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Well-Posedness Results for a Class of Nonlinear Reaction-Diffusion Equations with Memory

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Abstract

In this paper, we are interested to study the initial value problem for nonlinear heat equation with memory. For the our problem, we show that the local existence theory related to the finite time blow-up is also obtained for the problem with logarithmic nonlinearity. We also obtain the blowup continuation property of the mild solution. The principal techniques are based on some Sobolev embeddings with L^p spaces.

Keywords: nonlinear heat equation, reaction-diffusion equations, memory term, blow-up
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1 Introduction

1.1 The problem

Let T be a positive number, $\Omega \subset \mathbb{R}^N$, $(N \geq 1)$ a bounded open set with boundary Ω^c, and $\Omega_T \equiv \Omega \times (0, T)$. The main aim of the paper is to study the nonlinear reaction-diffusion equation with a nonlocal-nonlinearity term of memory type. More precisely, we consider the

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following problem:

$$\begin{cases} u_t = \Delta u + \int_0^t g(t-s, u(s))ds + F(x,t;u), & \text{in } \Omega_T, \\ \frac{\partial u}{\partial \mathrm{n}}(x,t) = 0, & \text{on } \Omega^c \times (0,T), \end{cases} \tag{1.1}$$

supplemented with the initial condition

$$u(x,0) = u_0(x), \quad \text{in } \Omega, \tag{1.2}$$

where Δ is the standard Laplacian with Dirichlet boundary conditions in $L^2(\Omega)$, two functions u_0 and f are given. The condition $\frac{\partial u}{\partial \mathrm{n}}(x,t) = 0$ is called the Neumann boundary condition. The source g, F are given nonlinear functions which will be defined later. Mathematical formulations modeled by problem (1.1) appear in several engineering models such as nonlocal reactive porous media [9], radioactive nuclear decay in fluid flow [39], etc. Recently, PDEs with a nonlocal nonlinearity (involving memory terms) have received significant attention due to their ability to capture long-term correlations, for example, material properties and memory effects. Many important physical models and practical problems require the consideration of physical models with memory effects [12, 18, 21, 25, 30, 31, 34, 38, 41] and some corresponding engineering problems with power-law memory (nonlocal effects) [4, 7, 13, 20, 24, 35] and references therein.

The existing literature on nonlocal nonlinearities (involving memory terms) is so separated. In 1988, Heard and Rakin [26] studied the following problem

$$\begin{cases} \partial_t u + A(t)u(t) = \int_0^t R(t,\tau)F(u(\tau))d\tau, & t \in (0,T), \\ u(0) = u_0. \end{cases} \tag{1.3}$$

Wave equations with memory with weak or strong damping see e.g. [13, 20, 24, 32, 33], plate equations with memory [28, 32, 33] and Navier-Stokes equation with memory [4, 7, 35]. For nonlinear perturbations that grow like $|u|^{p-1}u$ for $p \geq 1$, B. Andrade et al. [15] considered the parabolic model with memory

$$u_t = Au + \int_0^t g(t-s, u(s))ds + |u|^{p-1}u, \quad t > 0, \tag{1.4}$$

where $A : D(A) \subset X_0 \to X_0$ (X_0 is a Banach space) be a linear operator and $-A$ be the sectorial operator. The authors are concerned with the well-posedness of the mild solutions of (1.4). Cazenave et al. [8] considered the equation

$$u_t - \Delta u = \int_0^t (t-s)^{-\alpha}|u(s)|^{p-1}u(s)ds, \tag{1.5}$$

and proved sharp blow-up properties of the solutions and global existence results. The power-type model $|u|^{p-1}u$ for $p \geq 1$ has also been considered in [2, 3, 6, 14, 16, 22, 23] and references therein.

The main contributions of our article is to the study of local existence, unique continuation and a finite time blow-up of solution to Problem (1.1)-(1.2) where the soure function F is logarithmic form as follows

$$F(x,t;u) = V_{\eta,p}(u)\log|u|, \quad V_{\eta,p}(u) = \eta|u|^{p-1}u. \tag{1.6}$$

New and interesting novel in the model (1.1)-(1.2) is the appearance of $\log|u|$ in the source function quantity, and so, our problem becomes much more difficult. We found the problem with memory term and the source function class of the nonlinear powers of $|u|^{p-1}u$ (no logarithmic term) has been studied rarely in a number of interesting articles, e.g. B. Andrade et al. [15], Cazenave et al. [8]. Firstly, for the sake of providing our motivation for investigating problem (1.1)-(1.2) with the nonlinearity of logarithmic type, let us recall some well-known information about the power-type nonlinearity of the form $F(u) = |u|^{p-1}u$. The behavior of a solution to a considered Cauchy problem may be not the same when we study different cases of $F(u)$, especially when $p = 1$ and when $p > 1$. More precisely, taking the usual heat equation into account, when $F(u) = u$, it can be seen as a global Lipschitzian. Hence, the solution to the problem can be proved globally well-posed (in almost reasonable Banach spaces). On the other hand, when we consider the case $p > 1$, our solution may often blow up at a finite time. In view of these facts, the effects on the solution by the nonlinearity of logarithmic type whose expansion at infinity can be approximated as the case p near 1. In [10], the authors show that the solution to a Cauchy problem involving a heat equation with logarithmic nonlinearity blows up at the infinity time-point. Inspired by their spirit, we decided to find an answer to the question: How does a mild solution to problem $(\mathbb{P}_0)$ with the nonlinearity of logarithmic type in the generalized form $F(u) = |u|^{p-1}u\log|u|$ behave? Likewise, considering the application aspect, the logarithmic nonlinearity expresses slowly cumulative nonlinearity, providing thus another kind of description of the dynamic process. Studies of logarithmic nonlinearity have a long history in physics as it occurs naturally in inflation cosmology, quantum mechanics, and nuclear physics and has attracted many authors with many interesting results, for instance, [5, 11, 36, 40] and the references therein. The solution operator of our problem contains nonlocal-nonlinearity which makes some difficult techniques for our paper.

Let us discuss the main difficulties and our contribution to this paper. For our problem (1.1)-(1.2) with logarithmic nonlinearity source terms, our approach is to consider directly the mild solutions to Problem (1.1)-(1.2) taking the form of the integral equations. These equations contain the nonlinear types of source functions $F(x,t;u)$ and nonlocal type of memory. Our goal is to achieve properties of the solutions to Problem (1.1)-(1.2) in Sobolev spaces $W^{s,q}(\Omega)$, for

$$0 \le s < 2\nu_2, \quad 0 \le \nu_2 < \min\left\{1; \frac{N}{4}\right\}$$

and $1 \le q \le p^*$ with

$$p^* = \frac{2N}{N+2s-4\nu_2}, \; 2\nu_2 - s < \frac{N}{2}$$

such that

$$qs < N, q < \frac{\gamma m N}{N+\gamma m s}, \quad q \le \frac{p\gamma m N}{2N + p\gamma m s + Np},$$

for some $1 \le \gamma \le p$. Indeed, We describe in more detail as follows.

- The convolution (or memory) term $\int_0^t g(t-s,u(s))ds$, where g grows like in (H_1) and (H_2) are quite difficult to control and change the qualitative aspects of the problem (because important concepts of Kato and Fujita do not encounter for the problem (1.1)).

- The logarithmic nonlinearity source terms $F(x,t;u) = V_{\eta,p}(u)\log|u|$ and $V_{\eta,p}(u) = \eta|u|^{p-1}u$, with $\eta > 0$ and $p \geq 1$. To present the properties of the solutions in the Sobolev space $W^{s,q}(\Omega)$, we need to consider the local Lipschitz properties of this source function. Based on the conditions of the constants $s > 0, q \geq 1$ depending on the dimensions $N \geq 1$, we set up the Sobolev embeddings

$$D(\mathcal{A}^{\nu}) \hookrightarrow W^{2\nu,2}(\Omega) \hookrightarrow W^{s,q}(\Omega), \quad \text{for } 2\nu \geq s,\ 2\nu - s < \frac{N}{2}$$

and

$$1 \leq q < \frac{2N}{N+2s-4\nu},$$

see the definitions of the spaces $W^{s,q}(\Omega)$ and $D(\mathcal{A}^{\nu})$ in (2.2) and (2.3) below.

1.2 Outline of the paper

The notations and the functional setting are introduced in Subsection 2.1 and in Subsection 2.2 we give some related results. The main results of this paper are in Section 3, we present local existence in Subsection 3.1, uniqueness continuation of the solution is discussed in Subsection 3.2 and a finite time blow-up result is demonstrated in Subsection 3.3.

2 Preliminaries

2.1 Functional setting and notation

Throughout the paper, $C > 0$ C will represent a generic constant that is independent of t. Let us define

$$\mathbb{V} = \left\{\varphi \in H^1(\Omega),\ \frac{\partial\varphi}{\partial n} = 0 \quad \text{on } \partial\Omega\right\}, \tag{2.1}$$

the closed subspace of $C_c^1(\Omega)$ in $H^1(\Omega)$. The eigenvalues of the operator $-\Delta$ on the open, bounded and connected domain $\Omega \subset \mathbb{R}^N$ with a smooth boundary, subject to standard homogeneous boundary conditions (here we specify the zero Neumann conditions in (1.1)) have the property that there exists an orthonormal basis of $L^2(\Omega)$, denoted by $\{\phi_j\}_{j\in\mathbb{N}}$, satisfying

$$\begin{cases} -\Delta\varphi_j(x) = \lambda_j\varphi_j(x), & \text{in } \Omega, \\ \varphi_j \in \mathbb{V}\cap C^{\infty}(\overline{\Omega}), \end{cases} \qquad \text{(see [19], Section 6.5)},$$

admits a family of eigenvalues $0 < \lambda_1 \leq \lambda_2 \leq \lambda_3 \leq \ldots \leq \lambda_j \leq \ldots \nearrow \infty$, where $\{\lambda_j\}_{j\in\mathbb{N}}$ is the discrete spectrum of the operator. The notation $\|\cdot\|_B$ stands for the norm in the Banach space B. For $1 \leq p \leq \infty$, $T > 0$, consider the Banach space of real-valued measurable functions $f : (0,T) \to B$ with the following norm

$$\|f\|_{L^p(0,T;B)} = \left(\int_0^T \|f(t)\|_B^p dt\right)^{\frac{1}{p}}, \quad \text{for } 1 \leq p < \infty,$$

and

$$\|f\|_{L^p(0,T;B)} = ess \sup_{t\in(0,T)} \|f(t)\|_B, \quad \text{for } p = \infty.$$

Given a Banach space B, let $C^k([0,T];B)$ be the set of all continuous functions and its derivatives of order less than or equal to k are uniformly continuous which map $[0,T]$ into B. The norm of the function space $C^k([0,T];B)$, for $0 \le k \le \infty$ is denoted by

$$\|f\|_{C^k([0,T];B)} = \sum_{i=0}^{k} \sup_{t\in[0,T]} \left\|f^{(i)}(t)\right\|_B < \infty.$$

For any real numbers $s > 0$ and $1 \le p < \infty$, we recall the fractional Sobolev type spaces with $W^{s,p}(\Omega)$ via the Gagliardo approach (also known as Aronszajn or Slobodeckij space). Fix a number $s \in (0,1)$ and $p \in [1,\infty)$, define $W^{s,p}(\Omega)$ as follows

$$W^{s,p}(\Omega) = \left\{ f \in L^p(\Omega) \quad \text{s.t.} \quad \frac{|f(x)-f(y)|}{|x-y|^{\frac{N+ps}{p}}} \in L^p(\Omega\times\Omega) \right\}. \tag{2.2}$$

For $0 < s < 1$, it can be said that $W^{s,p}(\Omega)$ is an intermediate Banach space between $L^p(\Omega)$ and $W^{1,p}(\Omega)$, endowed the corresponding norm

$$\|f\|_{W^{s,p}(\Omega)} = \left(\int_\Omega |f|^p dx + \int_\Omega \int_\Omega \frac{|f(x)-f(y)|^p}{|x-y|^{N+sp}} dxdy \right)^{\frac{1}{p}},$$

where the seminorm

$$|f|_{W^{s,p}(\Omega)} = \left(\int_\Omega \int_\Omega \frac{|f(x)-f(y)|^p}{|x-y|^{N+sp}} dxdy \right)^{\frac{1}{p}},$$

denotes the Gagliardo (semi)norm of f. For $p = 2$ in (2.2), together with the norm $\|\cdot\|_{W^{s,2}(\Omega)}$ the space $W^{s,2}(\Omega)$ becomes a Hilbert space. Let us also set $W_0^{s,2}(\Omega) = \overline{C_c^\infty(\Omega)}^{W^{s,2}(\Omega)}$. It has long been known that if Ω is bounded then we have the following continuous embeddings:

$$W_0^{s,2}(\Omega) \hookrightarrow \begin{cases} L^{\frac{2N}{N-2s}}(\Omega), & \text{if} \quad s < \frac{N}{2}; \\ L^p(\Omega), & \text{if} \quad s = \frac{N}{2}; \\ C^{0,s-\frac{N}{2}}(\Omega), & \text{if} \quad s > \frac{N}{2}, \end{cases}$$

where we denote $C^{0,\beta}(\Omega)$ be the space of locally uniformly Hölder continuous functions in Ω with exponent β. For more details about the fractional Sobolev spaces, see [1, 17] and the references in it.

