

Regularization of Nonlocal Pseudo-Parabolic Equation with Random Noise

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Abstract

In this paper, we consider an inverse problem for a time-fractional diffusion equation with the inhomogeneous source. These problems have many applications in engineering such as image processing, geophysics, and biology. We get the result in random cases as follows

- This problem is ill-posed.
- We have used the nonlocal condition, instead of the final time condition.
- Using the IFT regularization method, constructing the regularized solution, the a-priori choice rule for the regularization parameter is discussed and yields the corresponding convergence rate.
- A numerical experiment is presented to illustrate the results in theory.

Keywords: Fractional Tikhonov, Regularization, Conformable time derivative, Discrete data, random noise

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1 Introduction

Mathematics is a useful tool for explaining physical phenomena, providing a working foundation for applied and computational science, and establishing relationships between well-known concepts in the field of physics [1], such as statistics, finance, [2–5], programming, and biology, see [6], medicine [7], and biochemistry [8]. When mathematical concepts are evaluated together with corresponding phenomena in applied science, a solution-oriented approach emerges by providing models and simulations for real-world problems. Time-segmented diffusion models are superior to integer-order diffusion models in modeling true superdiffusion and

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superdiffusion processes [9, 10]. This is due to the non-local nature of the fractional differential operator.

The derivative and integral operators, used as tools for understanding the operating principles of dynamic systems, form the basis of mathematical modeling. The fact that it is so important has led to in-depth studies on the subject and it has led to the discovery of fractional operators, which are generalized forms of integer ordinal operators. The most important difference between classical derivative and integral is that it has more than one definition to get the best solution depending on the type of problem. However, the fact that it has almost no practical applications has led to its acceptance as an abstract field consisting only of mathematical operations. In this work, we consider the problem

$$\begin{cases} \partial_t^\alpha (u(x, t) + k(-\Delta)u(x, t)) = (-\Delta)^\sigma u(x, t) + F(x, t), & (x, t) \in D \times (0, T), \\ u(x, t) = 0, & (x, t) \in \partial D \times (0, T), \\ \int_0^T u(x, t) dt = \rho(x), & x \in D. \end{cases}$$

where D is a bounded domain in $\mathbb{R}^d$, ($d = 1, 2, 3$) with sufficiently smooth boundary $\partial\Omega$, and $T > 0$, $F \in L^\infty(0, T; L^2(\mathcal{D}))$, and $\rho \in L^2(\mathcal{D})$ are given. $\partial_t^\alpha u$ refers to the Caputo derivative defined by

$$\partial_t^\alpha u := \frac{1}{\Gamma(1-\alpha)} \int_0^t (t-s)^{-\alpha} u'(s) ds, \quad 0 < \alpha < 1, \quad (1.1)$$

where $\Gamma(x)$ is a gamma function. Until the present time, the results for the problem (1.1) in this case are still very limited, we found only one research: Tuan and his group investigated the problem (1.1) in two cases: the nonlinear and the linear source function. With the non-linear source function, the main technique is used the Sobolev embeddings. With the linear source function, based on the Banach fixed-point theorem, they established the mild solution's existence, uniqueness, and certain regularity findings. Numerous works have examined the deterministic case, [11–15]. However, the (1.1) random noise case yields very few results. The function (ρ^δ, F^δ) is superseded by $(\tilde{\rho}^\delta, \tilde{F}^\delta)$ where $\tilde{\rho}^\delta$ and $\tilde{F}^\delta$ share the following observation model.

$$\tilde{\rho}^\delta(x) = \rho(x) + \delta\omega(x), \quad \tilde{F}^\delta(x, t) = F(x, t) + \delta\Xi(x), \quad (1.2)$$

where ω and Ξ are stochastic errors, $0 < \delta < 1$ is a small parameter (the noise level). For the results of the random noise case, the readers can refer to [16, 17]. In [17], using the filter regularization method: the random noise and the deterministic cases, as well as the convergence rates and examples of filters, the research team misrepresented the error between the exact solution and the regularized solution in both parameter choice rules, the a-priori and the a-posteriori. Another point in this research is the final time condition $\int_0^T u(x, t) dt = \rho(x)$ instead of $u(x, T) = \rho(x)$. The problem (1.1) is non-well-posed, examples of ill-posed can be found in the literature [18]. We would want to briefly recap the problem (1.1) history before moving on to adjustment.

For survey types for random cases, we would like to introduce some typical results as follows:

- In [19], the creators thought about a converse source issue for a period of fragmentary dissemination condition. We will use the filter regularization method to create a regularized solution for the random noise case because the problem in general is poorly posed. In order to guarantee that the approximate and exact solutions will converge, we will set up the right conditions. After that, we demonstrate various filters and provide error estimations

for their approximate solutions. We conclude by providing a numerical illustration of the method's effectiveness.

- In [20], the research group investigated the issue of deriving the final value data from a multidimensional time fractional reaction-diffusion equation with a nonlinear source. We demonstrate that the current issue is not properly posed. Then, regularized problems are built using the quasi-boundary value method for multi-dimensional problems and the truncated expansion method for two-dimensional problems. Finally, numerical research is done to determine the regularized solutions' convergence rates.
- In [21] Thach and his group considered a multi-dimensional fractional pseudo-parabolic problem with a nonlinear source in case the input data is measured on a discrete set of points instead of the whole domain. For any number of dimensions, the solution is not stable. This makes the problem we are interested in ill-posed. Here, we construct regularized solutions for this problem in two cases of the number of dimensions (denoted by m) including and m is arbitrary. In each case, we show the uniqueness of the regularized solution and give the error estimates. Finally, the convergence rate is also investigated numerically.
- In [22], the author considered the forward and the backward problems for fractional damped wave equations $u_{tt} + \alpha(-\Delta)^{s_1}u_t + \beta(-\Delta)^{s_2}u = F(x, t, u(x, t))$, subject to the random Gaussian white noise for initial and final data. First of all, they constructed the mild solutions to problems under all cases for parameters α, β, s_1 , and s_2 , and then investigate their stability and instability properties in the sense of Hadamard. Second, by establishing well-posedness for regularized solutions in the above unstable cases using the Fourier truncation method. Thirdly, we theoretically characterize the Fourier truncation approximation effect from regularized solutions to exact solutions and derive some error estimates between the exact solutions and their regularized solutions in the $\mathbb{E} \|\cdot\|_{H^{s_2}}^2$ -norm. Last but not least, we illustrate the regularization approximation effect of the above method with a number of numerical examples.
- In [23], this paper suggested the forward and inverse issues for the nonlinear partial pseudo-parabolic equation $u_t + (-\Delta)^{s_1}u_t + \beta(-\Delta)^{s_2}u = F(x, t, u(x, t))$, with Gaussian white noise and final data. Under the reasonable suspicions s_1, s_2 , and β , we first show the poorly presented ness of gentle arrangements, which are principally determined by the irregular commotion. Besides, we propose the Fourier truncation strategy for balancing out the above poorly presented issues. In an $\mathbb{E} \|\cdot\|_{H^{s_2}}^2$ norm, we estimate the error between the exact solution and its regularized solution and provide numerical examples to demonstrate the impact of the above method.

Concerning the matter of regularization: The Tikhonov method [24], Quasi Boundary Value method [25], and Fractional Tikhonov method [26]. In [27], T. Wei with the Tikhonov regularization method, in [28], a boundary element method combined with a generalized Tikhonov regularization. In this work, we used the Iterated fractional Tikhonov method, to better understand this method of alignment, readers can refer to the document [29]. Here, we talk a little more about how the researchers came up with this method. The previously studied Tikhonov and Tikhonov fraction regularization methods show that they have optimal ordering in their convergent properties, see references [26, 28]. Tikhonov's new repeated segmentation regularization methods are introduced in [30]. We show that these iterative methods have an optimal order and surpass the previous saturation results. Immediately, it was applied by Xiong's group, see [30], using the iterated fractional Tikhonov regularization method, they focus on a backward problem for an inhomogeneous time-fractional diffusion equation on a spherical symmetric domain.

- For the deterministic case, the priori and the posteriori choice rules for regularization

parameters are discussed.

- For random noise cases, the priori choice rule is considered.

The paper's sequel is structured as follows: Three sections make up the remaining portion of this essay. We offer some preliminary information that is necessary for the paper in Section 2 and we define a modest solution. In Section 3, two alternative regularization techniques are used to provide regularized solutions, and after that, convergence estimates are provided using an a priori parameter selection methodology.

