w is not smooth or

$$\lim_{t\to T_{\max}^-} \|w(\cdot,t)\|_{L^\infty(\Omega)} = +\infty.$$

The theorem is thus proved. □

4 Declarations

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Competing Interests

The authors declare that they have no competing interests.

Ethical Approval

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Authors' Contributions

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Availability Data and Materials

Not applicable.

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