For each number $\sigma \ge 0$, we define

$$D(\mathcal{A}^\sigma) := \left\{ f(x) = \sum_{j\in\mathbb{N}^*} (f,\varphi_j)\,\varphi_j(x) \in L^2(\Omega) : \|f\|^2_{D(\mathcal{A}^\sigma)} = \sum_{j\in\mathbb{N}^*} (f,\varphi_j)^2 \lambda_j^{2\sigma} < \infty \right\}, \tag{2.3}$$

where $(\cdot,\cdot)$ denotes the inner product in $L^2(\Omega)$. Let us denote by $H^\sigma(\Omega)$ the Sobolev-Slobodecki space $W^{\sigma,p}(\Omega)$ when $p = 2$, and by $H_0^\sigma(\Omega)$ is the closure of $C_c^\infty(\Omega)$ in $H^\sigma(\Omega)$. We identify the dual space $D(\mathcal{A}^{-\sigma}) = [D(\mathcal{A}^\sigma)]^*$ which corresponds to $(\cdot,\cdot)_{-\sigma,\sigma}$ be the inner product (dual type) between $D(\mathcal{A}^\sigma)$ and $D(\mathcal{A}^{-\sigma})$ and

$$\mathcal{A}^{-\sigma} f(x) = \sum_{j\in\mathbb{N}^*} \varphi_j(x)\,(f,\varphi_j)_{-\sigma,\sigma}\,\lambda_j^{-\sigma},$$

and
$$\|f\|^2_{D(\mathcal{A}^{-\sigma})} = \sum_{j\in\mathbb{N}^*} (f,\varphi_j)^2_{-\sigma,\sigma}\lambda_j^{-2\sigma} < \infty.$$

In the next lemmas, we present some useful embeddings between the previously mentioned spaces.

Lemma 2.1. *Let $0 \le \sigma \le \sigma' \le 2$, then the following embeddings hold*
$$D(\mathcal{A}^{\sigma}) \hookrightarrow L^2(\Omega) \hookrightarrow D(\mathcal{A}^{-\sigma}),$$
and
$$D(\mathcal{A}^{\sigma'}) \hookrightarrow D(\mathcal{A}^{\sigma}) \hookrightarrow H^{2\sigma}(\Omega) \hookrightarrow L^2(\Omega) \hookrightarrow H^{-2\sigma}(\Omega) \hookrightarrow D(\mathcal{A}^{-\sigma}) \hookrightarrow D(\mathcal{A}^{-\sigma'}).$$

Lemma 2.2. *(See [1, 17]) Let $\Omega \subset \mathbb{R}^N$, $k,s \in \mathbb{N}$ with $k \ge s$ satisfying $(k-s)p < N$ and $1 \le p < \infty$. Then the following Sobolev embeddings hold:*
$$\begin{cases} W^{k,p}(\Omega) \hookrightarrow W^{s,q}(\Omega), & \text{for} \quad 1 \le q < \frac{Np}{N-(k-s)p}, \\ D(\mathcal{A}^{\sigma}) \hookrightarrow W^{2\sigma,2}(\Omega), & \text{for} \quad \sigma > 0, \\ L^p(\Omega) \hookrightarrow D(\mathcal{A}^{\sigma}), & \text{for} \quad -\frac{N}{4} < \sigma \le 0, \quad p \ge \frac{2N}{N-4\sigma}, \\ D(\mathcal{A}^{\sigma}) \hookrightarrow L^p(\Omega), & \text{for} \quad 0 \le \sigma < \frac{N}{4}, \quad p \le \frac{2N}{N-4\sigma}. \end{cases}$$

2.2 Some related results

Lemma 2.3. *a) (See [29], Lemma 2.1) For $s > 0$, then there is a constant $\ell > 0$ depending on γ, we have*
$$\begin{cases} \log|s| \le \ell s^{-\gamma}, & \text{for} \quad \gamma > 0,\ 0 < s < 1, \\ \log|s| \le \ell s^{\gamma}, & \text{for} \quad \gamma > 0,\ s \ge 1. \end{cases} \tag{2.4}$$

b) (Hölder's inequality for negative exponents, see [27]) Let $k' < 0$, and $k \in \mathbb{R}$ be such that $\frac{1}{k'} + \frac{1}{k} = 1$. For $f(x) > 0, g(x) \ge 0$, $\forall x \in \Omega$ are Lebesgue measurable functions. Then
$$\int_\Omega fg dx \ge \left(\int_\Omega |f|^{k'} dx\right)^{\frac{1}{k'}} \left(\int_\Omega |g|^{k} dx\right)^{\frac{1}{k}},$$
which is equivalent to,
$$\left(\int_\Omega fg dx\right)^{k'} \le \left(\int_\Omega |f|^{k'} dx\right) \left(\int_\Omega |g|^{k} dx\right)^{\frac{k'}{k}}, \quad \text{for } k' < 0,\ \frac{1}{k'} + \frac{1}{k} = 1. \tag{2.5}$$

Proof. The proof of inequalities (2.4) and (2.5) are elementary, so we omit them here. □

Lemma 2.4. *(See [37]) For two real numbers a, b we have*
$$\int_a^b (b-s)^{m-1}(s-a)^{n-1} ds = (b-a)^{m+n-1} \mathrm{B}(m,n),$$
where $\mathrm{B}(m,n)$ is Beta function and
$$\mathrm{B}(m,n) = \frac{\Gamma(m)\Gamma(n)}{\Gamma(m+n)}.$$

Lemma 2.5. *For $0 \le s \le t \le T$, and $q \le 0 \le p, \mu \in (0,1)$ such that $p-q < 1-\mu$, we have the following*

$$a) \quad \left\| e^{\Delta(t-s)} f \right\|_{D(\mathcal{A}^p)} \le C_{\mu,p,q}(t-s)^{\mu-1} \|f\|_{D(\mathcal{A}^q)}, \quad \text{for } f \in D(\mathcal{A}^q);$$

$$b) \quad \left\| e^{\Delta(t-s)} \int_0^s f(\tau)d\tau \right\|_{D(\mathcal{A}^p)} \le C_{\mu,p,q}(t-s)^{\mu-1} s^{\frac{1}{2}} \left(\int_0^s \|f(\cdot,\tau)\|^2_{D(\mathcal{A}^q)} d\tau \right)^{1/2},$$

for $f \in C([0,T]; D(\mathcal{A}^q))$.

Proof.
Proof *a*. Since $0 < \mu < 1$, we know that there is a constant $C_\mu > 0$ and for any $z > 0$, we have $e^{-z} \le C_\mu z^{\mu-1}$. Let us take the constants p, q such that $q \le 0 \le p$. Let us choose $\mu > 0$ small enough such that $p - q < 1 - \mu$. Then

$$\begin{aligned} \left\| e^{\Delta(t-s)} f \right\|^2_{D(\mathcal{A}^p)} &= \sum_{j \in \mathbb{N}^*} \lambda_j^{2p} \left(e^{-\lambda_j(t-s)} (f, \varphi_j) \right)^2 \\ &\le C_\mu^2 (t-s)^{2\mu-2} \sum_{j\in\mathbb{N}^*} \lambda_j^{p-q+2\mu-2} (f,\varphi_j)^2 \lambda_j^{2q} \\ &\le \frac{C_\mu^2}{\lambda_1^{2+2q-2p-2\mu}} (t-s)^{2\mu-2} \|f\|^2_{D(\mathcal{A}^q)} \\ &\le C_{\mu,p,q}(t-s)^{2\mu-2} \|f\|^2_{D(\mathcal{A}^q)}. \end{aligned} \tag{2.6}$$

Proof *b*. By a similar argument,

$$\begin{aligned} \left\| e^{\Delta(t-s)} \int_0^s f(\tau)d\tau \right\|^2_{D(\mathcal{A}^p)} &= \sum_{j\in\mathbb{N}^*} \lambda_j^{2p} \left(e^{-\lambda_j(t-s)} \int_0^s (f(\cdot,\tau),\varphi_j)\, d\tau \right)^2 \\ &\le C_\mu^2 (t-s)^{2\mu-2} \sum_{j\in\mathbb{N}^*} \lambda_j^{2p-2q+2\mu-2} \lambda_j^{2q} \left(\int_0^s (f(\cdot,\tau),\varphi_j)\, d\tau \right)^2 \\ &\le \frac{C_\mu^2}{\lambda_1^{2+2q-2p-2\mu}} (t-s)^{2\mu-2} s \int_0^s \|f(\cdot,\tau)\|^2_{D(\mathcal{A}^q)} d\tau \\ &\le C_{\mu,p,q}(t-s)^{2\mu-2} s \int_0^s \|f(\cdot,\tau)\|^2_{D(\mathcal{A}^q)} d\tau. \end{aligned} \tag{2.7}$$

The proof of this lemma is completed. □

3 Local well-posedness results

In this section, we consider the semilinear Volterra integrodifferential equation is given by

$$\begin{cases} u_t = \Delta u + \displaystyle\int_0^t g(t-s, u(s))ds + V_{\eta,p}(u)(s) \log|u(s)|, & \text{in } \Omega_T, \\ \frac{\partial u}{\partial n}(x,t) = 0, & \text{on } \Omega^c \times (0,T), \\ u(x,0) = u_0(x), & \text{in } \Omega. \end{cases} \tag{$\mathbb{P}_0$}$$

The nonlinear function $V_{\eta,p}(u) = \eta|u|^{p-1}u$ for $p \ge 1, \eta > 0$. We assume that g grows like

$$|g(u) - g(v)| \le k_1 t^{-\alpha} |u-v|(1 + |u|^{p-1} + |v|^{redp-1}), \quad t > 0, \tag{H_1}$$

and

$$|g(u)| \leq k_1 t^{-\alpha}|u|(1+|u|^{p-1}), \quad t>0, \tag{H_2}$$

for $\alpha \in (0,1), k_1>0, p \geq 1$.

Lemma 3.1. *For $V_{\eta,p}(u)=\eta|u|^{p-1}u,\ p\geq 1, \eta>0$. For $u,v \in \mathbb{R}$, there are the positive constants c, k_2 such that*

$$\left|V_{\eta,p}(u)-V_{\eta,p}(v)\right| \leq c(1+|u|^{p-1}+|v|^{p-1})|u-v|, \tag{H_3}$$

and

$$\begin{aligned}\left|V_{\eta,p}(u)\log|u|-V_{\eta,p}(v)\log|v|\right| \leq k_2\left(\left|\log|u|\right||u-v|+|u|^{p-1}\left|\log|u|\right||u-v|\right)\\ +k_2\left(|v|^{p-1}\left|\log|u|\right||u-v|+|v|^{p-1}|u-v|\right).\end{aligned} \tag{H_4}$$

Proof. For proof, see Appendix [A1, Section 4.1].

Definition 3.2. A function u be a mild solution of Problem ($\mathbb{P}_0$) if $u \in C([0,T];L^2(\Omega))$ and satisfying

$$\begin{aligned}u(t)=e^{\Delta t}u_0+\int_0^t e^{\Delta(t-s)}\int_0^s g(s-\tau,u(\tau))d\tau ds\\ +\int_0^t e^{\Delta(t-s)}V_{\eta,p}(u)(s)\log|u(s)|ds.\end{aligned} \tag{3.1}$$

for all $0<t<T,\ p\geq 1,\ \eta>0$.

3.1 Local existence of solutions to Problem ($\mathbb{P}_0$)

Theorem 3.3 (Local existence). *For $N \geq 1, p \geq 1, -\frac{N}{4} < \nu_1 \leq 0 \leq \nu_2 < \min\left\{1;\frac{N}{4}\right\}$ and for $0 \leq s < 2\nu_2$ and $m=\frac{2N}{N-4\nu_1}$. Let $1 \leq q \leq p^*$ with $p^*=\frac{2N}{N+2s-4\nu_2}$ such that*

$$qs<N, q<\frac{\gamma m N}{N+\gamma m s}, q \leq \frac{p\gamma m N}{2N+p\gamma m s+Np},$$

for some $1 \leq \gamma \leq p$. If $u_0 \in D(\mathcal{A}^{\nu_2}) \cap W^{s,q}(\Omega)$, then there is a time $T>0$ (depending only on u_0) such that Problem ($\mathbb{P}_0$) possesses a unique mild solution $u \in C([0,T];W^{s,q}(\Omega))$.

Proof. For $N \geq 1, p \geq 1$, we put

$$0 \leq \nu_2 < \min\left\{1;\frac{N}{4}\right\}, \quad \nu_1 = p\nu_2-\iota^*, \tag{3.2}$$

$$s \in \mathbb{N} \text{ satisfies } s<2\nu_2, \quad 1 \leq q \leq \min\left\{p^*;\frac{\gamma N}{N+\gamma s}\right\}, \quad sq<N, \quad 1 \leq \gamma \leq p, \tag{3.3}$$

$$\max\left\{p\nu_2;\mathcal{T}(\phi_1,\phi_2)\right\} < \iota^* \leq \min\left\{p\nu_2+\frac{N}{4};1+(p-1)\nu_2\right\}, \tag{3.4}$$

where $p^*=\frac{2N}{N+2s-4\nu_2}$, and let $\mathcal{T}(\phi_1,\phi_2)$ be defined by

$$\mathcal{T}(\phi_1,\phi_2)=\frac{N(2p\phi_1-q\phi_2)+2pq(2\nu_2\phi_2-s\phi_1)}{4q\phi_2}, \tag{3.5}$$

with the pairs (ϕ_1, ϕ_2) satisfying

$$(\phi_1, \phi_2) \in \left\{(1,1), (\gamma, 1), (p-1, 1), (\gamma, p-1), (p, p-1), \left(\frac{1}{p}, 1\right)\right\}.$$

The following lemmas are necessary for the proof of this theorem.