2 Preliminaries

The spectral problem

$$\begin{cases} (-\Delta)e_n(x) = \lambda_n e_n(x), & x \in D, \\ e_n(x) = 0, & x \in \partial D, \end{cases}$$

admits the eigenvalues

$$0 < \lambda_1 \leq \lambda_2 \leq \lambda_3 \leq \dots \leq \lambda_n \leq \dots \quad (2.1)$$

and $\lambda_n \rightarrow \infty$ as $n \rightarrow \infty$, $e_n \in \mathcal{H}_0^1(D)$. The space $\mathcal{H}^s(\mathcal{D})$ is defined as follows

$$\mathcal{H}^s(\mathcal{D}) = \left\{ v \in L^2(\mathcal{D}); \sum_{n=1}^{\infty} \lambda_n^{2s} |\langle v, e_n \rangle|^2 < +\infty \right\}, \quad (2.2)$$

with the norm

$$\|v\|_{\mathcal{H}^s(\mathcal{D})} = \left(\sum_{n=1}^{\infty} \lambda_n^{2s} |\langle v, e_n \rangle|^2 \right)^{\frac{1}{2}}. \quad (2.3)$$

Problem (1.1) has a solution u as follows

$$u_n(t) = E_{\alpha,1} \left(-\frac{\lambda_n^\sigma}{1+k\lambda_n} t^\alpha \right) u_n(0) + \frac{1}{1+k\lambda_n} \int_0^t (t-s)^{\alpha-1} E_{\alpha,\alpha} \left(-\frac{\lambda_n^\sigma}{1+k\lambda_n} (t-s)^\alpha \right) F_n(s) ds, \quad (2.4)$$

where $u_n(0) = \langle u(\cdot, 0), e_n \rangle$, and $F_n(s) = \langle F(\cdot, s), e_n \rangle$. Letting the nonlocal-condition, we have

$$\begin{aligned} \rho_n &= \int_0^T u_n(t) dt = u_n(0) \int_0^T E_{\alpha,1} \left(-\frac{\lambda_n^\sigma}{1+k\lambda_n} t^\alpha \right) dt \\ &\quad + \left(\frac{1}{1+k\lambda_n} \right) \int_0^T \left(\int_0^t (t-s)^{\alpha-1} E_{\alpha,\alpha} \left(-\frac{\lambda_n^\sigma}{1+k\lambda_n} (t-s)^\alpha \right) F_n(s) ds \right) dt. \end{aligned} \quad (2.5)$$

It gives

$$\begin{aligned} u_n(0) &= \left[\int_0^T E_{\alpha,1} \left(-\frac{\lambda_n^\sigma}{1+k\lambda_n} t^\alpha \right) dt \right]^{-1} \\ &\quad \times \left(\rho_n - \frac{1}{1+k\lambda_n} \int_0^T \left(\int_0^t (t-s)^{\alpha-1} E_{\alpha,\alpha} \left(-\frac{\lambda_n^\sigma}{1+k\lambda_n} (t-s)^\alpha \right) F_n(s) ds \right) dt \right), \end{aligned} \quad (2.6)$$

whereupon

$$u(x, 0) = \sum_{n=1}^{\infty} \left(\int_0^T E_{\alpha,1} \left(-\frac{\lambda_n^\sigma}{1+k\lambda_n} t^\alpha \right) dt \right)^{-1} \\ \times \left(\rho_n - \frac{1}{1+k\lambda_n} \int_0^T \left(\int_0^t (t-s)^{\alpha-1} E_{\alpha,\alpha} \left(-\frac{\lambda_n^\sigma}{1+k\lambda_n} (t-s)^\alpha \right) F_n(s) ds \right) dt \right) e_n(x). \quad (2.7)$$

Definition 2.1. [31]. The Mittag-Leffler function is

$$E_{\alpha,\beta}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha k + \beta)}, z \in \mathbb{C}, \alpha > 0, \beta \in \mathbb{R}. \quad (2.8)$$

Lemma 2.2. [32] For $\lambda > 0, \alpha > 0$ and positive integer $m \in \mathbb{N}$, we have

$$\frac{d^m}{dt^m} E_{\alpha,1}(-\lambda t^\alpha) = -\lambda t^{\alpha-m} E_{\alpha,\alpha-m+1}(-\lambda t^\alpha), t > 0. \quad (2.9)$$

Lemma 2.3. [31] Let $0 < \alpha_0 < \alpha_1 < 1, \forall \alpha \in [\alpha_0, \alpha_1]$, then there exist constants $A_1, A_2, A_3, A_4 > 0$ depending only on α_0, α_1 such that

$$a) \frac{A_1}{\alpha} e^{x^{\frac{1}{\alpha}}} \leq E_{\alpha,1}(x) \leq \frac{A_2}{\alpha} e^{x^{\frac{1}{\alpha}}}, \quad \forall x \geq 0, \\ b) \frac{A_3}{\Gamma(1-\alpha)} \frac{1}{1-x} \leq E_{\alpha,1}(x) \leq \frac{A_4}{\Gamma(1-\alpha)} \frac{1}{1-x}, \quad \forall x \leq 0. \quad (2.10)$$

Lemma 2.4. Let $0 < \alpha < 1, \sigma > 1$ and $\lambda_n > 0$, then we have

$$\frac{A_3}{\lambda_n^{\sigma-1}} \leq \int_0^T E_{\alpha,1} \left(-\frac{\lambda_n^\sigma}{1+k\lambda_n} t^\alpha \right) dt \leq \frac{A_4}{\lambda_n^{\sigma-1}}. \quad (2.11)$$

whereby $A_3 = \left(\frac{A_1(\lambda_1^{-1}+k)T}{\frac{\lambda_1^{-1}+k}{\lambda_1^{\sigma-1}} + T^\alpha} \right), A_4 = \left(\frac{A_2(\lambda_1^{-1}+k)T^{1-\alpha}}{1-\alpha} \right).$

Proof. From Lemma 2.3, we have

$$\int_0^T E_{\alpha,1} \left(-\frac{\lambda_n^\sigma}{1+k\lambda_n} t^\alpha \right) dt \leq A_2 \int_0^T \frac{dt}{1 + \frac{\lambda_n^\sigma}{1+k\lambda_n} t^\alpha} \\ \leq \frac{A_2(\lambda_1^{-1}+k)}{\lambda_n^{\sigma-1}} \int_0^T \frac{dt}{t^\alpha} = \frac{1}{\lambda_n^{\sigma-1}} \left(\frac{A_2(\lambda_1^{-1}+k)T^{1-\alpha}}{1-\alpha} \right). \quad (2.12)$$

Next, because of Lemma 2.3, one has

$$\int_0^T E_{\alpha,1} \left(-\frac{\lambda_n^\sigma}{1+k\lambda_n} t^\alpha \right) dt \geq A_1 \int_0^T \frac{dt}{1 + \frac{\lambda_n^\sigma}{1+k\lambda_n} t^\alpha} \geq \frac{A_1}{\lambda_n^{\sigma-1}} \int_0^T \frac{dt}{\frac{1}{\lambda_n^{\sigma-1}} + \frac{1}{\lambda_n^{-1}+k} T^\alpha} \\ \geq \frac{A_1}{\lambda_n^{\sigma-1}} \int_0^T \frac{dt}{\frac{1}{\lambda_1^{\sigma-1}} + \frac{1}{\lambda_1^{-1}+k} T^\alpha} \\ \geq \frac{A_1}{\lambda_n^{\sigma-1}} \int_0^T \frac{(\lambda_1^{-1}+k)dt}{\frac{\lambda_1^{-1}+k}{\lambda_1^{\sigma-1}} + T^\alpha} = \frac{1}{\lambda_n^{\sigma-1}} \left(\frac{A_1(\lambda_1^{-1}+k)T}{\frac{\lambda_1^{-1}+k}{\lambda_1^{\sigma-1}} + T^\alpha} \right). \quad (2.13)$$

From (2.12) and (2.13), we conclude that

$$\frac{1}{\lambda_n^{\sigma-1}} \left(\frac{A_1(\lambda_1^{-1} + k)T}{\frac{\lambda_1^{-1} + k}{\lambda_1^{\sigma-1}} + T^\alpha} \right) \leq \int_0^T E_{\alpha,1} \left(-\frac{\lambda_n^\sigma}{1 + k\lambda_n} t^\alpha \right) dt \leq \frac{1}{\lambda_n^{\sigma-1}} \left(\frac{A_2(\lambda_1^{-1} + k)T^{1-\alpha}}{1 - \alpha} \right). \quad (2.14)$$