Lemma 3.4. *We have the following Sobolev embeddings:*
a) For $-\frac{N}{4} < \nu_1 \leq 0$, *we have the Sobolev embedding*

$$L^{\frac{2N}{N-4\nu_1}}(\Omega) \hookrightarrow D(\mathcal{A}^{\nu_1}). \tag{3.6}$$

b) For $0 \leq \nu_2 < \frac{N}{4}, 0 \leq s < 2\nu_2, 1 \leq q < p^* = \frac{2N}{N+2s-4\nu_2}$ *then we get*

$$D(\mathcal{A}^{\nu_2}) \hookrightarrow W^{2\nu_2,2}(\Omega) \hookrightarrow W^{s,q}(\Omega). \tag{3.7}$$

c) For $sq < N$ *and* $1 \leq r \leq q_s^*$ *for* $q_s^* = \frac{qN}{N-sq}$, *we have the following Sobolev embedding*

$$W^{s,q}(\Omega) \hookrightarrow L^r(\Omega). \tag{3.8}$$

Proof of Lemma 3.4. For proof, see Appendix [A2, Section 4.2].

Lemma 3.5. *For g satisfies the hypotheses* (H_1) *and* (H_2). *For* $1 < m < 2$, $p \geq 1$ *and for all* $0 \leq \tau \leq s$, *we have the following*

$$a) \quad \|g(s-\tau, u(\tau))\|_{L^m(\Omega)} \leq C\tau^{-\alpha}\left(\|u(\tau)\|_{L^m(\Omega)} + \|u(\tau)\|^p_{L^{pm}(\Omega)}\right), \tag{3.9}$$

and

$$\begin{aligned} b) \quad \Big\|g(s-\tau, u(\tau)) - g(s-\tau, v(\tau))\Big\|_{L^m(\Omega)} &\leq C\tau^{-\alpha}\|u(\tau) - v(\tau)\|_{L^m(\Omega)} \\ &\quad + C\tau^{-\alpha}\|u(\tau)\|^{p-1}_{L^{pm}(\Omega)}\|u(\tau) - v(\tau)\|_{L^{pm}(\Omega)} \\ &\quad + C\tau^{-\alpha}\|v(\tau)\|^{p-1}_{L^{pm}(\Omega)}\|u(\tau) - v(\tau)\|_{L^{pm}(\Omega)}. \end{aligned} \tag{3.10}$$

Proof of Lemma 3.5. For proof, see Appendix [A3, Section 4.3].

Lemma 3.6. *For g satisfies the hypotheses* (H_3) *and* (H_4). *For* $p \geq 1$, $1 < m < 2$, *we have the following*

$$a) \quad \|V_{\eta,p}(u)\log|u|\|_{L^m(\Omega)} \leq C\left(\|u\|^{-\gamma}_{L^\gamma(\Omega)} + \|u\|^\gamma_{L^{p\gamma m}(\Omega)}\right)\|u\|^p_{L^{\frac{p^2 m}{p-1}}(\Omega)}, \tag{3.11}$$

and

$$\begin{aligned} b) \quad \Big\|V_{\eta,p}(u)\log|u| - V_{\eta,p}(v)\log|v|\Big\|_{L^m(\Omega)} &\leq C\left(\|u\|^{-\gamma}_{L^\gamma(\Omega)} + \|u\|^\gamma_{L^{\frac{p\gamma m}{p-1}}(\Omega)}\right)\|u-v\|_{L^{pm}(\Omega)} \\ &\quad + C\left(\|u\|^{-\gamma}_{L^\gamma(\Omega)} + \|u\|^\gamma_{L^{\frac{p\gamma m}{p-1}}(\Omega)}\right)\|u\|^{p-1}_{L^{p(p-1)m}(\Omega)}\|u-v\|_{L^{pm}(\Omega)} \\ &\quad + C\|v\|^{p-1}_{L^{pm}(\Omega)}\|u-v\|_{L^{pm}(\Omega)}. \end{aligned} \tag{3.12}$$

Proof of Lemma 3.6. For proof, see Appendix [A4, Section 4.4].

Now we are ready to prove Theorem 3.3. Let $T > 0$ and $\rho > 0$ be chosen later and we consider the functional space

$$\mathscr{A}_\rho := \left\{ u \in C([0,T]; W^{s,q}(\Omega)) : u(\cdot,0) = u_0, \text{ and } \|u - u_0\|_{C([0,T];W^{s,q}(\Omega))} \le \rho \right\}, \tag{3.13}$$

and without the restriction of generality we can assume there exists the constant $\epsilon(\rho)$ such that

$$\sup_{t\in[0,T]} \|u(\cdot,t)\|_{W^{s,q}(\Omega)} \ge \epsilon(\rho). \tag{3.14}$$

From now on, we set $P_{0,\rho} = \rho + \|u_0\|_{W^{s,q}(\Omega)}$, and we define the mapping $\mathbf{X}$ on $\mathscr{A}_\rho$ by

$$\begin{aligned}\mathbf{X}u(t) &= e^{\Delta t}u_0 + \int_0^t e^{\Delta(t-s)} \int_0^s g(s-\tau, u(\tau))d\tau ds \\ &\quad + \int_0^t e^{\Delta(t-s)} V_{\eta,p}(u)(s) \log|u(s)|ds \\ &:= e^{\Delta t}u_0 + J_1(u)(t) + J_2(u)(t).\end{aligned} \tag{3.15}$$

we show that $\mathbf{X}$ is invariant in $\mathscr{A}_\rho$ and $\mathbf{X}$ is a contraction.

- *Claim I: If $u_0 \in D(\mathcal{A}^{\nu_2}) \cap W^{s,q}(\Omega)$, then $\mathbf{X}$ is $\mathscr{A}_\rho$-invariant*. In fact, we have

$$\begin{aligned}\left\|e^{\Delta t}u_0 - u_0\right\|^2_{D(\mathcal{A}^{\nu_2})} &= \left\|(e^{\Delta t} - I)u_0\right\|^2_{D(\mathcal{A}^{\nu_2})} \\ &= \sum_{j\in\mathbb{N}^*} (u_0,\varphi_j)^2 \left(1 - e^{-\lambda_j t}\right)^2 \lambda_j^{2\nu_2} \\ &\le \sum_{j\in\mathbb{N}^*} (u_0,\varphi_j)^2 \lambda_j^{2\nu_2} \le \|u_0\|^2_{D(\mathcal{A}^{\nu_2})}, \quad \forall t \in [0,T].\end{aligned} \tag{3.16}$$

From (3.7) we have that $D(\mathcal{A}^{\nu_2}) \hookrightarrow W^{s,q}(\Omega)$ implies $\|\cdot\|_{W^{s,q}(\Omega)} \le C_{s,q,\nu_2}\|\cdot\|_{D(\mathcal{A}^{\nu_2})}$ and then, we conclude from (3.16) that

$$\left\|e^{\Delta t}u_0 - u_0\right\|_{C([0,T];W^{s,q}(\Omega))} \le C_{s,q,\nu_2}\|u_0\|_{D(\mathcal{A}^{\nu_2})} \le C\|u_0\|_{D(\mathcal{A}^{\nu_2})}, \quad t \in [0,T].$$

From (3.6), this implies that $0 < \nu_2 - \nu_1 \le 1$ and we assume that there exists $\mu > 0$, small enough, such that $\nu_2 - \nu_1 \le 1 - \mu$. Thanks to Lemma 2.5b),

$$\begin{aligned}\|J_1(u)(t)\|_{D(\mathcal{A}^{\nu_2})} &\le \int_0^t \left\|e^{\Delta(t-s)} \int_0^s g(s-\tau,u(\tau))d\tau\right\|_{D(\mathcal{A}^{\nu_2})} ds \\ &\le C_{\mu,p,q}\int_0^t (t-s)^{\mu-1}s^{\frac12}\left(\int_0^s \|g(s-\tau,u(\tau))\|^2_{D(\mathcal{A}^{\nu_1})}d\tau\right)^{\frac12} ds.\end{aligned} \tag{3.17}$$

Taking $m = \frac{2N}{N-4\nu_1}$ and combining with (3.6), we obtain $L^m(\Omega) \hookrightarrow D(\mathcal{A}^{\nu_1})$ which implies $\|\cdot\|_{D(\mathcal{A}^{\nu_2})} \le C_{N,\nu_2}\|\cdot\|_{L^m(\Omega)}$, and we have for $t \in [0,T]$

$$\begin{aligned}\text{The (RHS) of (3.17)} &\le C_{\mu,p,q}C_{N,\nu_2}\int_0^t (t-s)^{\mu-1}s^{\frac12}\left(\int_0^s \|g(s-\tau,u(\tau))\|^2_{L^m(\Omega)}d\tau\right)^{\frac12} ds \\ &\le C\int_0^t (t-s)^{\mu-1}s^{\frac12}\left(\int_0^s \|g(s-\tau,u(\tau))\|^2_{L^m(\Omega)}d\tau\right)^{\frac12} ds.\end{aligned} \tag{3.18}$$

From Lemma 3.5 a), we conclude that

$$\|g(s-\tau,u(\tau))\|_{L^m(\Omega)} \leq C\tau^{-\alpha}\left(\|u(\tau)\|_{L^m(\Omega)} + \|u(\tau)\|^p_{L^{pm}(\Omega)}\right). \tag{3.19}$$

From (3.2)-(3.5) it follows for $\iota^* > \mathcal{T}(1,1)$ that

$$\begin{aligned} pm = \frac{2pN}{N-4\nu_1} = \frac{2pN}{N+4\iota^*-4p\nu_2} &< \frac{2pN}{N+4\mathcal{T}(1,1)-4p\nu_2} \\ &= \frac{qN}{N-qs} := q_s^*, \end{aligned}$$

and for $\iota^* > \mathcal{T}(\frac{1}{p},1)$, we have the following bound

$$\begin{aligned} m = \frac{2N}{N-4\nu_1} = \frac{2N}{N+4\iota^*-4p\nu_2} &< \frac{2N}{N+4\mathcal{T}(\frac{1}{p},1)-4p\nu_2} \\ &= \frac{Nq}{N-sq} = q_s^*, \end{aligned}$$

then we deduce from (3.8) that the following Sobolev embeddings are fulfilled

$$W^{s,q}(\Omega) \hookrightarrow L^m(\Omega), \quad W^{s,q}(\Omega) \hookrightarrow L^{pm}(\Omega).$$

Thus, from (3.19), there is a constant C_{N,s,p,q,ν_1} such that

$$\begin{aligned} \|g(s-\tau,u(\tau))\|_{L^m(\Omega)} &\leq CC_{N,s,p,q,\nu_1}\tau^{-\alpha}\left(\|u(\tau)\|_{W^{s,q}(\Omega)} + \|u(\tau)\|^p_{W^{s,q}(\Omega)}\right) \\ &\leq C\tau^{-\alpha}\left(\rho + \|u_0\|_{W^{s,q}(\Omega)} + \left(\rho + \|u_0\|_{W^{s,q}(\Omega)}\right)^p\right), \quad \forall t \in [0,T], \end{aligned} \tag{3.20}$$

Inserting (3.20) into (3.18) and recalling that $P_{0,\rho} = \rho + \|u_0\|_{W^{s,q}(\Omega)}$, we obtain

$$\begin{aligned} \underline{\text{The (RHS) of (3.17)}} &\leq C\left(P_{0,\rho} + P^p_{0,\rho}\right)\int_0^t (t-s)^{\mu-1}s^{\frac{1}{2}}\left(\int_0^s \tau^{-2\alpha}d\tau\right)^{\frac{1}{2}} ds \\ &\leq C\left(P_{0,\rho} + P^p_{0,\rho}\right)\int_0^t (t-s)^{\mu-1}s^{\frac{1}{2}}\frac{s^{\frac{1-2\alpha}{2}}}{(1-2\alpha)^{\frac{1}{2}}}ds \\ &\leq C\left(P_{0,\rho} + P^p_{0,\rho}\right)\int_0^t (t-s)^{\mu-1}s^{1-\alpha}ds \\ &\leq C\left(P_{0,\rho} + P^p_{0,\rho}\right)\mathrm{B}(\mu,2-\alpha)t^{\mu-\alpha+1}, \end{aligned} \tag{3.21}$$

where we have used Lemma 2.4 in order to derive

$$\begin{aligned} \int_0^t (t-s)^{\mu-1}s^{1-\alpha}ds &= \int_0^t (t-s)^{\mu-1}s^{(2-\alpha)-1}ds \\ &= \frac{\Gamma(\mu)\Gamma(2-\alpha)}{\Gamma(\mu-\alpha+2)}t^{\mu-\alpha+1} = \mathrm{B}(\mu,2-\alpha)t^{\mu-\alpha+1}. \end{aligned}$$

From (3.7) we have that $D(\mathcal{A}^{\nu_2}) \hookrightarrow W^{s,q}(\Omega)$ and then, we conclude from (3.17) and (3.21) that

$$\|J_1(u)(t)\|_{C([0,T];W^{s,q}(\Omega))} \leq C\left(P_{0,\rho} + P^p_{0,\rho}\right)\mathrm{B}(\mu,2-\alpha)T^{\mu-\alpha+1}, \tag{3.22}$$

where $C_2 := C_{N,s,p,q,\mu,k_1,\nu_1} > 0$. We continue with the estimate on the third term of (3.15), thanks to Lemma 2.5a) and (3.6) to obtain the Sobolev embedding $L^m(\Omega) \hookrightarrow D(\mathcal{A}^{\nu_1})$, we have

$$\begin{aligned}\left\|J_2(u)(t)\right\|_{D(\mathcal{A}^{\nu_2})} &\leq \int_0^t \left\|e^{\Delta(t-s)} V_{\eta,p}(u)(s) \log|u(s)|\right\|_{D(\mathcal{A}^{\nu_2})} ds \\ &\leq C_{\mu,p,q} \int_0^t (t-s)^{\mu-1} \|V_{\eta,p}(u)(s) \log|u(s)|\|_{D(\mathcal{A}^{\nu_1})} ds \\ &\leq C_{\mu,p,q} \int_0^t (t-s)^{\mu-1} \|V_{\eta,p}(u)(s) \log|u(s)|\|_{L^m(\Omega)} ds. \end{aligned} \tag{3.23}$$

From Lemma 3.6a), we conclude that

$$\|V_{\eta,p}(u) \log|u|\|_{L^m(\Omega)} \leq C \left(\|u\|_{L^\gamma(\Omega)}^{-\gamma} + \|u\|_{L^{p\gamma m}(\Omega)}^{\gamma} \right) \|u\|^p_{L^{\frac{p^2 m}{p-1}}(\Omega)}. \tag{3.24}$$

From (3.3), $q \leq \frac{\gamma N}{N+\gamma s}$ and $qs < N$, we deduce that $q_s^* \leq \gamma$ with $q_s^* = \frac{qN}{N-qs}$. By using Lemma 2.2 we obtain the embedding $L^\gamma(\Omega) \hookrightarrow W^{s,q}(\Omega)$ which implies immediately that $\|u\|_{W^{s,q}(\Omega)} \leq C_{s,q,\gamma}\|u\|_{L^\gamma(\Omega)}$. From (3.14), one obtains that

$$\|u\|_{L^\gamma(\Omega)}^{-\gamma} \leq C\|u\|_{W^{s,q}(\Omega)}^{-\gamma} \leq C\left(\epsilon(\rho)\right)^{-\gamma}, \quad \text{for } \gamma > 0 \text{ and for all } t \in [0,T].$$

From (3.5), the fact

$$\iota^* > \mathcal{T}(p,p-1) = \frac{N(2p-q) + 2pq(2\nu_2 - s)}{4q},$$

and

$$\begin{aligned}\frac{p^2 m}{p-1} = \frac{2Np}{N-4\nu_1} = \frac{2Np}{N+4\iota^*-4p\nu_2} &< \frac{2Np}{N+4\mathcal{T}(p,p-1)-4p\nu_2} \\ &= \frac{Nq}{N-sq} = q_s^*, \end{aligned} \tag{3.25}$$

from (3.8), we can also obtain the following Sobolev embedding

$$W^{s,q}(\Omega) \hookrightarrow L^{pm}(\Omega).$$

For $\iota^* > \mathcal{T}(\gamma,1)$ ($\mathcal{T}(\gamma,1)$ is defined in (3.5)), we have $p\gamma m \leq \frac{Nq}{N-sq}$, from (3.8) we infer that the following Sobolev embedding holds

$$W^{s,q}(\Omega) \hookrightarrow L^{p\gamma m}(\Omega).$$

From (3.24), we get that

$$\|V_{\eta,p}(u) \log|u|\|_{L^m(\Omega)} \leq C \left(\|u\|_{W^{s,q}(\Omega)}^{-\gamma} + \|u\|_{W^{s,q}(\Omega)}^{\gamma} \right) \|u\|^p_{W^{s,q}(\Omega)}.$$

The right hand side of (3.23) becomes (there exists $\vartheta \in (0,1)$ such that $\vartheta < \gamma \leq p$)

$$\begin{aligned}\underline{\text{The (RHS) of (3.23)}} &\leq C \int_0^t (t-s)^{\gamma-1} \left(\|u(s)\|_{W^{s,q}(\Omega)}^{p-\gamma} + \|u(s)\|_{W^{s,q}(\Omega)}^{p+\gamma} \right) ds \\ &\leq CT^\vartheta \int_0^t (t-s)^{\gamma-1} s^{-\vartheta} \left(\|u(s)\|_{W^{s,q}(\Omega)}^{p-\gamma} + \|u(s)\|_{W^{s,q}(\Omega)}^{p+\gamma} \right) ds \\ &\leq CT^\vartheta \left(P_{0,\rho}^{p-\gamma} + P_{0,\rho}^{p+\gamma} \right) \int_0^t (t-s)^{\gamma-1} s^{-\vartheta} ds \\ &\leq CT^\vartheta \left(P_{0,\rho}^{p-\gamma} + P_{0,\rho}^{p+\gamma} \right) \mathrm{B}(\gamma, 1-\vartheta) t^{\gamma-\vartheta}, \end{aligned} \tag{3.26}$$

where it follows from (3.13) that $\|u\|_{W^{s,q}(\Omega)} \le \rho + \|u_0\|_{W^{s,q}(\Omega)} = P_{0,\rho}$ and we have used Lemma 2.4 to obtain

$$\int_0^t (t-s)^{\gamma-1} s^{-\vartheta} ds = \int_0^t (t-s)^{\gamma-1} s^{(1-\vartheta)-1} ds$$
$$= \frac{\Gamma(\gamma)\Gamma(1-\vartheta)}{\Gamma(\gamma+1-\vartheta)} t^{\gamma-\vartheta} = \mathrm{B}(\gamma, 1-\vartheta) t^{\gamma-\vartheta},$$

where B be a Beta function. From (3.18), (3.26), we deduce that for $t \in [0,T]$

$$\|J_2(u)(t)\|_{D(\mathcal{A}^{\nu_2})} \le CT^{\vartheta} \left(P_{0,\rho}^{p-\gamma} + P_{0,\rho}^{p+\gamma} \right) \mathrm{B}(\gamma, 1-\vartheta) T^{\gamma-\vartheta}.$$

From (3.7), we have that $D(\mathcal{A}^{\nu_2}) \hookrightarrow W^{s,q}(\Omega)$ implies $\|\cdot\|_{W^{s,q}(\Omega)} \le C_{s,q,\nu_2} \|\cdot\|_{D(\mathcal{A}^{\nu_2})}$, and we derive for all $t \in [0,T]$, $0 < \vartheta < \gamma < p-1$ that

$$\|J_2(u)(t)\|_{W^{s,q}(\Omega)} \le CT^{\vartheta} \left(P_{0,\rho}^{p-\gamma} + P_{0,\rho}^{p+\gamma} \right) \mathrm{B}(\gamma, 1-\vartheta) T^{\gamma-\vartheta}. \tag{3.27}$$

Hence, from (3.15), (3.16), (3.22) and (3.27), for every $t \in [0,T]$,

$$\|\mathbf{X}u(t) - u_0\|_{W^{s,q}(\Omega)} \le C\|u_0\|_{D(\mathcal{A}^{\nu_2})} + C\left(P_{0,\rho} + P_{0,\rho}^{p} \right) \mathrm{B}(\mu, 2-\alpha) T^{\mu-\alpha+1}$$
$$+ CT^{\gamma} \left(P_{0,\rho}^{p-\gamma} + P_{0,\rho}^{p+\gamma} \right) \mathrm{B}(\gamma, 1-\vartheta).$$

Therefore, if $\rho = 2C\|u_0\|_{D(\mathcal{A}^{\nu_2})}$ and

$$4C\left(P_{0,\rho} + P_{0,\rho}^{p} \right) \mathrm{B}(\mu, 2-\alpha) T^{\mu-\alpha+1} \le \rho, \qquad 4C\left(P_{0,\rho}^{p-\gamma} + P_{0,\rho}^{p+\gamma} \right) \mathrm{B}(\gamma, 1-\vartheta) T^{\gamma} \le \rho,$$

then $\mathbf{X}$ is invariant in $\mathscr{A}_\rho$.

• *Claim II: $\mathbf{X} : \mathscr{A}_\rho \to \mathscr{A}_\rho$ is a contraction map*. Let $u, v \in \mathscr{A}_\rho$, then

$$\|\mathbf{X}u(t) - \mathbf{X}v(t)\|_{D(\mathcal{A}^{\nu_2})} \le \int_0^t \left\| e^{\Delta(t-s)} \int_0^s \left(g(s-\tau, u(\tau)) - g(s-\tau, v(\tau)) \right) d\tau \right\|_{D(\mathcal{A}^{\nu_2})} ds$$
$$+ \int_0^t \left\| e^{\Delta(t-s)} \left(V_{\eta,p}(u)(s) \log|u(s)| - V_{\eta,p}(v)(s) \log|v(s)| \right) \right\|_{D(\mathcal{A}^{\nu_2})} ds$$
$$=: J_3(u,v)(t) + J_4(u,v)(t). \tag{3.28}$$

By an argument analogous to that used in (3.17), we deduce the following estimate

$$J_3(u,v)(t) \le C_{\mu,p,q} \int_0^t (t-s)^{\mu-1} s^{\frac{1}{2}} \left(\int_0^s \|g(s-\tau, u(\tau)) - g(s-\tau, v(\tau))\|_{D(\mathcal{A}^{\nu_1})}^2 d\tau \right)^{\frac{1}{2}} ds$$
$$\le C_{\mu,p,q} C_{N,\nu_1} \int_0^t (t-s)^{\mu-1} s^{\frac{1}{2}} \left(\int_0^s \|g(s-\tau, u(\tau)) - g(s-\tau, v(\tau))\|_{L^m(\Omega)}^2 d\tau \right)^{\frac{1}{2}} ds, \tag{3.29}$$

where, from (3.6) we have used $L^m(\Omega) \hookrightarrow D(\mathcal{A}^{\nu_1})$. From Lemma 3.5 b), we have that

$$\left\| g(s-\tau, u(\tau)) - g(s-\tau, v(\tau)) \right\|_{L^m(\Omega)} \le C\tau^{-\alpha} \|u(\tau) - v(\tau)\|_{L^m(\Omega)}$$
$$+ C\tau^{-\alpha} \|u(\tau)\|_{L^{pm}(\Omega)}^{p-1} \|u(\tau) - v(\tau)\|_{L^{pm}(\Omega)}$$
$$+ C\tau^{-\alpha} \|v(\tau)\|_{L^{pm}(\Omega)}^{p-1} \|u(\tau) - v(\tau)\|_{L^{pm}(\Omega)}. \tag{3.30}$$

From (3.2)-(3.5), for $q_s^* = \frac{qN}{N-qs}$ and $qs < N$, we have the following:
For $\iota^* > \mathcal{T}(1,1)$, a similar argument with (3.25) and observing that $q_s^* \geq pm$, then (3.8) implies the following Sobolev embedding

$$W^{s,q}(\Omega) \hookrightarrow L^{pm}(\Omega).$$

For $\iota^* > \mathcal{T}(\frac{1}{p},1) = \frac{N(2-q)+2pq(2\nu_1-\frac{s}{q})}{4q}$, we have the following bound

$$m = \frac{2N}{N-4\nu_1} = \frac{2N}{N+4\iota^*-4p\nu_2} < \frac{2N}{N+4\mathcal{T}(\frac{1}{p},1)-4p\nu_2} = \frac{Nq}{N-sq} = q_s^*,$$

and we deduce from (3.8) that the following Sobolev embedding is fulfilled

$$W^{s,q}(\Omega) \hookrightarrow L^{m}(\Omega).$$

We can now combine the previous results together with (3.30) to obtain

$$\left\|g(s-\tau,u(\tau)) - g(s-\tau,v(\tau))\right\|_{L^m(\Omega)} \leq CL_1(\rho,u_0)\tau^{-\alpha}\left\|u(\tau)-v(\tau)\right\|_{W^{s,q}(\Omega)}, \tag{3.31}$$

for all $t \in [0,T]$, where we have used that $\max\left\{\|u\|_{W^{s,q}(\Omega)}; \|v\|_{W^{s,q}(\Omega)}\right\} \leq \rho + \|u_0\|_{W^{s,q}(\Omega)}$, and the constant $L_1(\rho,u_0) := L_1(N,p,\gamma,\nu_1,\rho,\|u_0\|_{W^{s,q}(\Omega)})$ but is independent of t. From (3.29), (3.31) and Lemma 2.4,

$$\begin{aligned}
J_3(u,v)(t) &\leq CL_1(\rho,u_0)\|u-v\|_{C([0,T];W^{s,q}(\Omega))}\int_0^t (t-s)^{\mu-1}s^{\frac{1}{2}}\left(\int_0^s \tau^{-2\alpha}d\tau\right)^{\frac{1}{2}}ds \\
&\leq CL_1(\rho,u_0)\|u-v\|_{C([0,T];W^{s,q}(\Omega))}\int_0^t (t-s)^{\mu-1}s^{\frac{1}{2}}\frac{s^{\frac{-2\alpha+1}{2}}}{(1-2\alpha)^{\frac{1}{2}}}ds \\
&\leq CL_1(\rho,u_0)\|u-v\|_{C([0,T];W^{s,q}(\Omega))}\int_0^t (t-s)^{\mu-1}s^{1-\alpha}ds \\
&\leq CL_1(\rho,u_0)\|u-v\|_{C([0,T];W^{s,q}(\Omega))}\mathrm{B}(\mu,2-\alpha)t^{\mu-\alpha+1}.
\end{aligned} \tag{3.32}$$

We continue estimating the second term in (3.28), using Lemma 2.5a) and (3.8) to obtain the Sobolev embedding $L^m(\Omega) \hookrightarrow D(\mathcal{A}^{\nu_1})$ that

$$\begin{aligned}
J_4(u,v)(t) &\leq C_{\mu,p,q}\int_0^t (t-s)^{\mu-1}\left\|V_{\eta,p}(u)(s)\log|u(s)| - V_{\eta,p}(v)(s)\log|v(s)|\right\|_{D(\mathcal{A}^{\nu_1})}ds \\
&\leq C_{\mu,p,q}C_{N,\nu_1}\int_0^t (t-s)^{\mu-1}\left\|V_{\eta,p}(u)(s)\log|u(s)| - V_{\eta,p}(v)(s)\log|v(s)|\right\|_{L^m(\Omega)}ds.
\end{aligned} \tag{3.33}$$