Denoting $A_3 = \left(\frac{A_1(\lambda_1^{-1} + k)T}{\frac{\lambda_1^{-1} + k}{\lambda_1^{\sigma-1}} + T^\alpha} \right)$, $A_4 = \left(\frac{A_2(\lambda_1^{-1} + k)T^{1-\alpha}}{1 - \alpha} \right)$ which leads to (2.11) holds. $\square$

Lemma 2.5. For constants $\beta > 0; T > 0; j > 0$, we have

$$\sup_{n \in \mathbb{N}} \mathbf{G}_{\beta, \zeta}^{(j)}(\zeta_n) \left[\int_0^T E_{\alpha,1} \left(-\frac{\lambda_n^\sigma}{1 + k\lambda_n} t^\alpha \right) dt \right]^{-1} \leq \beta^{-\frac{1}{2}} j A_5. \quad (2.15)$$

Proof. By Proposition 18 in [33], we have

$$\mathbf{G}_{\beta, \zeta}^{(j)}(\zeta_n) \leq j \mathbf{G}_{\beta, \zeta}(\zeta_n), \quad (2.16)$$

where $\mathbf{G}_{\beta, \zeta}(\zeta_n)$ is the fractional Tikhonov method filter function

$$\mathbf{G}_{\beta, \zeta}(\zeta_n) = \zeta_n^{2\zeta} (\zeta_n^2 + \beta)^{-\zeta}. \quad (2.17)$$

From (2.16) and (2.17), we obtain

$$\begin{aligned} & \sup_{n \in \mathbb{N}} \mathbf{G}_{\beta, \zeta}^{(j)}(\zeta_n) \left[\int_0^T E_{\alpha,1} \left(-\frac{\lambda_n^\sigma}{1 + k\lambda_n} t^\alpha \right) dt \right]^{-1} \\ & \leq \sup_{n \in \mathbb{N}} j \mathbf{G}_{\beta, \zeta}(\zeta_n) \left[\int_0^T E_{\alpha,1} \left(-\frac{\lambda_n^\sigma}{1 + k\lambda_n} t^\alpha \right) dt \right]^{-1} \\ & = \sup_{n \in \mathbb{N}} j \left[\int_0^T E_{\alpha,1} \left(-\frac{\lambda_n^\sigma}{1 + k\lambda_n} t^\alpha \right) dt \right]^{-1} \left(\frac{\left| \int_0^T E_{\alpha,1} \left(-\frac{\lambda_n^\sigma}{1 + k\lambda_n} t^\alpha \right) dt \right|^2}{\beta + \left| \int_0^T E_{\alpha,1} \left(-\frac{\lambda_n^\sigma}{1 + k\lambda_n} t^\alpha \right) dt \right|^2} \right)^\zeta \\ & = \sup_{n \in \mathbb{N}} \frac{j \left| \int_0^T E_{\alpha,1} \left(-\frac{\lambda_n^\sigma}{1 + k\lambda_n} t^\alpha \right) dt \right|^{2\zeta-1}}{\left(\beta + \left| \int_0^T E_{\alpha,1} \left(-\frac{\lambda_n^\sigma}{1 + k\lambda_n} t^\alpha \right) dt \right|^2 \right)^\zeta} \leq \sup_{n \in \mathbb{N}} \frac{j \left(\frac{A_4}{\lambda_n^{\sigma-1}} \right)^{2\zeta-1}}{\left(\beta + \left(\frac{A_3}{\lambda_n^{\sigma-1}} \right)^2 \right)^\zeta}. \end{aligned} \quad (2.18)$$

Putting $z = \lambda_n^{1-\sigma}$, then we receive

$$\mathcal{B}(z) = j A_4^{2\zeta-1} \frac{z^{2\zeta-1}}{(\beta + A_3^2 z^2)^\zeta}. \quad (2.19)$$

For $\zeta > \frac{1}{2}$, to solve $\mathcal{B}'(z_0) = 0$, we find that $z_0 = A_3^{-1}((2\zeta - 1)\beta)^{\frac{1}{2}}$. Inserting z_0 in $\mathcal{B}(z)$ yields the maximum

$$\mathcal{B}(z) \leq \mathcal{B}(z_0) = \beta^{-\frac{1}{2}} \left(j A_4^{2\zeta-1} A_3^{1-2\zeta} (2\zeta - 1)^{1-\frac{1}{2}} (2\zeta)^{-\zeta} \right). \quad (2.20)$$

By denoting $A_5 = A_4^{2\zeta-1} A_3^{1-2\zeta} (2\zeta-1)^{1-\frac{1}{2}} (2\zeta)^{-\zeta}$, combining (2.20), we have

$$\sup_{n \in \mathbb{N}} \frac{\mathbf{G}_{\beta, \zeta}^{(j)}(\zeta_n)}{\int_0^T E_{\alpha, 1} \left(\frac{\lambda_n^\sigma}{1+k\lambda_n} \right) dt} \leq \beta^{-\frac{1}{2}} j A_5. \quad (2.21)$$

□

Lemma 2.6. For constants $\beta > 0; j > 0$, we have

$$\sup_{n \in \mathbb{N}} \left(1 - \mathbf{G}_{\beta, \zeta}^{(j)}(\zeta_n) \right)^2 \lambda_n^{-s} \leq \begin{cases} A_3^{-2} \beta^{2j}, & s \geq (4\sigma - 4)j, \\ A_6 \beta^{\frac{s}{2\sigma-2}}, & 0 < s < (4\sigma - 4)j. \end{cases} \quad (2.22)$$

Proof. By Proposition 18 in [33], we obtain

$$1 - \mathbf{G}_{\beta, \zeta}^{(j)}(\zeta_n) = (1 - \mathbf{G}_{\beta, \zeta}(\zeta_n))^j.$$

See in [30], we get

$$\mathbf{G}_{\beta, \zeta}(\zeta_n) \geq \mathbf{G}_{\beta, 1}(\zeta_n) > 0, \quad (2.23)$$

where $\mathbf{G}_{\beta, 1}(\zeta_n)$ is the classical Tikhonov method filter function

$$\mathbf{G}_{\beta, 1}(\zeta_n) = \frac{\zeta_n^2}{\zeta_n^2 + \beta}. \quad (2.24)$$

Combining (2.20)-(2.23), we get

$$\begin{aligned} \sup_{n \in \mathbb{N}} \left(1 - \mathbf{G}_{\beta, \zeta}^{(j)}(\zeta_n) \right)^2 \lambda_n^{-s} &= \sup_{n \in \mathbb{N}} \left(1 - \mathbf{G}_{\beta, \zeta}(\zeta_n) \right)^{2j} \lambda_n^{-s} \leq \sup_{n \in \mathbb{N}} \left(1 - \mathbf{G}_{\beta, 1}(\zeta_n) \right)^{2j} \lambda_n^{-s} \\ &= \sup_{n \in \mathbb{N}} \left(\frac{\beta}{\zeta_n^2 + \beta} \right)^{2j} \lambda_n^{-s} \leq \sup_{n \in \mathbb{N}} \left(\frac{\beta}{\left| \int_0^T E_{\alpha, 1} \left(\frac{\lambda_n^\sigma}{1+k\lambda_n} \right) dt \right|^2 + \beta} \right)^{2j} \lambda_n^{-s} \\ &\leq \sup_{n \in \mathbb{N}} \left(\frac{\beta \lambda_n^{-\frac{s}{2j}}}{\left(\frac{A_3}{\lambda_n^{\sigma-1}} \right)^2 + \mu} \right)^{2j} = \sup_{n \in \mathbb{N}} \left(\frac{\beta \lambda_n^{-\frac{s}{2j} + 2\sigma - 2}}{A_3^2 + \beta \lambda_n^{2\sigma - 2}} \right)^{2j}. \end{aligned} \quad (2.25)$$

Denoting $z = \lambda_n$. Let $\mathcal{S}(z) = \frac{\beta z^{-\frac{s}{2j} + 2\sigma - 2}}{A_3^2 + \beta z^{2\sigma - 2}}$. Then we have