From Lemma 3.6b), we get that

$$\begin{aligned}
&\left\|V_{\eta,p}(u)\log|u| - V_{\eta,p}(v)\log|v|\right\|_{L^m(\Omega)} \\
&\qquad \leq C\left(\|u\|^{-\gamma}_{L^\gamma(\Omega)} + \|u\|^{\gamma}_{L^{\frac{p\gamma m}{p-1}}(\Omega)}\right)\|u-v\|_{L^{pm}(\Omega)} \\
&\qquad + C\left(\|u\|^{-\gamma}_{L^\gamma(\Omega)} + \|u\|^{\gamma}_{L^{\frac{p\gamma m}{p-1}}(\Omega)}\right)\|u\|^{p-1}_{L^{p(p-1)m}(\Omega)}\|u-v\|_{L^{pm}(\Omega)} \\
&\qquad + C\|v\|^{p-1}_{L^{pm}(\Omega)}\|u-v\|_{L^{pm}(\Omega)}.
\end{aligned} \tag{3.34}$$

From (3.2)-(3.5), for $q_s^* = \frac{qN}{N-qs}$, $qs < N$, and (3.8) we have the following:
For q_s^* satisfying $\frac{1}{q_s^*} = \frac{1}{q} - \frac{s}{N}$, for $q \le \frac{N\gamma}{N+\gamma s}$ and $sq < N$, then $q_s^* \le \gamma$ and we deduce from Lemma 2.2 that $L^\gamma(\Omega) \hookrightarrow W^{s,q}(\Omega)$. From (3.14), one obtains that

$$\|u\|_{L^\gamma(\Omega)}^{-\gamma} \lesssim \|u\|_{W^{s,q}(\Omega)}^{-\gamma} \lesssim (\epsilon(\rho))^{-\gamma}, \quad \text{for } \gamma > 0 \text{ and for all } t \in [0,T].$$

For $\iota^* > \mathcal{T}(1,1)$, a similar argument with (3.25) and noticing that $q_s^* \ge pm$, then we deduce that the following Sobolev embedding holds

$$W^{s,q}(\Omega) \hookrightarrow L^{pm}(\Omega).$$

For $\iota^* > \mathcal{T}(p-1,1)$, we have

$$p(p-1)m = \frac{2p(p-1)N}{N-4\nu_1} < \frac{2p(p-2)N}{N+4\mathcal{T}(p-1,1)-4p\nu_2} = q_s^*,$$

and the following Sobolev embedding

$$W^{s,q}(\Omega) \hookrightarrow L^{p(p-1)m}(\Omega).$$

For $\iota^* > \mathcal{T}(\gamma, p-1)$, we easily observe that $\frac{p\gamma m}{p-1} \le q_s^*$, which allows us to obtain

$$W^{s,q}(\Omega) \hookrightarrow L^{\frac{p\gamma m}{p-1}}(\Omega).$$

This follows from (3.34) that

$$\Big\| V_{\eta,p}(u)\log|u| - V_{\eta,p}(v)\log|v| \Big\|_{L^m(\Omega)} \le CL_2(\rho,u_0)\|u-v\|_{W^{s,q}(\Omega)}, \tag{3.35}$$

for all $t \in [0,T]$, we note that $\max\Big\{\|u\|_{W^{s,q}(\Omega)}; \|v\|_{W^{s,q}(\Omega)}\Big\} \le \rho + \|u_0\|_{W^{s,q}(\Omega)}$, and $L_2(\rho,u_0) := L_2(N,p,\gamma,\nu_1,\rho,\|u_0\|_{W^{s,q}(\Omega)})$. From (3.33), (3.35), and using Lemma 2.4, one observes that

$$\begin{aligned} J_4(u,v)(t) &\le CL_2(\rho,u_0)\int_0^t (t-s)^{-\mu-1}\|u(s)-v(s)\|_{W^{s,q}(\Omega)}ds \\ &\le CL_2(\rho,u_0)T^\vartheta\|u-v\|_{C([0,T];W^{s,q}(\Omega))}\int_0^t (t-s)^{\mu-1}s^{-\vartheta}ds \\ &\le CL_2(\rho,u_0)T^\vartheta\|u-v\|_{C([0,T];W^{s,q}(\Omega))}\mathrm{B}(\mu,1-\vartheta)t^{\mu-\vartheta}. \end{aligned} \tag{3.36}$$

Inserting the results of (3.32) and (3.36) into (3.28),

$$\begin{aligned} \|\mathbf{X}u(t)-\mathbf{X}v(t)\|_{D(\mathcal{A}^{\nu_2})} \le\ & CL_1(\rho,u_0)\|u-v\|_{C([0,T];W^{s,q}(\Omega))}\mathrm{B}(\mu,2-\alpha)t^{\mu-\alpha+1} \\ &+ CL_2(\rho,u_0)T^\vartheta\|u-v\|_{C([0,T];W^{s,q}(\Omega))}\mathrm{B}(\mu,1-\vartheta)t^{\mu-\vartheta}. \end{aligned}$$

From (3.7), we have that $D(\mathcal{A}^{\nu_2}) \hookrightarrow W^{s,q}(\Omega)$, and

$$\begin{aligned} \|\mathbf{X}u-\mathbf{X}v\|_{C([0,T];W^{s,q}(\Omega))} \le\ & CL_1(\rho,u_0)\mathrm{B}(\mu,2-\alpha)T^{\mu-\alpha+1} \\ &+ CL_2(\rho,u_0)\mathrm{B}(\mu,1-\vartheta)T^\mu\|u-v\|_{C([0,T];W^{s,q}(\Omega))}. \end{aligned}$$

Choosing $T, L_1(\rho,u_0), L_2(\rho,u_0)$ small enough such that

$$CL_1(\rho,u_0)\mathrm{B}(\mu,2-\alpha)T^{\mu-\alpha+1} + CL_2(\rho,u_0)\mathrm{B}(\mu,1-\vartheta)T^\mu < 1,$$

it follows that $\mathbf{X}$ is a contraction map on $\mathscr{A}_\rho$. Invoking now the contraction mapping principle we conclude that the map $\mathbf{X}$ has a unique fixed point u in $\mathscr{A}_\rho$. The proof of Theorem 3.3 is completed. $\square$

Remark 3.1. *In Theorem 3.3, for $N \geq 1$, and $0 \leq \nu_2 < N/4$ let us choose $N = 5, \nu_2 = 1$. From the conditions*

$$\begin{cases} s < \nu_2, s \in \mathbb{N}, \\ 1 \leq q \leq \frac{2N}{N+2s-4\nu_2}, \end{cases} \tag{3.37}$$

this implies that

$$\begin{cases} s = 0, \quad q \in [1, 10], \\ s = 1, \quad q \in [1, \frac{10}{3}]. \end{cases} \tag{3.38}$$

Then, the Problem (1.1) has a unique mild solution $u \in C([0,T]; L^q(\Omega))$, $1 \leq q \leq 10$, or $u \in C([0,T]; W^{1,q}(\Omega))$, $1 \leq q \leq \frac{10}{3}$.

3.2 Unique continuation of the solution to Problem ($\mathbb{P}_0$)

Definition 3.7. Given a mild solution $u \in C([0,T]; W^{s,q}(\Omega))$ of ($\mathbb{P}_0$), the function $u^\star$ is a *continuation* of u in $[0, T^\star]$ for $T^\star > T$ if

$$\begin{cases} u^\star \in C([0,T^\star]; W^{s,q}(\Omega)) \;\text{ is a mild solution of } (\mathbb{P}_0) \;\text{ in } [T, T^\star], \\ u^\star(t) = u(t) \quad \text{whenever} \quad t \in [0,T]. \end{cases}$$

Theorem 3.8 (Continuation)**.** *Let us assume that the conditions of Theorem 3.3 are true. Then, the (unique) mild solution on $[0,T]$ of Problem ($\mathbb{P}_0$) can be extended to the interval $[0, T^\star]$, for some $T^\star > T$.*

Proof. Let $u : [0,T] \to W^{s,q}(\Omega)$ be a mild solution of Problem ($\mathbb{P}_0$) (T is the time from Theorem 3.3). Fix $\rho^\star > 0$, and for $T^\star > T$ ($T^\star$ depending on $\rho^\star$), we shall prove that $u^\star : [0, T^\star] \to W^{s,q}(\Omega)$ is a mild solution of Problem ($\mathbb{P}_0$). Assume the following estimates hold $\max\{Q_i\} \leq \frac{\rho^\star}{7}, \quad Q_i > 0, \quad i = 1, \ldots, 7.$ with the constants $Q_i, (i = 1, \ldots, 7)$ are defined as follows:

$$CT^{-1}T^\star \|u_0\|_{D(\mathcal{A}^{\nu_2})} =: Q_1; \tag{3.39a}$$
$$C\left(P_{T,\rho^\star} + P^p_{T,\rho^\star}\right) \mathrm{B}(\mu, 2-\alpha)(T^\star)^{\mu-\alpha+1} =: Q_2; \tag{3.39b}$$
$$C(T^\star)^\vartheta \mathrm{B}(\gamma, 1-\vartheta)(T^\star)^{\gamma-\vartheta}\left(P^{p-\gamma}_{T,\rho^\star} + P^{p+\gamma}_{T,\rho^\star}\right) =: Q_3; \tag{3.39c}$$
$$C\left(P_{0,\rho} + P^p_{0,\rho}\right) \mathrm{B}(\mu, 2-\alpha)(T^\star)^{\mu-\alpha+1} =: Q_4; \tag{3.39d}$$
$$C(T^\star)^\vartheta \mathrm{B}(\gamma, 1-\vartheta)(T^\star)^{\gamma-\vartheta}\left(P^{p-\gamma}_{0,\rho} + P^{p+\gamma}_{0,\rho}\right) =: Q_5; \tag{3.39e}$$
$$CL_1(\rho, u_0)\mathrm{B}(\mu, 2-\alpha)(T^\star)^{\mu-\alpha+1} =: Q_6; \tag{3.39f}$$
$$CL_2(\rho, u_0)\mathrm{B}(\mu, 1-\vartheta)(T^\star)^{\mu} =: Q_7, \tag{3.39g}$$

where $\vartheta, \gamma \in (0,1)$, such that $\vartheta \leq \gamma \leq p-1$, $P_{0,\rho} = \rho + \|u_0\|_{W^{s,q}(\Omega)}$, $P_{T,\rho^\star} = \rho^\star + \|u(\cdot,T)\|_{W^{s,q}(\Omega)}$ and $L_1(\rho, u_0), L_2(\rho, u_0)$ are defined in the proof of Theorem 3.3. For $T^\star \geq T > 0$ and $\rho^\star > 0$, let us define

$$\widetilde{\mathscr{A}_{\rho^\star}} := \left\{ u^\star \in C([0,T^\star]; W^{s,q}(\Omega)) : \left| \begin{array}{ll} u^\star(t) = u(t), & \forall t \in [0,T], \\ \|u^\star(t) - u(T)\|_{C([T,T^\star]; W^{s,q}(\Omega))} \leq \rho^\star, & \forall t \in [T, T^\star]. \end{array} \right. \right\} \tag{3.40}$$

• *Step I: Claim that* $\mathbf{X}$ *defined as in* (3.15) *is the operator on* $\widetilde{\mathscr{A}_{\rho^\star}}$. Let $u^\star \in \widetilde{\mathscr{A}_{\rho^\star}}$ and we consider two cases.

If $t \in [0,T]$, then by virtue of Theorem 3.3, we have that Problem ($\mathbb{P}_0$) possesses a unique solution and we also have $u^\star(t) = u(t)$. Thus $\mathbf{X}u^\star(t) = \mathbf{X}u(t) = u(t)$ for all $t \in [0,T]$.

If $t \in [T, T^\star]$, we have

$$
\begin{aligned}
\mathbf{X}u^\star(t) - u(\cdot,T) \le{}& \left(e^{\Delta(t-T)} - I\right) e^{\Delta T} u_0 \\
&+ \int_T^t e^{\Delta(t-s)} \int_0^s g(s-\tau, u^\star(\tau)) d\tau ds + \int_T^t e^{\Delta(t-s)} V_{\eta,p}(u^\star)(s) \log|u^\star(s)| ds \\
&+ \left(e^{\Delta(t-T)} - I\right) \int_0^T e^{\Delta(T-s)} \int_0^s g(s-\tau, u^\star(\tau)) d\tau ds \\
&+ \left(e^{\Delta(t-T)} - I\right) \int_0^T e^{\Delta(T-s)} V_{\eta,p}(u^\star)(s) \log|u^\star(s)| ds \\
=:{}& (A_1) + (A_2) + (A_3) + (A_4) + (A_5), \qquad (\text{“respectively”}).
\end{aligned}
$$

Estimating the term (A_1), we have for all $t \in [T, T^\star]$,

$$
\|(A_1)\|^2_{D(\mathcal{A}^{\nu_2})} = \sum_{j \in \mathbb{N}^*} \lambda_j^{2\nu_2} \left(u_0, \varphi_j\right)^2 \left(e^{-\lambda_j T}\left(e^{-\lambda_j(t-T)} - 1\right)\right)^2.
$$

For $z > 0$, using the inequality $1 - e^{-z} \le z$, and $ze^{-z} \le 1$, one obtains

$$
\begin{aligned}
\|(A_1)\|^2_{D(\mathcal{A}^{\nu_2})} &\le \sum_{j \in \mathbb{N}^*} \lambda_j^{2\nu_2} \left(u_0, \varphi_j\right)^2 \left(\lambda_j(t-T)\left(\lambda_j T\right)^{-1}\right)^2 \\
&\le \sum_{j \in \mathbb{N}^*} \lambda_j^{2\nu_2} \left(u_0, \varphi_j\right)^2 (t-T)^2 T^{-2} \\
&\le (t-T)^2 T^{-2} \|u_0\|^2_{D(\mathcal{A}^{\nu_2})}.
\end{aligned}
$$