- If $-\frac{s}{2j} + 2\sigma - 2 \leq 0$, that is $s > (4\sigma - 4)j$, for $z \geq 1$, then we have

$$\mathcal{S}(z) \leq A_3^{-2} \beta.$$

- If $-\frac{s}{2j} + 2\sigma - 2 > 0$, that is $0 < s < (4\sigma - 4)j$. To solve $\mathcal{S}'(z) = 0$, we obtain $z_0 = \left(\frac{A_3^2(-s+4j(\sigma-1))}{\beta s} \right)^{\frac{1}{2(\sigma-1)}}$. Inserting z_0 in $\mathcal{D}(z)$ yields the maximum

$$\mathcal{S}(z) \leq \mathcal{S}(z_0) = \left(\frac{A_3^2(-s+4j(\sigma-1))}{\beta s} \right)^{-\frac{s}{2j(2\sigma-2)}} \leq \left(\frac{\beta s}{A_3^2(-s+4j(\sigma-1))} \right)^{\frac{s}{2j(2\sigma-2)}}.$$

This leads to

$$\sup_{n \in \mathbb{N}} (1 - \mathbf{G}_{\beta, \zeta}^{(j)}(\zeta_n))^2 \lambda_n^{-s} \leq \begin{cases} (A_3^{-2} \beta)^{2j}, & s \geq (4\sigma - 4)j, \\ \left(\left(\frac{\beta s}{A_3^2(-s+4j(\sigma-1))} \right)^{\frac{s}{2j(2\sigma-2)}} \right)^{2j}, & 0 < s < (4\sigma - 4)j. \end{cases}$$

Putting $\left(\frac{\beta s}{A_3^2(-s+4j(\sigma-1))} \right)^{\frac{s}{2j(2\sigma-2)}} = A_6$, we receive

$$\sup_{n \in \mathbb{N}} (1 - \mathbf{G}_{\beta, \zeta}^{(j)}(\zeta_n))^2 \lambda_n^{-s} \leq \begin{cases} A_3^{-4j} \beta^{2j}, & s \geq (4\sigma - 4)j, \\ A_6 \beta^{\frac{s}{2\sigma-2}}, & 0 < s < (4\sigma - 4)j. \end{cases}$$

□

Lemma 2.7. Let $F(x, t) \in L^\infty(0, T; L^2(\mathcal{D}))$. Then a positive constant A_6 exists such that

$$\sum_{n=1}^{\infty} \left| \int_0^T \left(\int_0^t (t-s)^{\alpha-1} E_{\alpha, \alpha} \left(-\frac{\lambda_n^\sigma}{1+k\lambda_n} (t-s)^\alpha \right) F_n(s) ds \right) dt \right|^2 \leq A_6 \|F\|_{L^\infty(0, T; L^2(\mathcal{D}))}^2. \quad (2.26)$$

Proof. For any $t > 0, \sigma > 1$, we have

$$\int_0^t s^{\alpha-1} E_{\alpha, \alpha} \left(-\frac{\lambda_n^\sigma}{1+k\lambda_n} s^\alpha \right) ds = \frac{1+k\lambda_n}{\lambda_n^\sigma} \left(1 - E_{\alpha, 1} \left(-\frac{\lambda_n^\sigma}{1+k\lambda_n} t^\alpha \right) \right).$$

Hence

$$0 \leq \int_0^t (t-s)^{\alpha-1} E_{\alpha, \alpha} \left(-\frac{\lambda_n^\sigma}{1+k\lambda_n} (t-s)^\alpha \right) ds \leq \frac{1}{\lambda_n^{\sigma-1}} (\lambda_1^{-1} + k), t \geq 0.$$

Noting that $|F_n(t)|^2 \leq \sum_{n=1}^{\infty} |\langle F(\cdot, t), e_n \rangle|^2 \leq \|F\|_{L^\infty(0, T; L^2(\mathcal{D}))}^2$. Therefore, we conclude that

$$\begin{aligned} & \sum_{n=1}^{\infty} \left| \int_0^t (t-s)^{\alpha-1} E_{\alpha, \alpha} \left(-\frac{\lambda_n^\sigma}{1+k\lambda_n} (t-s)^\alpha \right) F_n(s) ds \right|^2 \\ & \leq \|F\|_{L^\infty(0, T; L^2(\mathcal{D}))}^2 \sum_{n=1}^{\infty} \left| \int_0^t (t-s)^{\alpha-1} E_{\alpha, \alpha} \left(-\frac{\lambda_n^\sigma}{1+k\lambda_n} (t-s)^\alpha \right) ds \right|^2 \\ & \leq \|F\|_{L^\infty(0, T; L^2(\mathcal{D}))}^2 \sum_{n=1}^{\infty} \left(\frac{\lambda_1^{-1} + k}{\lambda_n^{\sigma-1}} \right)^2 = \|F\|_{L^\infty(0, T; L^2(\mathcal{D}))}^2 (\lambda_1^{-1} + k)^2 \sum_{n=1}^{\infty} \frac{1}{\lambda_n^{2\sigma-2}}. \end{aligned} \quad (2.27)$$

Letting $C_5 = (\lambda_1^{-1} + k)^2 \sum_{n=1}^{\infty} \frac{1}{\lambda_n^{2\sigma-2}}$, then

$$\sum_{n=1}^{\infty} \left| \int_0^T \left(\int_0^t (t-s)^{\alpha-1} E_{\alpha, \alpha} \left(-\frac{\lambda_n^\sigma}{1+k\lambda_n} (t-s)^\alpha \right) F_n(s) ds \right) dt \right|^2 \leq \underbrace{C_5 T}_{:A_6} \|F\|_{L^\infty(0, T; L^2(\mathcal{D}))}^2. \quad (2.28)$$

□

3 Regularization solution

The iterated version of the fractional Tikhonov method [34]

$$f_{\beta,\zeta}^0 := 0, \quad (\mathcal{P}^* \mathcal{P} + \beta I)^\zeta f_{\beta,\zeta}^j := (\mathcal{P}^* \mathcal{P})^{\zeta-1} \mathcal{P}^* \ell + \left((\mathcal{P}^* \mathcal{P} + \beta I)^\zeta - (\mathcal{P}^* \mathcal{P})^\zeta \right) f_{\beta,\zeta}^{j-1}, \quad (3.1)$$

with $\frac{1}{2} \leq \zeta \leq 1$. If $\zeta = 1$ we get the standard iterative Tikhonov method, by choosing $\frac{1}{2} < \zeta < 1$ one can get more precise numerical results. for discontinuous solutions, please refer to [?]. With the n iteration presumably fixed in this section, β is the normalization parameter. The IFTRM in [?] is a filter-based normalization method, for each given $j \in \mathbb{N}^*$ and $\frac{1}{2} < \zeta < 1$.

$$\mathbf{G}_{\beta,\zeta}^{(j)}(\zeta) = \frac{(\zeta^2 + \beta)^{\zeta j} - \left[(\zeta^2 + \beta)^\zeta - \zeta^{2\zeta} \right]^j}{(\zeta^2 + \beta)^{\zeta j}}. \quad (3.2)$$

For $t = 0$, the solution $u(x, t)$ is unstable. The main idea is to replace

$$\frac{1}{\zeta_n} = \left(\int_0^T E_{\alpha,1} \left(-\frac{\lambda_n^\sigma}{1+k\lambda_n} t^\alpha \right) dt \right)^{-1} \text{ by the } \left(\int_0^T E_{\alpha,1} \left(-\frac{\lambda_n^\sigma}{1+k\lambda_n} t^\alpha \right) dt \right)^{-1} \mathbf{G}_{\beta,\zeta}^{(j)}(\zeta_n),$$

then $u(x, 0)$ is approximated by $u_{\beta,\zeta}^{j,\delta}(x, 0)$ as follows:

$$\begin{aligned} u_{\beta,\zeta}^{j,\delta}(x, 0) &= \sum_{n=1}^{\infty} \mathbf{G}_{\beta,\zeta}^{(j)}(\zeta_n) \left[\int_0^T E_{\alpha,1} \left(-\frac{\lambda_n^\sigma}{1+k\lambda_n} t^\alpha \right) dt \right]^{-1} \\ &\times \left(\rho_n^\delta - \frac{1}{1+k\lambda_n} \int_0^T \left(\int_0^t (t-s)^{\alpha-1} E_{\alpha,\alpha} \left(-\frac{\lambda_n^\sigma}{1+k\lambda_n} (t-s)^\alpha \right) F_n^\delta(s) ds \right) dt \right) e_n(x). \end{aligned} \quad (3.3)$$

Afterthat, we will prove the estimate for $\|u_{\beta,\zeta}^{j,\delta}(\cdot, 0) - u(\cdot, 0)\|_{L^2(D)}$.

4 The random noise

From (1.2), the coefficients of $\tilde{\rho}^\delta(x)$, $\tilde{F}^\delta(x, t)$ with respect to e_n are

$$\begin{aligned} \tilde{\rho}_n^\delta &= \langle \tilde{\rho}^\delta(x), e_n(x) \rangle = \langle \rho(x), e_n \rangle + \delta \langle \omega, e_n \rangle = \rho_n + \delta \omega_n, \\ \tilde{F}_n^\delta(t) &= \langle \tilde{F}^\delta(x, t), e_n \rangle = \langle F(x, t), e_n \rangle + \delta \langle \Xi, e_n \rangle = F_n(t) + \delta \Xi_n. \end{aligned} \quad (4.1)$$

Since ω is white noise, Ξ is a sequence of standard Gaussian random variables $\mathcal{N}(0, 1)$. It gives

$$\tilde{\rho}_n^\delta = \rho_n + \delta \omega_n, \quad \tilde{F}_n^\delta(t) = F_n(t) + \delta \Xi_n. \quad (4.2)$$

Definition 4.1. The mean *integrated squared risk* (MISE) of $v_{\gamma(\delta)}^\delta$ is defined as

$$R(u_{\beta,\zeta}^{j,\delta}, u) = \mathbb{E} \|u_{\beta,\zeta}^{j,\delta}(\cdot, 0) - u(\cdot, 0)\|_{L^2(D)}^2 = \mathbb{E} \sum_{n=1}^{\infty} \left(u_{\beta,\zeta,n}^{j,\delta}(0) - u_n(0) \right)^2, \quad (4.3)$$

where $u_{\beta,\zeta,n}^{j,\delta}(0) = \langle u_{\beta,\zeta}^{j,\delta}(\cdot, 0), e_n \rangle$, $u_n(0) = \langle u(\cdot, 0), e_n \rangle$. A regularized solution for $u_{\beta,\zeta}^{j,\delta}(x, 0)$ with the filter function $\mathbf{G}_{\beta,\zeta}^{(j)}(\zeta_n)$ as follows

$$\begin{aligned} u_{\beta,\zeta}^{j,\delta}(x, 0) &= \sum_{n=1}^{\infty} \mathbf{G}_{\beta,\zeta}^{(j)}(\zeta_n) \left(\int_0^T E_{\alpha,1} \left(-\frac{\lambda_n^\sigma}{1+k\lambda_n} t^\alpha \right) dt \right)^{-1} \\ &\times \left(\tilde{\rho}_{\delta,n} - \frac{1}{1+k\lambda_n} \int_0^T \left(\int_0^t (t-s)^{\alpha-1} E_{\alpha,\alpha} \left(-\frac{\lambda_n^\sigma}{1+k\lambda_n} (t-s)^\alpha \right) \tilde{F}_{\delta,n}(s) ds \right) dt \right) e_n(x), \end{aligned} \quad (4.4)$$

where $\tilde{\rho}_{\delta,n} = \langle \tilde{\rho}_\delta, e_n \rangle$, $\tilde{F}_{\delta,n}(s) = \langle \tilde{F}_\delta(\cdot, s), e_n \rangle$.

Lemma 4.2. *Let*

$$\tilde{J}_{1,n} = \langle \omega, e_n \rangle + \langle \Xi, e_n \rangle \frac{1}{1 + k\lambda_n} \int_0^T \left(\int_0^t (t-s)^{\alpha-1} E_{\alpha,\alpha} \left(-\frac{\lambda_n^\sigma}{1 + k\lambda_n} (t-s)^\alpha \right) ds \right) dt. \quad (4.5)$$

Then one has $\mathbb{E}\tilde{J}_{1,n} = 0$ and

$$\mathbb{E}\tilde{J}_{1,n}^2 = \langle \omega, e_n \rangle^2 + \langle \Xi, e_n \rangle^2 \left(\frac{1}{1 + k\lambda_n} \left(\int_0^T \left(\int_0^t (t-s)^{\alpha-1} E_{\alpha,\alpha} \left(-\frac{\lambda_n^\sigma}{1 + k\lambda_n} (t-s)^\alpha \right) ds \right) dt \right) \right)^2. \quad (4.6)$$

Proof. We receive

$$\mathbb{E}\langle \omega, e_n \rangle = \mathbb{E}\langle \Xi, e_n \rangle = 0, \quad \mathbb{E}\langle \Xi, e_n \rangle^2 = 1.$$

Hence, we obtain

$$\mathbb{E}\tilde{J}_{1,n} = \mathbb{E}\langle \omega, e_n \rangle + \mathbb{E}\langle \Xi, e_n \rangle \frac{1}{1 + k\lambda_n} \int_0^T \int_0^t ((t-s)^{\alpha-1} E_{\alpha,\alpha} \left(-\frac{\lambda_n^\sigma}{1 + k\lambda_n} (t-s)^\alpha \right) ds) dt = 0.$$

Then, we have

$$\begin{aligned} \tilde{J}_{1,n}^2 &= \langle \omega, e_n \rangle^2 \\ &+ 2\langle \omega, e_n \rangle \langle \Xi, e_n \rangle \left(\frac{1}{1 + k\lambda_n} \right) \left(\int_0^T \left(\int_0^t (t-s)^{\alpha-1} E_{\alpha,\alpha} \left(-\frac{\lambda_n^\sigma}{1 + k\lambda_n} (t-s)^\alpha \right) ds \right) dt \right) \\ &+ \langle \Xi, e_n \rangle^2 \left(\frac{1}{1 + k\lambda_n} \right)^2 \left(\int_0^T \left(\int_0^t (t-s)^{\alpha-1} E_{\alpha,\alpha} \left(-\frac{\lambda_n^\sigma}{1 + k\lambda_n} (t-s)^\alpha \right) ds \right) dt \right)^2. \end{aligned}$$

Because of $\mathbb{E}(\langle \omega, e_n \rangle \langle \Xi, e_n \rangle) = 0$. The Lemma 4.2 is proven. $\square$

4.1 A-priori parameter choice rule

Theorem 4.3. *For any $\sigma > 1$, assume that (1.2) holds, then we get:*

- If $s \geq (4\sigma - 4)j$ and choose $\beta(\delta) = \left(\frac{\delta}{E} \right)^{\frac{2}{(2\sigma-2)j+1}}$, then we have

$$\|u_{\beta,\zeta}^{j,\delta}(\cdot, 0) - u(\cdot, 0)\|_{L^2(\mathcal{D})} \text{ is of order } \max \left\{ \delta^{\frac{4(\sigma-1)j}{2(\sigma-1)j+1}}, \delta^{\frac{4j}{(2\sigma-2)j+1}} \right\}. \quad (4.7)$$

- If $0 < s < (4\sigma - 4)j$ and choose $\beta(\delta) = \left(\frac{\delta}{E} \right)^{\frac{4}{s+2}}$, then we have

$$\|u_{\beta,\zeta}^{j,\delta}(\cdot, 0) - u(\cdot, 0)\|_{L^2(\mathcal{D})} \text{ is of order } \max \left\{ \delta^{\frac{2s}{s+2}}, \delta^{\frac{2s}{(s+2)(\sigma-1)}} \right\}. \quad (4.8)$$