For the constants s, q satisfying (3.3), we have that $D(\mathcal{A}^{\nu_2}) \hookrightarrow W^{s,q}(\Omega)$. Hence,

$$
\|(A_1)\|_{W^{s,q}(\Omega)} \le C(t-T) T^{-1} \|u_0\|_{D(\mathcal{A}^{\nu_2})} \le C T^\star T^{-1} \|u_0\|_{D(\mathcal{A}^{\nu_2})}.
$$

From (3.39a), this implies that the following estimate holds

$$
\|(A_1)\|_{C([0,T^\star];W^{s,q}(\Omega))} \le \frac{\rho^\star}{7}. \tag{3.41}
$$

Similar to (3.22), we have the following estimate for all $t \in [T, T^\star]$ (note that we can choose $T^\star > T$ and close enough to T)

$$
\|(A_2)\|_{W^{s,q}(\Omega)} \le C\left(P_{T,\rho^\star} + P^p_{T,\rho^\star}\right) \mathrm{B}(\mu, 2-\alpha)(t-T)^{\mu-\alpha+1},
$$

where from (3.40), for all $t \in [T, T^\star]$, we have used that

$$
\|u(\cdot,t)\|_{W^{s,q}(\Omega)} \le \rho^\star + \|u(\cdot,T)\|_{W^{s,q}(\Omega)} = P_{T,\rho^\star}.
$$

Using (3.39b),

$$
\|(A_2)\|_{W^{s,q}(\Omega)} \le C\left(P_{T,\rho^\star} + P^p_{T,\rho^\star}\right) \mathrm{B}(\mu, 2-\alpha)(T^\star)^{\mu-\alpha+1} \le \frac{\rho^\star}{7}. \tag{3.42}
$$

Similar to (3.27), we have the following estimate for all $t \in [T, T^\star]$ (note that we again can choose $T^\star > T$ and close enough to T)

$$\left\|(A_3)\right\|_{W^{s,q}(\Omega)} \leq C(T^\star)^{\vartheta}\left(P_{T,\rho^\star}^{p-\gamma} + P_{T,\rho^\star}^{p+\gamma}\right)\mathrm{B}(\gamma, 1-\vartheta)(t-T)^{\gamma-\vartheta}.$$

Using (3.39c), we infer that

$$\left\|(A_3)\right\|_{C([0,T^\star];W^{s,q}(\Omega))} \leq C(T^\star)^{\vartheta}\left(P_{T,\rho^\star}^{p-\gamma} + P_{T,\rho^\star}^{p+\gamma}\right)\mathrm{B}(\gamma, 1-\vartheta)(T^\star)^{\gamma-\vartheta} \leq \frac{\rho^\star}{7}. \tag{3.43}$$

Since $0 < \mu < 1$, we know that there is a constant $C_\mu > 0$ such that for any $z > 0$, we have the basic inequality $e^{-z} < C_\mu z^{\mu-1}$, and we assume that $\nu_2 - \nu_1 \leq 1 - \mu$. Hence, we deduce that

$$\begin{aligned}
\left\|(A_4)\right\|_{D(\mathcal{A}^{\nu_2})} &\leq \int_0^T \left\|\left(e^{\Delta(t-T)} - 1\right)e^{\Delta(T-s)}\int_0^s g(s-\tau, u^\star(\tau))d\tau\right\|_{D(\mathcal{A}^{\nu_2})} ds \\
&\leq \int_0^T \left(\sum_{j\in\mathbb{N}^*}\lambda_j^{2\nu_2}e^{-2\lambda_j(T-s)}\left|e^{-\lambda_j(t-T)} - 1\right|^2 s\int_0^s \left|g_j(s-\tau, u^\star(\tau))\right|^2 d\tau\right)^{\frac{1}{2}} ds \\
&\leq C_\mu \int_0^T (T-s)^{\mu-1}s^{\frac{1}{2}}\left(\sum_{j\in\mathbb{N}^*}\lambda_j^{2\nu_2+2\mu-2\nu_1-2}\lambda_j^{2\nu_1}\int_0^s \left(g_j(s-\tau, u^\star(\tau))\right)^2 d\tau\right)^{\frac{1}{2}} ds \\
&\leq \frac{C_\mu}{\lambda_1^{1+\nu_1-\nu_2-\mu}}\int_0^T (T-s)^{\mu-1}s^{\frac{1}{2}}\left(\int_0^s \left\|g(s-\tau, u^\star(\tau))\right\|_{D(\mathcal{A}^{\nu_1})}^2 d\tau\right)^{\frac{1}{2}} ds,
\end{aligned} \tag{3.44}$$

where

$$g_j(s-\tau, u^\star(\tau)) = \int_\Omega g(s-\tau, u^\star(\tau))\varphi_j(x)dx.$$

From (3.6), we obtain the Sobolev embedding $L^m(\Omega) \hookrightarrow D(\mathcal{A}^{\nu_1})$. In the same way as in (3.21), and from (3.40), one obtains

$$\begin{aligned}
\underline{\text{The (RHS) of (3.44)}} &\leq C\int_0^T (T-s)^{\mu-1}s^{\frac{1}{2}}\left(\int_0^s \left\|g(s-\tau, u^\star(\tau))\right\|_{L^m(\Omega)}^2 d\tau\right)^{\frac{1}{2}} ds \\
&\leq C\left(P_{0,\rho} + P_{0,\rho}^p\right)\mathrm{B}(\mu, 2-\alpha)T^{\mu-\alpha+1},
\end{aligned}$$

where we have used Lemma 2.4

$$\int_0^T (T-s)^{\mu-1}s^{1-\alpha}ds = \int_0^T (T-s)^{\mu-1}s^{(2-\alpha)-1}ds = \frac{\Gamma(\mu)\Gamma(2-\alpha)}{\Gamma(\mu-\alpha+2)}T^{\mu-\alpha+1} = \mathrm{B}(\mu, 2-\alpha)T^{\mu-\alpha+1}.$$

From 3.7, we have the Sobolev embedding $D(\mathcal{A}^{\nu_2}) \hookrightarrow W^{s,q}(\Omega)$, and we find the following bound

$$\left\|(A_4)\right\|_{W^{s,q}(\Omega)} \leq C\left(P_{0,\rho} + P_{0,\rho}^p\right)\mathrm{B}(\mu, 2-\alpha)T^{\mu-\alpha+1}.$$

From (3.39d), for $t \in (T, T^\star]$, we obtain

$$\left\|(A_4)\right\|_{C([0,T^\star];W^{s,q}(\Omega))} \leq C\left(P_{0,\rho} + P_{0,\rho}^p\right)\mathrm{B}(\mu, 2-\alpha)(T^\star)^{\mu-\alpha+1} < \frac{\rho^\star}{7}. \tag{3.45}$$

By an argument analogous to that one used in (3.26) and (3.44), we infer that (for $0 < \vartheta \le \gamma \le p-1$)

$$\begin{aligned}\left\|(A_5)\right\|_{D(\mathcal{A}^{\nu_2})} &\le \int_0^T \left\|\left(e^{\Delta(t-T)} - I\right) e^{\Delta(T-s)} V_{\eta,p}(u^\star)(s) \log|u^\star(s)|\right\|_{D(\mathcal{A}^{\nu_2})} ds \\ &\le C\int_0^T (T-s)^{\mu-1}\left\|V_{\eta,p}(u^\star)(s)\log|u^\star(s)|\right\|_{L^m(\Omega)} ds \\ &\le CT^{\vartheta}\left(P_{0,\rho}^{p-\gamma} + P_{0,\rho}^{p+\gamma}\right) \mathrm{B}(\gamma, 1-\vartheta) T^{\gamma-\vartheta}.\end{aligned}$$

From (3.7), we have that $D(\mathcal{A}^{\nu_2}) \hookrightarrow W^{s,q}(\Omega)$, and using (3.39e),

$$\left\|(A_5)\right\|_{C([0,T^\star];W^{s,q}(\Omega))} \le C(T^\star)^{\vartheta}\left(P_{0,\rho}^{p-\gamma} + P_{0,\rho}^{p+\gamma}\right)\mathrm{B}(\gamma,1-\vartheta)(T^\star)^{\gamma-\vartheta} \le \frac{\rho^\star}{7}. \tag{3.46}$$

It follows from (3.41), (3.42), (3.43), (3.45), (3.46) that, for every $t \in [0, T^\star]$

$$\left\|\mathbf{X}u^\star - u(\cdot,T)\right\|_{C([0,T^\star];W^{s,q}(\Omega))} < \frac{\rho^\star}{7} + \frac{\rho^\star}{7} + \frac{\rho^\star}{7} + \frac{\rho^\star}{7} + \frac{\rho^\star}{7} = \frac{5\rho^\star}{7} \le \rho^\star.$$

Let us now to claim that $\mathbf{X}$ is a map $\widetilde{\mathscr{A}_{\rho^\star}}$ into $\widetilde{\mathscr{A}_{\rho^\star}}$.

• *Step II: We show that $\mathbf{X}$ is a contraction on $\widetilde{\mathscr{A}_{\rho^\star}}$.* Let $u, v \in \widetilde{\mathscr{A}_{\rho^\star}}$, and we have that, for $0 \le t \le T^\star$,

$$\begin{aligned}\mathbf{X}u(t) - \mathbf{X}v(t) &= \int_0^t e^{\Delta(t-s)} \int_0^s \left(g(s-\tau, u(\tau)) - g(s-\tau, v(\tau))\right) d\tau ds \\ &\quad + \int_0^t e^{\Delta(t-s)}\left(V_{\eta,p}(u)(s)\log|u(s)| - V_{\eta,p}(v)(s)\log|v(s)|\right) ds,\end{aligned}$$

where we note that $\mathbf{X}u(t) - \mathbf{X}v(t) = 0$, vanishes in $\widetilde{\mathscr{A}_{\rho^\star}}$ for all $t \in (0,T]$. Then, for all $t \in [0, T^\star]$, proceeding as in Claim (II) of the last theorem, we have

$$\begin{aligned}&\left\|\mathbf{X}u(t) - \mathbf{X}v(t)\right\|_{D(\mathcal{A}^{\nu_2})} \\ &\le C\left(L_1(\rho,u_0)\mathrm{B}(\mu, 2-\alpha)(T^\star)^{\mu-\alpha+1} + L_2(\rho,u_0)\mathrm{B}(\mu,1-\vartheta)(T^\star)^{\mu}\right)\|u-v\|_{C([0,T^\star];W^{s,q}(\Omega))}.\end{aligned}$$

Thus, from (3.7) we have the Sobolev embedding $D(\mathcal{A}^{\nu_2}) \hookrightarrow W^{s,q}(\Omega)$, and from (3.39f)-(3.39g), we deduce

$$\left\|\mathbf{X}u - \mathbf{X}v\right\|_{C([0,T^\star];W^{s,q}(\Omega))} \le \frac{2\rho^\star}{7}\|u-v\|_{C([0,T^\star];W^{s,q}(\Omega))}.$$

Let us choose $0 \le \rho^\star \le \frac{7}{2}$ then we deduce that $\mathbf{X}$ is a $\frac{2\rho^\star}{7}$-contraction. By the Banach contraction principle, it follows that $\mathbf{X}$ has a unique fixed point $u^\star$ of $\mathbf{X}$ in $\widetilde{\mathscr{A}_{\rho^\star}}$, which is a continuation of u. This finishes the proof. □

3.3 Finite time blow-up or global existence to Problem ($\mathbb{P}_0$)

The next results are on the global existence or non-continuation by a blow-up.

Definition 3.9. Let $u(x,t)$ is a solution of ($\mathbb{P}_0$). We define the maximal existence time $T_{\max}$ of $u(x,t)$ as follows:

(i) If $u(x,t)$ exists for all $0 \le t < \infty$, then $T_{\max} = \infty$.

(ii) If there exists $T \in [0,\infty)$ such that $u(x,t)$ exists for $0 \le t < T$, but does not exist at $t = T$, then $T_{\max} = T$.