Proof. From (2.7) and (3.3), we have

$$\begin{aligned}
 & u_n(0) - u_{\beta, \zeta, n}^{j, \delta}(0) \\
 &= \frac{1 - \mathbf{G}_{\beta, \zeta}^{(j)}(\zeta_n)}{\int_0^T E_{\alpha, 1}\left(-\frac{\lambda_n^\sigma}{1+k\lambda_n} t^\alpha\right) dt} \left(\rho_n - \frac{1}{1+k\lambda_n} \int_0^T \left(\int_0^t (t-s)^{\alpha-1} E_{\alpha, \alpha} \left(-\frac{\lambda_n^\sigma}{1+k\lambda_n} (t-s)^\alpha \right) F_n(s) ds \right) dt \right) \\
 &\quad - \delta \frac{\mathbf{G}_{\beta, \zeta}^{(j)}(\zeta_n)}{\int_0^T E_{\alpha, 1}\left(-\frac{\lambda_n^\sigma}{1+k\lambda_n} t^\alpha\right) dt} \left(\omega_n - \Xi_n \frac{1}{1+k\lambda_n} \int_0^T \left(\int_0^t (t-s)^{\alpha-1} E_{\alpha, \alpha} \left(-\frac{\lambda_n^\sigma}{1+k\lambda_n} (t-s)^\alpha \right) ds \right) dt \right) \\
 &:= \tilde{J}_{2,n} - \delta \frac{\mathbf{G}_{\beta, \zeta}^{(j)}(\zeta_n)}{\int_0^T E_{\alpha, 1}\left(-\frac{\lambda_n^\sigma}{1+k\lambda_n} t^\alpha\right) dt} \tilde{J}_{1,n}, \tag{4.9}
 \end{aligned}$$

where $\tilde{J}_{1,n}$ is defined in Lemma 4.2 and

$$\tilde{J}_{2,n} = \frac{1 - \mathbf{G}_{\beta, \zeta}^{(j)}(\zeta_n)}{\int_0^T E_{\alpha, 1}\left(-\frac{\lambda_n^\sigma}{1+k\lambda_n} t^\alpha\right) dt} \left(\rho_n - \frac{1}{1+k\lambda_n} \int_0^T \left(\int_0^t (t-s)^{\alpha-1} E_{\alpha, \alpha} \left(-\frac{\lambda_n^\sigma}{1+k\lambda_n} (t-s)^\alpha \right) F_n(s) ds \right) dt \right). \tag{4.10}$$

This leads to

$$\begin{aligned}
 & (u_n(\cdot, 0) - u_{\beta, \zeta, n}^{j, \delta}(\cdot, 0))^2 \\
 &= |\tilde{J}_{2,n}|^2 - 2\delta \frac{\mathbf{G}_{\beta, \zeta}^{(j)}(\zeta_n)}{\int_0^T E_{\alpha, 1}\left(-\frac{\lambda_n^\sigma}{1+k\lambda_n} t^\alpha\right) dt} \tilde{J}_{2,n} \tilde{J}_{1,n} + \delta^2 \left(\frac{\mathbf{G}_{\beta, \zeta}^{(j)}(\zeta_n)}{\int_0^T E_{\alpha, 1}\left(-\frac{\lambda_n^\sigma}{1+k\lambda_n} t^\alpha\right) dt} \right)^2 |\tilde{J}_{1,n}|^2
 \end{aligned}$$

Since $\mathbb{E}\tilde{J}_{1,n} = 0$ and using Lemma 4.2, one has

$$\begin{aligned}
& \mathbb{E}(u_n(0) - u_{\beta,\zeta,n}^{j,\delta}(0))^2 \\
&= \delta^2 \left(\frac{\mathbf{G}_{\beta,\zeta}^{(j)}(\zeta_n)}{\int_0^T E_{\alpha,1} \left(-\frac{\lambda_n^\sigma}{1+k\lambda_n} t^\alpha \right) dt} \right)^2 \mathbb{E}|\tilde{J}_{1,n}|^2 - 2\delta \frac{\mathbf{G}_{\beta,\zeta}^{(j)}(\zeta_n)}{\int_0^T E_{\alpha,1} \left(-\frac{\lambda_n^\sigma}{1+k\lambda_n} t^\alpha \right) dt} \left(\mathbb{E}\tilde{J}_{1,n} \right) \tilde{J}_{2,n} + |\tilde{J}_{2,n}|^2 \\
&= |\tilde{J}_{2,n}|^2 + \delta^2 \left(\frac{\mathbf{G}_{\beta,\zeta}^{(j)}(\zeta_n)}{\int_0^T E_{\alpha,1} \left(-\frac{\lambda_n^\sigma}{1+k\lambda_n} t^\alpha \right) dt} \right)^2 \mathbb{E}|\tilde{J}_{1,n}|^2 \\
&= |\tilde{J}_{2,n}|^2 + \delta^2 \frac{\mathbf{G}_{\beta,\zeta}^{(j)}(\zeta_n)}{\int_0^T E_{\alpha,1} \left(-\frac{\lambda_n^\sigma}{1+k\lambda_n} t^\alpha \right) dt} \mathbb{E} \langle \omega(x), e_n(x) \rangle^2 \\
&\quad + \delta^2 \frac{\mathbf{G}_{\beta,\zeta}^{(j)}(\zeta_n)}{\int_0^T E_{\alpha,1} \left(-\frac{\lambda_n^\sigma}{1+k\lambda_n} t^\alpha \right) dt} \left(\frac{1}{1+k\lambda_n} \int_0^T \left(\int_0^t (t-s)^{\alpha-1} E_{\alpha,\alpha} \left(-\frac{\lambda_n^\sigma}{1+k\lambda_n} (t-s)^\alpha \right) dt \right) dt \right)^2 \\
&= |\tilde{J}_{2,n}|^2 + \delta^2 \tilde{J}_{3,n} + \delta^2 \tilde{J}_{4,n}, \tag{4.11}
\end{aligned}$$

where

$$\begin{aligned}
\tilde{J}_{3,n} &= \frac{\mathbf{G}_{\beta,\zeta}^{(j)}(\zeta_n)}{\int_0^T E_{\alpha,1} \left(-\frac{\lambda_n^\sigma}{1+k\lambda_n} t^\alpha \right) dt} \mathbb{E} \langle \omega(x), e_n(x) \rangle^2, \text{ and} \\
\tilde{J}_{4,n} &= \frac{\mathbf{G}_{\beta,\zeta}^{(j)}(\zeta_n)}{\int_0^T E_{\alpha,1} \left(-\frac{\lambda_n^\sigma}{1+k\lambda_n} t^\alpha \right) dt} \left(\frac{1}{1+k\lambda_n} \int_0^T \left(\int_0^t (t-s)^{\alpha-1} E_{\alpha,\alpha} \left(-\frac{\lambda_n^\sigma}{1+k\lambda_n} (t-s)^\alpha \right) dt \right) dt \right)^2. \tag{4.12}
\end{aligned}$$

Estimate $\sum_{n=1}^{\infty} |\tilde{J}_{2,n}|^2$ as follows

$$\begin{aligned}
\sum_{n=1}^{\infty} |\tilde{J}_{2,n}|^2 &= \sum_{n=1}^{\infty} \left(1 - \mathbf{G}_{\beta,\zeta}^{(j)}(\zeta_n) \right)^2 |\langle u(\cdot, 0), e_n(\cdot) \rangle|^2 \\
&= \sum_{n=1}^{\infty} \left(1 - \mathbf{G}_{\beta,\zeta}^{(j)}(\zeta_n) \right)^2 \lambda_n^{-2s} \lambda_n^{2s} |\langle u(\cdot, 0), e_n(\cdot) \rangle|^2 \\
&\leq \begin{cases} A_3^{-2} \beta^{2j} E^2, & s \geq (4\sigma - 4)j, \\ A_6 \beta^{\frac{s}{2\sigma-2}} E^2, & 0 < s < (4\sigma - 4)j. \end{cases} \tag{4.13}
\end{aligned}$$

By Lemma 2.6, estimation for $\sum_{n=1}^{\infty} |\tilde{J}_{3,n}|^2$ as follows

$$\begin{aligned}
\sum_{n=1}^{\infty} |\tilde{J}_{3,n}|^2 &= \sum_{n=1}^{\infty} \left(\frac{\mathbf{G}_{\beta,\zeta}^{(j)}(\zeta_n)}{\int_0^T E_{\alpha,1} \left(-\frac{\lambda_n^\sigma}{1+k\lambda_n} t^\alpha \right) dt} \right)^2 \mathbb{E} \omega_n^2 \\
&\leq \left(\frac{jA_5}{\beta^{\frac{1}{2}}} \right)^2 \sum_{n=1}^{\infty} \mathbb{E} \omega_n^2 = \left(\frac{jA_5}{\beta^{\frac{1}{2}}} \right)^2 \mathbb{E} \|\omega\|_{L^2(\Omega)}^2 = \frac{j^2 A_5^2}{\beta}. \tag{4.14}
\end{aligned}$$

By Lemma 2.6, we get the estimation for $\sum_{n=1}^{\infty} |\tilde{J}_{4,n}|^2$ as follows

$$\begin{aligned} \sum_{n=1}^{\infty} |\tilde{J}_{4,n}|^2 &= \sum_{n=1}^{\infty} \left(\frac{\mathbf{G}_{\beta,\zeta}^{(j)}(\sigma_n)}{\int_0^T E_{\alpha,1} \left(-\frac{\lambda_n^\sigma}{1+k\lambda_n} t^\alpha \right) dt} \right)^2 \left(\frac{1}{1+k\lambda_n} \int_0^T \left(\int_0^t (t-s)^{\alpha-1} E_{\alpha,\alpha} \left(-\frac{\lambda_n^\sigma}{1+k\lambda_n} (t-s)^\alpha \right) dt \right) dt \right)^2 \\ &\leq \frac{1}{(k\lambda_1)^2} \left(\frac{jA_5}{\beta^{\frac{1}{2}}} \right)^2 \sum_{n=1}^{\infty} \left(\int_0^T \left(\int_0^t (t-s)^{\alpha-1} E_{\alpha,\alpha} \left(-\frac{\lambda_n^\sigma}{1+k\lambda_n} (t-s)^\alpha \right) dt \right) dt \right)^2 \\ &= \frac{1}{(k\lambda_1)^2} \frac{j^2 A_5^2 A_6^2}{\beta}. \end{aligned} \quad (4.15)$$

Combining (4.9), (4.13), (4.14), and (4.15), we conclude that

$$\begin{aligned} \mathbb{E} \|\tilde{u}_{\beta,\zeta}^{j,\delta}(\cdot, 0) - u(\cdot, 0)\|_{L^2(\mathcal{D})}^2 &= \mathbb{E} \sum_{n=1}^{\infty} (u_n(\cdot, 0) - \tilde{u}_{\beta,\zeta,n}^{j,\delta}(\cdot, 0))^2 = \sum_{n=1}^{\infty} \mathbb{E} (u_n(\cdot, 0) - \tilde{u}_{\beta,\zeta,n}^{j,\delta}(\cdot, 0))^2 \\ &= \sum_{n=1}^{\infty} (|\tilde{J}_{2,n}|^2 + \delta^2 \tilde{J}_{3,n} + \delta^2 \tilde{J}_{4,n}) \\ &\leq j^2 \left(A_5^2 + \frac{(A_5 A_6)^2}{(k\lambda_1)^2} \right) \frac{\delta^2}{\beta} + \begin{cases} A_3^{-2} \beta^{2j} E^2, & s \geq (4\sigma - 4)j, \\ A_6 \beta^{\frac{s}{2\sigma-2}} E^2, & 0 < s < (4\sigma - 4)j. \end{cases} \end{aligned} \quad (4.16)$$

$\beta(\delta)$ is chosen by

$$\beta(\delta) = \begin{cases} \left(\frac{\delta}{E} \right)^{\frac{2}{(2\sigma-2)j+1}}, & s \geq (4\sigma - 4)j, \\ \left(\frac{\delta}{E} \right)^{\frac{4}{s+2}}, & 0 < s < (4\sigma - 4)j. \end{cases}$$

- If $s \geq (4\sigma - 4)j$, then we have

$$\begin{aligned} \mathbb{E} \|\tilde{u}_{\beta,\zeta}^{j,\delta}(\cdot, 0) - u(\cdot, 0)\|_{L^2(\mathcal{D})}^2 &\leq j^2 \left(A_5^2 + \frac{(A_5 A_6)^2}{(k\lambda_1)^2} \right) E^{\frac{2}{(2\sigma-2)j+1}} \delta^{\frac{4(\sigma-1)j}{2(\sigma-1)j+1}} \\ &\quad + A_3^{-2} \delta^{\frac{4j}{(2\sigma-2)j+1}} E^{\frac{4(\sigma-1)j}{(2\sigma-2)j+1}}. \end{aligned} \quad (4.17)$$

- If $0 < s \leq (4\sigma - 4)j$, then we have

$$\mathbb{E} \|\tilde{u}_{\beta,\zeta}^{j,\delta}(\cdot, 0) - u(\cdot, 0)\|_{L^2(\mathcal{D})}^2 \leq j^2 \left(A_5^2 + \frac{(A_5 A_6)^2}{(k\lambda_1)^2} \right) \delta^{\frac{2s}{s+2}} E^{\frac{4}{s+2}} + A_6 \delta^{\frac{2s}{(s+2)(\sigma-1)}} E^{\frac{2s}{s+2}}. \quad (4.18)$$

□

5 Simulation

In this section, we choice $\Omega = (0, \pi)$, $T = 1$, and $\lambda_n = n^2$ to complete our numerical experiment. Assume that the function (ρ^δ, F^δ) is replaced by $(\tilde{\rho}^\delta, \tilde{F}^\delta)$ where $\tilde{\rho}^\delta$ and $\tilde{F}^\delta$ have the following noise model

$$\tilde{\rho}^\delta(x) = \rho(x) + \delta\omega, \quad \tilde{F}^\delta(x, t) = F(x, t) + \delta\Xi, \quad (5.1)$$

where ω and Ξ are the white noises, we can find that $\mathbb{E}\omega_n^2 = 1$, $\mathbb{E}\Xi_n^2 = 1$. Let $F(x, t) = \frac{\pi}{2} E_{\alpha,1}(-t^\alpha) \sin(2x)$, $\rho(x) = \frac{\pi}{2} E_{\alpha,1}(-1) \sin(2x)$. In order to calculate the integral in (5.4), we

use the following facts:

$$\int_0^x u^{\rho-1} E_{\alpha,\rho}(yu^\alpha)(x-u)^{\beta-1} E_{\alpha,\beta}[z(x-u)^\alpha] du = \frac{yE_{\alpha,\rho+\beta}(yx^\alpha) - zE_{\alpha,\rho+\beta}(zx^\alpha)}{y-z} x^{\beta+\rho+1}. \quad (5.2)$$

From (4.11), we receive

$$\mathbb{E} \|\tilde{u}_{\beta,\zeta}^{j,\delta}(\cdot, 0) - u(\cdot, 0)\|_{L^2(\mathcal{D})}^2 = |\tilde{J}_{2,2}|^2 + \delta^2 \tilde{J}_{3,2} + \delta^2 \tilde{J}_{4,2}, \quad (5.3)$$

whereby

$$\begin{aligned} \tilde{J}_{2,2} &= \frac{\pi}{2} \frac{(1 - \mathbb{G}_{\beta,\zeta}^{(j)}(\zeta_2))}{\int_0^1 E_{\alpha,1}\left(-\frac{n^{2\sigma}}{1+kn^2}t^\alpha\right)dt} \left(E_{\alpha,1}(-1) - \frac{1}{1+kn^2} \right. \\ &\quad \times \left. \int_0^1 \left(\int_0^t (t-s)^{\alpha-1} E_{\alpha,\alpha}\left(-\frac{n^{2\sigma}}{1+kn^2}(t-s)^\alpha\right) E_{\alpha,1}(-s^\alpha) ds \right) dt \right), \\ \tilde{J}_{3,2} &= \left(\frac{\mathbb{G}_{\beta,\zeta}^{(j)}(\zeta_2)}{\int_0^1 E_{\alpha,1}\left(-\frac{n^{2\sigma}}{1+kn^2}t^\alpha\right)dt} \right)^2, \\ \tilde{J}_{4,2} &= \left(\frac{\mathbb{G}_{\beta,\zeta}^{(j)}(\zeta_2)}{\int_0^1 E_{\alpha,1}\left(-\frac{n^{2\sigma}}{1+kn^2}t^\alpha\right)dt} \right)^2 \left(\frac{1}{1+kn^2} \int_0^1 \left(\int_0^t (t-s)^{\alpha-1} E_{\alpha,\alpha}\left(-\frac{n^{2\sigma}}{1+kn^2}(t-s)^\alpha\right) ds \right) dt \right)^2, \\ \mathbb{G}_{\beta,\zeta}^{(j)}(\zeta_2) &= 1 - \frac{\left[\left(\left| \int_0^1 E_{\alpha,1}\left(-\frac{n^{2\sigma}}{1+kn^2}t^\alpha\right)dt \right|^2 + \beta \right)^\zeta - \left| \int_0^1 E_{\alpha,1}\left(-\frac{n^{2\sigma}}{1+kn^2}t^\alpha\right)dt \right|^{2\zeta} \right]^j}{\left(\left| \int_0^1 E_{\alpha,1}\left(-\frac{n^{2\sigma}}{1+kn^2}t^\alpha\right)dt \right|^2 + \beta \right)^{\zeta j}} \\ &= 1 - \frac{\left[\left(\left| E_{\alpha,2}\left(-\frac{n^{2\sigma}}{1+kn^2}\right) \right|^2 + \beta \right)^\zeta - \left| E_{\alpha,2}\left(-\frac{n^{2\sigma}}{1+kn^2}\right) \right|^{2\zeta} \right]^j}{\left(\left| E_{\alpha,2}\left(-\frac{n^{2\sigma}}{1+kn^2}\right) \right|^2 + \beta \right)^{\zeta j}}. \end{aligned} \quad (5.4)$$