Definition 3.10. Let $u(x,t)$ be a solution of ($\mathbb{P}_0$). We say $u(x,t)$ blows up in finite time if the maximal existence time $T_{\max}$ is finite and

$$\limsup_{t\to T_{\max}^-} \|u(\cdot,t)\|_{W^{s,q}(\Omega)} = \infty.$$

Theorem 3.11 (Global existence or finite time blow-up)**.** *Let $N \ge 1, 0 \le s < 2\nu_2$, for $0 \le \nu_2 < N/2$ and $1 \le q \le p^*$ with $p^* = \frac{2N}{N-2(\nu_2-s)}$. Then there exists a maximal time $T_{\max} > 0$ such that $u \in C([0,T_{\max}]; W^{s,q}(\Omega))$ is the mild solution of ($\mathbb{P}_0$). Thus, either Problem ($\mathbb{P}_0$) has a unique global mild solution on $[0,\infty)$, or there is a maximal time $T_{\max} < \infty$ such that*

$$\limsup_{t\to T_{\max}^-} \|u(\cdot,t)\|_{W^{s,q}(\Omega)} = \infty.$$

Proof. Define

$$T_{\max} := \sup\left\{T > 0 : \text{there exists a solution on } [0,T]\right\}.$$

Assume that $T_{\max} < \infty$, and $\|u(\cdot,t)\|_{W^{s,q}(\Omega)} \le \rho_0$, for some $\rho_0 > 0$. Now suppose there exists a sequence $\{t_k\}_{k\in\mathbb{N}} \subset [0,T_{\max})$ such that $t_k \to T_{\max}$ and $\{u(\cdot,t_k)\}_{k\in\mathbb{N}} \subset W^{s,q}(\Omega)$. Let us show that $\{u(\cdot,t_k)\}_{k\in\mathbb{N}}$ is a Cauchy sequence in $W^{s,q}(\Omega)$. Indeed, given $\epsilon > 0$, fix $N \in \mathbb{N}$ such that for all $k,n > N$, $0 < t_k < t_n < T_{\max}$, we have

$$\begin{aligned}
u(\cdot,t_n) - u(\cdot,t_k) &= \left(e^{\Delta(t_n-t_k)} - I\right) e^{\Delta t_k} u_0 \\
&\quad + \int_{t_k}^{t_n} e^{\Delta(t_n-s)} \int_0^s g(s-\tau,u(\tau))d\tau ds + \int_{t_k}^{t_n} e^{\Delta(t_n-s)} V_{\eta,p}(u)(s)\log|u(s)|ds \\
&\quad + \left(e^{\Delta(t_n-t_k)} - I\right) \int_0^{t_k} e^{\Delta(t_k-s)} \int_0^s g(s-\tau,u(\tau))d\tau ds \\
&\quad + \left(e^{\Delta(t_n-t_k)} - I\right) \int_0^{t_k} e^{\Delta(t_k-s)} V_{\eta,p}(u)(s)\log|u(s)|ds \\
&= \left(e^{\Delta(t_n-t_k)} - I\right) u(\cdot,t_k) + \int_{t_k}^{t_n} e^{\Delta(t_n-s)} \int_0^s g(s-\tau,u(\tau))d\tau ds \\
&\quad + \int_{t_k}^{t_n} e^{\Delta(t_n-s)} V_{\eta,p}(u)(s)\log|u(s)|ds.
\end{aligned}$$

Thus, we have

$$\begin{aligned}
\|u(\cdot,t_n) - u(\cdot,t_k)\|_{W^{s,q}(\Omega)} &\le \left\|\left(e^{\Delta(t_n-t_k)} - I\right) u(\cdot,t_k)\right\|_{W^{s,q}(\Omega)} \\
&\quad + \int_{t_k}^{t_n} \left\|e^{\Delta(t_n-s)} \int_0^s g(s-\tau,u(\tau))d\tau\right\|_{W^{s,q}(\Omega)} ds \\
&\quad + \int_{t_k}^{t_n} \left\|e^{\Delta(t_n-s)} V_{\eta,p}(u)(s)\log|u(s)|\right\|_{W^{s,q}(\Omega)} ds \\
&=: (O_1) + (O_2) + (O_3).
\end{aligned}$$

First, we estimate the term (O_1) as follows

$$\begin{aligned}
\|(O_1)\|_{W^{s,q}(\Omega)} &= \left\|\left(e^{\Delta(t_n-t_k)} - I\right) u(\cdot,t_k)\right\|_{W^{s,q}(\Omega)} \\
&\le \left\|e^{\Delta(t_n-t_k)} - I\right\|_{\mathbb{L}(W^{s,q}(\Omega))} \|u(\cdot,t_k)\|_{W^{s,q}(\Omega)}.
\end{aligned}$$

By an argument analogous to that one used in (3.21),

$$\begin{aligned}\|(O_2)\|_{D(\mathcal{A}^{\nu_2})} &\leq C\left(\rho_0+\rho_0^p\right)\int_{t_k}^{t_n}(t_n-s)^{\mu-1}s^{1-\alpha}ds\\ &\leq Ct_n^{\mu-\alpha}\left(\rho_0+\rho_0^p\right)\int_{\frac{t_k}{t_n}}^{1}(1-s)^{\mu-1}s^{1-\alpha}ds.\end{aligned}$$

From (3.7), implies $D(\mathcal{A}^{\nu_2})\hookrightarrow W^{s,q}(\Omega)$, and therefore we obtain

$$\|(O_2)\|_{W^{s,q}(\Omega)}\leq CT_{\max}^{\mu-\alpha}\left(\rho_0+\rho_0^p\right)\int_{\frac{t_k}{t_n}}^{1}(1-s)^{\mu-1}s^{1-\alpha}ds.$$

Reasoning in the same way as in (3.26),

$$\begin{aligned}\|(O_3)\|_{D(\mathcal{A}^{\nu_2})} &\leq CT_{\max}^{\vartheta}\left(\rho_0^{p-\gamma}+\rho_0^{p+\gamma}\right)\int_{t_k}^{t_n}(t_n-s)^{\gamma-1}s^{-\vartheta}ds\\ &\leq CT_{\max}^{\vartheta}t_n^{\gamma+\vartheta-1}\left(\rho_0^{p-\gamma}+\rho_0^{p+\gamma}\right)\int_{\frac{t_k}{t_n}}^{1}(1-s)^{\gamma-1}s^{-\vartheta}ds.\end{aligned}$$

Using (3.7) we imply $D(\mathcal{A}^{\nu_2})\hookrightarrow W^{s,q}(\Omega)$, we have

$$\|(O_3)\|_{W^{s,q}(\Omega)}\leq CT_{\max}^{\vartheta}t_n^{\gamma-\vartheta-1}\left(\rho_0^{p-\gamma}+\rho_0^{p+\gamma}\right)\int_{\frac{t_k}{t_n}}^{1}(1-s)^{\gamma-1}s^{-\vartheta}ds.$$

Thus, since $\{t_k\}_{k\in\mathbb{N}^*}$ is convergent we can take $N:=N(\epsilon)\in\mathbb{N}^*$ with $n\geq k\geq N$ such that $|t_n-t_k|$ is sufficiently small. Since the semigroup $\{e^{\Delta t}\}_{t\geq 0}$ is strongly continuous in $W^{s,q}(\Omega)$, given $\epsilon>0$, we have

$$\rho_0\left\|e^{\Delta(t_n-t_k)}-I\right\|_{\mathbb{L}(W^{s,q}(\Omega))}<\frac{\epsilon}{3},$$

$$Ct_n^{\mu-\alpha}\left(\rho_0+\rho_0^p\right)\int_{\frac{t_k}{t_n}}^{1}(1-s)^{\mu-1}s^{1-\alpha}ds<\frac{\epsilon}{3},$$

and

$$CT_{\max}^{\vartheta}t_n^{\gamma-\vartheta-1}\left(\rho_0^{p-\gamma}+\rho_0^{p+\gamma}\right)\int_{\frac{t_k}{t_n}}^{1}(1-s)^{\gamma-1}s^{-\vartheta}ds<\frac{\epsilon}{3}.$$

Consequently, given $\epsilon>0$ there exists the number $N\in\mathbb{N}$ such that

$$\|u(\cdot,t_n)-u(\cdot,t_k)\|_{W^{s,q}(\Omega)}<\epsilon,\quad \text{for } n,k\geq N.$$

It follows that $\{u(\cdot,t_k)\}_{k\in\mathbb{N}}\subset W^{s,q}(\Omega)$ is a Cauchy sequences and for $\{t_k\}_{k\in\mathbb{N}^*}$ arbitrary then the following limit is exist

$$\lim_{t\to T_{\max}^-}\|u(\cdot,t)\|_{W^{s,q}(\Omega)}<\infty.$$

From our previous result we deduce that the solution can extend to some larger interval (u can be continued beyond $T_{\max}$), and this contradicts the definition of $T_{\max}$. Thus, either $T_{\max}=\infty$ or if $T_{\max}<\infty$ then $\lim_{t\to T_{\max}^-}\|u(\cdot,t)\|_{W^{s,q}(\Omega)}=\infty$. The proof of Theorem 3.11 is finished. □

4 Appendix

4.1 A1. Proof of Lemma 3.1

Let $u, v \in \mathbb{R}$, thanks to the results in [2], (H_3) holds true. Next, we prove (H_4). Indeed,

$$\begin{aligned}\big|V_{\eta,p}(u)\log|u| - V_{\eta,p}(v)\log|v|\big| \leq& \big|V_{\eta,p}(u) - V_{\eta,p}(v)\big|\big|\log|u|\big| \\ &+ \big|V_{\eta,p}(v)\big|\,\big|\log|u| - \log|v|\big|. \end{aligned} \tag{4.1}$$

From (H_3),

$$\begin{aligned}\big|V_{\eta,p}(u) - V_{\eta,p}(v)\big| &= \eta\Big||u|^{p-1}u - |v|^{p-1}v\Big| \\ &\leq \eta c(1 + |u|^{p-1} + |v|^{p-1})|u-v|. \end{aligned} \tag{4.2}$$

Using the basic inequality $\log(1+z) < z$ for $z > 0$,

$$\begin{aligned}\big|\log|u| - \log|v|\big| &= \left|\log\left|1 + \frac{|u|-|v|}{|v|}\right|\right| \\ &< \left|\log\left(1 + \frac{|u-v|}{|v|}\right)\right| < \frac{|u-v|}{|v|}. \end{aligned} \tag{4.3}$$

From (4.1)-(4.3), we have the proof of Lemma 3.1.

4.2 A2. Proof of Lemma 3.4

From (3.2)-(3.5) and Lemmas 2.1 and 2.2, we have the following conclusions:

a) Proof (3.6). From (3.4), we have $p\nu_2 < \iota^* \leq p\nu_2 + \frac{N}{4}$, implies that $-\frac{N}{4} < \nu_1 \leq 0$, we invoke Lemma 2.2 to deduce the Sobolev embedding

$$L^{\frac{2N}{N-4\nu_1}}(\Omega) \hookrightarrow D(\mathcal{A}^{\nu_1}).$$

Moreover, from $p\nu_2 < \iota^* \leq 1 + (p-1)\nu_2$, we have that $0 < \nu_2 - \nu_1 \leq 1$.

b) Proof (3.7). From (3.2)-(3.3), we have $0 \leq \nu_2 < \frac{N}{4}, 0 \leq s < \nu_2, p^* = \frac{2N}{N+2s-4\nu_2}$ then we get

$$\begin{cases} 4\nu_2 - 2s < N, \\ 1 \leq q \leq \frac{2N}{N+2s-4\nu_2}, \end{cases}$$

Lemmas 2.1 and 2.2 imply

$$D(\mathcal{A}^{\nu_2}) \hookrightarrow H^{2\nu_2}(\Omega) \hookrightarrow W^{s,q}(\Omega).$$

c) Proof (3.8). From $sq < N$ and $1 \leq r \leq q_s^*$ for $q_s^* = \frac{qN}{N-sq}$,from Lemma 2.2 we have the following Sobolev embedding

$$W^{s,q}(\Omega) \hookrightarrow L^r(\Omega).$$

The proof of the lemma is completed.

4.3 A3. Proof of Lemma 3.5

a) Proof (3.9). From hypothesis (H_2), using the inequality $(a+b)^q \leq 2^q a^q + 2^q b^q$, for $a, b > 0$, $q > 1$, we deduce for $1 < m \leq 2$ and $\alpha \in (0,1)$,

$$\begin{aligned}\int_\Omega |g(s-\tau, u(\tau))|^m dx &\leq k_1 \tau^{-\alpha m} \int_\Omega (|u(\tau)| + |u(\tau)|^p)^m \, dx \\ &\leq k_1 2^m \tau^{-\alpha m} \left(\int_\Omega |u(\tau)|^m dx + \int_\Omega |u(\tau)|^{pm} dx \right) \\ &\leq C\tau^{-\alpha m} \left(\|u(\tau)\|^m_{L^m(\Omega)} + \|u(\tau)\|^{pm}_{L^{pm}(\Omega)} \right).\end{aligned}$$

Then, we get

$$\|g(s-\tau, u(\tau))\|_{L^m(\Omega)} \leq C\tau^{-\alpha} \left(\|u(\tau)\|_{L^m(\Omega)} + \|u(\tau)\|^p_{L^{pm}(\Omega)} \right),$$

where we have used $a^m + b^m \leq (a+b)^m$ for $m > 1$.

b) Proof (3.10). By recalling hypothesis (H_1), we arrive at

$$\begin{aligned}&\|g(s-\tau, u(\tau)) - g(s-\tau, v(\tau))\|_{L^m(\Omega)} \\ &\leq k_1 \tau^{-\alpha} \|u(\tau) - v(\tau)\|_{L^m(\Omega)} \\ &\quad + k_1 \tau^{-\alpha} \left\| |u(\tau)|^{p-1} |u(\tau) - v(\tau)| \right\|_{L^m(\Omega)} \\ &\quad + k_1 \tau^{-\alpha} \left\| |v(\tau)|^{p-1} |u(\tau) - v(\tau)| \right\|_{L^m(\Omega)}.\end{aligned} \tag{4.4}$$

For the constant $1 < m \leq 2$, using Hölder's inequality to obtain

$$\begin{aligned}\left\| |u|^{p-1} |u-v| \right\|^m_{L^m(\Omega)} &= \int_\Omega \left(|u|^{p-1} |u-v| \right)^m dx \\ &\leq \left(\int_\Omega |u|^{pm} dx \right)^{\frac{p-1}{p}} \left(\int_\Omega |u-v|^{pm} dx \right)^{\frac{1}{p}} \\ &\leq \|u\|^{(p-1)m}_{L^{pm}(\Omega)} \|u-v\|^m_{L^{pm}(\Omega)}.\end{aligned}$$

Hence

$$\left\| |u|^{p-1} |u-v| \right\|_{L^m(\Omega)} \leq \|u\|^{p-1}_{L^{pm}(\Omega)} \|u-v\|_{L^{pm}(\Omega)}. \tag{4.5}$$

Similar to (4.5),

$$\left\| |v|^{p-1} |u-v| \right\|_{L^m(\Omega)} \leq \|v\|^{p-1}_{L^{pm}(\Omega)} \|u-v\|_{L^{pm}(\Omega)}. \tag{4.6}$$

Combining (4.4), (4.5) and (4.6), we get that

$$\begin{aligned}&\left\| g(s-\tau, u(\tau)) - g(s-\tau, v(\tau)) \right\|_{L^m(\Omega)} \\ &\leq C\tau^{-\alpha} \|u(\tau) - v(\tau)\|_{L^m(\Omega)} \\ &\quad + C\tau^{-\alpha} \|u(\tau)\|^{p-1}_{L^{pm}(\Omega)} \|u(\tau) - v(\tau)\|_{L^{pm}(\Omega)} \\ &\quad + C\tau^{-\alpha} \|v(\tau)\|^{p-1}_{L^{pm}(\Omega)} \|u(\tau) - v(\tau)\|_{L^{pm}(\Omega)}.\end{aligned}$$

This ends the proof.

4.4 A4. Proof of Lemma 3.6

a) Proof (3.11). We first set $B_x^- := \{x \in \Omega : |u(x)| < 1\}$ and $B_x^+ := \{x \in \Omega : |u(x)| \geq 1\}$. Using the Hölder's inequality, we have

$$\begin{aligned}
&\int_\Omega |V_{\eta,p}(u)\log|u||^m dx \leq \eta^m \int_\Omega |u|^{pm} |\log|u||^m dx \\
&\leq \eta^m \left(\int_\Omega |\log|u||^{pm} dx\right)^{\frac{1}{p}} \left(\int_\Omega |u|^{\frac{p^2 m}{p-1}} dx\right)^{\frac{p-1}{p}} \\
&\leq \eta^m \left(\int_{B_x^-} |\log|u||^{pm} dx + \int_{B_x^+} |\log|u||^{pm} dx\right)^{\frac{1}{p}} \left(\int_\Omega |u|^{\frac{p^2 m}{p-1}} dx\right)^{\frac{p-1}{p}} \\
&\leq \eta^m \left(\left(\int_{B_x^-} |\log|u||^{pm} dx\right)^{\frac{1}{p}} + \left(\int_{B_x^+} |\log|u||^{pm} dx\right)^{\frac{1}{p}}\right) \left(\int_\Omega |u|^{\frac{p^2 m}{p-1}} dx\right)^{\frac{p-1}{p}}, \qquad (4.7)
\end{aligned}$$

where we have used the elementary inequality $(a+b)^q \leq a^q + b^q$, for $a, b > 0$, $0 < q < 1$. From the inequality (2.4) for $\gamma > 0$ and $|u(x)| < 1$, $x \in B_x^-$, we obtain

$$\begin{aligned}
\left(\int_{B_x^-} |\log|u||^{pm} dx\right)^{\frac{1}{p}} &\leq \ell \left(\int_{B_x^-} |u|^{-p\gamma m} dx\right)^{\frac{1}{p}} \leq \ell \left(\left(\int_{B_x^-} |u|^{-p\gamma m} dx\right)^{-\frac{1}{pm}}\right)^{-m} \\
&\leq \ell \left(\left(\int_{B_x^-} |u|^{\gamma} dx\right) \left(\int_{B_x^-} 1 dx\right)^{-\frac{1+pm}{pm}}\right)^{-m} \\
&\leq \ell \, \|u(\cdot,t)\|_{L^\gamma(\Omega)}^{-\gamma m} |\Omega|^{\frac{1+pm}{p}}, \qquad (4.8)
\end{aligned}$$

where we have chosen $k' = -\frac{1}{pm} < 0$ in the inequality (2.5) as follows

$$\left(\int_{B_x^-} |u|^\gamma dx\right)^{-\frac{1}{pm}} \leq \left(\int_{B_x^-} |u|^\gamma dx\right) \left(\int_{B_x^-} 1 dx\right)^{-\frac{1+pm}{pm}}.$$

From inequality (2.4) for $\gamma > 0$ and $|u(x)| \geq 1$, $x \in B_x^+$, we have

$$\left(\int_{B_x^+} |\log|u||^{pm} dx\right)^{\frac{1}{p}} \leq \ell \left(\int_{B_x^+} |u|^{p\gamma m} dx\right)^{\frac{1}{p}} \leq \ell \|u(\cdot,t)\|_{L^{p\gamma m}(\Omega)}^{\gamma m}. \qquad (4.9)$$

From (4.7), (4.8) and (4.9), we conclude that

$$\|V_{\eta,p}(u)\log|u|\|_{L^m(\Omega)} \leq C \left(\|u\|_{L^\gamma(\Omega)}^{-\gamma} + \|u\|_{L^{p\gamma m}(\Omega)}^{\gamma}\right) \|u\|_{L^{\frac{p^2 m}{p-1}}(\Omega)}^p.$$

b) Proof (3.12). By recalling hypothesis (H_4), we arrive at

$$\begin{aligned}
&\|V_{\eta,p}(u)\log|u| - V_{\eta,p}(v)\log|v|\|_{L^m(\Omega)} \\
&\qquad \leq k_2 \Big\| |\log|u|| \, |u-v| \Big\|_{L^m(\Omega)} \\
&\qquad + k_2 \Big\| |u|^{p-1} |\log|u|| \, |u-v| \Big\|_{L^m(\Omega)} \\
&\qquad + k_2 \Big\| |v|^{p-1} |\log|u|| \, |u-v| \Big\|_{L^m(\Omega)} \\
&\qquad + k_2 \Big\| |v|^{p-1} |u-v| \Big\|_{L^m(\Omega)}. \qquad (4.10)
\end{aligned}$$

For the constant $1 < m \le 2$, using the Hölder inequality,

$$\Big\| |\log|u||\,|u-v|\,\Big\|^m_{L^m(\Omega)} = \int_\Omega |\log|u||^m|u-v|^m dx$$
$$\le \left(\int_\Omega |\log|u||^{\frac{pm}{p-1}}dx\right)^{\frac{p-1}{p}}\left(\int_\Omega |u-v|^{pm}dx\right)^{\frac{1}{p}}.$$

For $B_x^- := \{x\in\Omega : |u(x)|<1\}$ and $B_x^+ := \{x\in\Omega : |u(x)|\ge 1\}$, we have

$$\Big\| |\log|u||\,|u-v|\,\Big\|^m_{L^m(\Omega)}$$
$$\le \left(\int_{B_x^-} |\log|u||^{\frac{pm}{p-1}}dx + \int_{B_x^+} |\log|u||^{\frac{pm}{p-1}}dx\right)^{\frac{p-1}{p}}\left(\int_\Omega |u-v|^{pm}dx\right)^{\frac{1}{p}}$$
$$\le \left(\left(\int_{B_x^-} |\log|u||^{\frac{pm}{p-1}}dx\right)^{\frac{p-1}{p}} + \left(\int_{B_x^+} |\log|u||^{\frac{pm}{p-1}}dx\right)^{\frac{p-1}{p}}\right)\left(\int_\Omega |u-v|^{pm}dx\right)^{\frac{1}{p}}, \tag{4.11}$$

where we have used the elementary $(a+b)^q < a^q + b^q$ for $a,b>0,\ 0<q<1$.
From inequality (2.4) for $|u(x,t)|<1,\ x\in B_x^-, t\in[0,T]$, we have

$$\left(\int_{B_x^-} |\log|u||^{\frac{pm}{p-1}}dx\right)^{\frac{p-1}{p}} \le \ell^m\left(\int_{B_x^-} |u|^{-\frac{p\gamma m}{p-1}}dx\right)^{\frac{p-1}{p}}$$
$$\le \ell^m\left(\left(\int_{B_x^-} |u|^{-\frac{p\gamma m}{p-1}}dx\right)^{-\frac{p-1}{pm}}\right)^{-m}$$
$$\le \ell^m\left(\left(\int_{B_x^-} |u|^{\gamma}dx\right)\left(\int_{B_x^-} 1dx\right)^{-\frac{p-1+pm}{pm}}\right)^{-m}$$
$$\le \ell^m\|u(\cdot,t)\|^{-\gamma m}_{L^\gamma(\Omega)}|\Omega|^{\frac{p-1+pm}{p}} \le C\ell^m\|u(\cdot,t)\|^{-\gamma m}_{L^\gamma(\Omega)}, \tag{4.12}$$

where we have chosen $k' = -\frac{p-1}{pm} < 0$ in Lemma 2.3b) as follows

$$\left(\int_{B_x^-} |u|^{-\frac{p\gamma m}{p-1}}dx\right)^{-\frac{p-1}{pm}} \le \left(\int_{B_x^-} |u|^{\gamma}dx\right)\left(\int_{B_x^-} 1dx\right)^{-\frac{p-1+pm}{pm}}.$$

From inequality (2.4) for $|u(x,t)|\ge 1,\ x\in B_x^+, t\in[0,T]$, we have

$$\left(\int_{B_x^+} |\log|u||^{\frac{pm}{p-1}}dx\right)^{\frac{p-1}{p}} \le \ell^m\left(\int_{B_x^+} |u|^{\frac{p\gamma m}{p-1}}dx\right)^{\frac{p-1}{p}}$$
$$\le \ell^m\|u\|^{\gamma m}_{L^{\frac{p\gamma m}{p-1}}(\Omega)}. \tag{4.13}$$

From (4.11), (4.12) and (4.13), we conclude that

$$\Big\| |\log|u||\,|u-v|\Big\|_{L^m(\Omega)} \le C\ell\left(\|u\|^{-\gamma}_{L^\gamma(\Omega)} + \|u\|^{\gamma}_{L^{\frac{p\gamma m}{p-1}}(\Omega)}\right)\|u-v\|_{L^{pm}(\Omega)}. \tag{4.14}$$

Thanks to Hölder's inequality,

$$
\begin{aligned}
&\int_\Omega \left(|u|^{p-1}|\log|u|||u-v|\right)^m dx \\
&\quad = \int_\Omega |u|^{(p-1)m}|\log|u||^m|u-v|^m dx \\
&\quad \le \left(\int_\Omega |\log|u||^{\frac{pm}{p-1}}dx\right)^{\frac{p-1}{p}} \left(\int_\Omega |u|^{p(p-1)m}|u-v|^{pm}dx\right)^{\frac{1}{p}}. \qquad (4.15)
\end{aligned}
$$

Using inequality $(a+b)^q \le a^q + b^q$, for $0<q<1$, similarly to (4.12) and (4.13), we have the following estimate

$$
\begin{aligned}
\left(\int_\Omega |\log|u||^{\frac{pm}{p-1}}dx\right)^{\frac{p-1}{p}} &\le \left(\int_{B_x^-} |\log|u||^{\frac{pm}{p-1}}dx\right)^{\frac{p-1}{p}} \\
&\quad + \left(\int_{B_x^+} |\log|u||^{\frac{pm}{p-1}}dx\right)^{\frac{p-1}{p}} \\
&\quad \le \ell^m \left(\|u\|_{L^\gamma(\Omega)}^{-\gamma m} + \|u\|^{\gamma m}_{L^{\frac{p\gamma m}{p-1}}(\Omega)}\right). \qquad (4.16)
\end{aligned}
$$

Combining (4.15) and (4.16) we get that

$$
\begin{aligned}
&\left\||u|^{p-1}|\log|u|||u-v|\right\|_{L^m(\Omega)} \\
&\quad \le \ell\left(\|u\|_{L^\gamma(\Omega)}^{-\gamma} + \|u\|^{\gamma}_{L^{\frac{p\gamma m}{p-1}}(\Omega)}\right)\|u\|^{p-1}_{L^{p(p-1)m}(\Omega)}\|u-v\|_{L^{pm}(\Omega)}. \qquad (4.17)
\end{aligned}
$$

In a similar way as above,

$$
\begin{aligned}
&\left\||v|^{p-1}|\log|u|||u-v|\right\|_{L^m(\Omega)} \\
&\quad \le \ell\left(\|u\|_{L^\gamma(\Omega)}^{-\gamma} + \|u\|^{\gamma}_{L^{\frac{p\gamma m}{p-1}}(\Omega)}\right)\|v\|^{p-1}_{L^{p(p-1)m}(\Omega)}\|u-v\|_{L^{pm}(\Omega)}. \qquad (4.18)
\end{aligned}
$$

By applying the Hölder inequality, we obtain that

$$
\begin{aligned}
\left\||v|^{p-1}|u-v|\right\|^m_{L^m(\Omega)} &= \int_\Omega \left(|v|^{p-1}|u-v|\right)^m dx \\
&\le \left(\int_\Omega |v|^{pm}dx\right)^{\frac{p-1}{p}}\left(\int_\Omega |u-v|^{pm}dx\right)^{\frac{1}{p}} \\
&\le \|v\|^{(p-1)m}_{L^{pm}(\Omega)}\|u-v\|^m_{L^{pm}(\Omega)}. \qquad (4.19)
\end{aligned}
$$

Combining the results obtained in (4.10), (4.14), (4.17), (4.18) and (4.19), we find that

$$
\begin{aligned}
&\left\|V_{\eta,p}(u)\log|u| - V_{\eta,p}(v)\log|v|\right\|_{L^m(\Omega)} \\
&\quad \le C\left(\|u\|_{L^\gamma(\Omega)}^{-\gamma} + \|u\|^{\gamma}_{L^{\frac{p\gamma m}{p-1}}(\Omega)}\right)\|u-v\|_{L^{pm}(\Omega)} \\
&\quad + C\left(\|u\|_{L^\gamma(\Omega)}^{-\gamma} + \|u\|^{\gamma}_{L^{\frac{p\gamma m}{p-1}}(\Omega)}\right)\|u\|^{p-1}_{L^{p(p-1)m}(\Omega)}\|u-v\|_{L^{pm}(\Omega)} \\
&\quad + C\|v\|^{p-1}_{L^{pm}(\Omega)}\|u-v\|_{L^{pm}(\Omega)}.
\end{aligned}
$$

The proof of the lemma is completed.

5 Declarations

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