From (5.2), we passed some calculations to get

$$\begin{aligned} \int_0^t (t-s)^{\alpha-1} E_{\alpha,\alpha}\left(-\frac{n^{2\sigma}}{1+kn^2}(t-s)^\alpha\right) ds &= \frac{1+kn^2}{n^{2\sigma}} \left(1 - E_{\alpha,1}\left(-\frac{n^{2\sigma}}{1+kn^2}t^\alpha\right) \right), \\ \int_0^1 E_{\alpha,1}\left(-\frac{n^{2\sigma}}{1+kn^2}t^\alpha\right) dt &= E_{\alpha,2}\left(-\frac{n^{2\sigma}}{1+kn^2}\right). \end{aligned} \quad (5.5)$$

This leads to

$$\begin{aligned} & \frac{1}{1+kn^2} \int_0^1 \left(\int_0^t (t-s)^{\alpha-1} E_{\alpha,\alpha} \left(-\frac{n^{2\sigma}}{1+kn^2} (t-s)^\alpha \right) ds \right) dt, \\ &= \frac{1}{1+kn^2} \int_0^1 \left(\frac{1+kn^2}{n^{2\sigma}} \left(1 - E_{\alpha,1} \left(-\frac{n^{2\sigma}}{1+kn^2} t^\alpha \right) \right) \right) dt \\ &= \frac{1}{n^{2\sigma}} \int_0^1 1 - E_{\alpha,1} \left(-\frac{n^{2\sigma}}{1+kn^2} t^\alpha \right) dt = \frac{1}{n^{2\sigma}} - E_{\alpha,2} \left(-\frac{n^{2\sigma}}{1+kn^2} \right). \end{aligned} \quad (5.6)$$

Next, we can find that

$$\tilde{J}_{4,2} = \left(\frac{\mathbb{G}_{\beta,\zeta}^j(\zeta_2)}{E_{\alpha,2} \left(-\frac{n^{2\sigma}}{1+kn^2} \right)} \right)^2 \left(\frac{1}{n^{2\sigma}} - E_{\alpha,2} \left(-\frac{n^{2\sigma}}{1+kn^2} \right) \right)^2. \quad (5.7)$$

Using (5.2), it gives

$$\int_0^t E_{\alpha,1}(-s^\alpha) (t-s)^{\alpha-1} E_{\alpha,\alpha}(z(t-s)^\alpha) ds = \frac{E_{\alpha,\alpha+1}(-t^\alpha) - z E_{\alpha,\alpha+1} \left(-\frac{n^{2\sigma} t^\alpha}{1+kn^2} \right)}{1-z} t^{\alpha+2}. \quad (5.8)$$

Replacing z to $\frac{-n^{2\sigma}}{1+kn^2}$ into (5.8), one has

$$\begin{aligned} & \int_0^1 \left(\int_0^t E_{\alpha,1}(-s^\alpha) (t-s)^{\alpha-1} E_{\alpha,\alpha} \left(-\frac{n^{2\sigma}}{1+kn^2} (t-s)^\alpha \right) ds \right) dt \\ &= \frac{1}{1 + \frac{n^{2\sigma}}{1+kn^2}} \int_0^1 E_{\alpha,\alpha+1}(-t^\alpha) t^{\alpha+2} dt + \frac{\frac{n^{2\sigma}}{1+kn^2}}{1 + \frac{n^{2\sigma}}{1+kn^2}} \int_0^1 E_{\alpha,\alpha+1} \left(-\frac{n^{2\sigma}}{1+kn^2} t^\alpha \right) t^{\alpha+2} dt. \end{aligned} \quad (5.9)$$

Substituting (5.8) into (5.4), we obtain

$$\begin{aligned} \tilde{J}_{2,2} &= \frac{\pi (1 - \mathbb{G}_{\beta,\zeta}^{(j)}(\zeta_2))}{2 E_{\alpha,2} \left(-\frac{n^{2\sigma}}{1+kn^2} \right)} \left(E_{\alpha,1}(-1) - \frac{1}{1+kn^2 + n^{2\sigma}} \underbrace{\left(\int_0^1 E_{\alpha,\alpha+1}(-t^\alpha) t^{\alpha+2} dt \right)}_{\mathcal{I}_1: \text{Calculate by Simson's rule}} \right. \\ &\quad \left. - \frac{n^{2\sigma}}{(1+kn^2)(1+kn^2 + n^{2\sigma})} \underbrace{\int_0^1 E_{\alpha,\alpha+1} \left(-\frac{n^{2\sigma}}{1+kn^2} t^\alpha \right) t^{\alpha+2} dt}_{\mathcal{I}_2: \text{Calculate by Simson's rule}} \right). \end{aligned} \quad (5.10)$$

In general, the calculation of this number will be through the following calculation steps:

Step 1: The domain $(0, \pi)$ into N_x sub-intervals of equal length Δ_x , where $\Delta_x = \frac{\pi}{N_x}$, (M is the numbers of partitions on the x -axis). Then we denote by $u_i^k = u(x_i, 0)$, where $x_i = (i-1)\Delta_x$ for $i = 1, 2, \dots, N_x + 1$.

Step 2: The Simpson's rule method as follows

$$\begin{aligned} \varepsilon_n = \frac{b-a}{n}, \quad \int_a^b f(\zeta) d\zeta \approx \frac{\varepsilon_n}{3} f(\zeta_0) + \frac{2\varepsilon_n}{3} \sum_{j=1}^{n/2-1} f(\zeta_{2j}) \\ + \frac{4\varepsilon_n}{3} \sum_{j=1}^{n/2} f(\zeta_{2j-1}) + \frac{\varepsilon_n}{3} f(\zeta_n), \end{aligned} \quad (5.11)$$

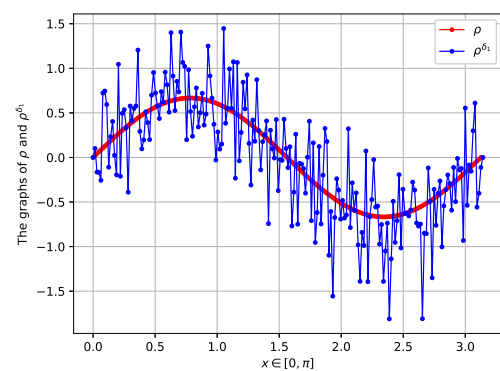
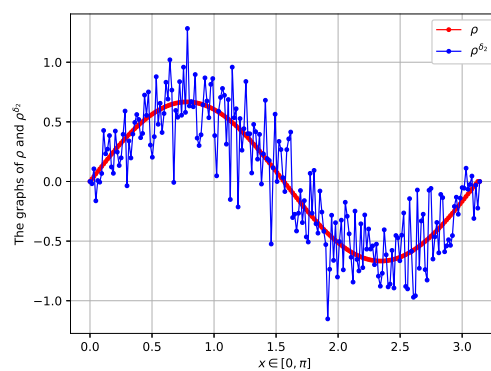
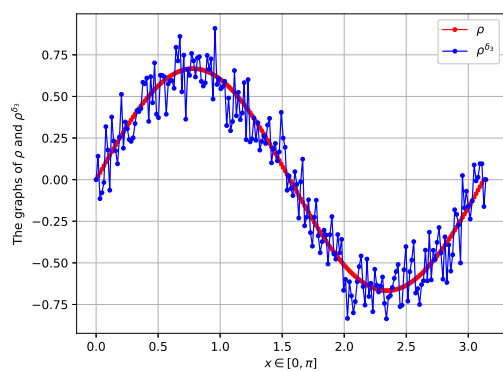
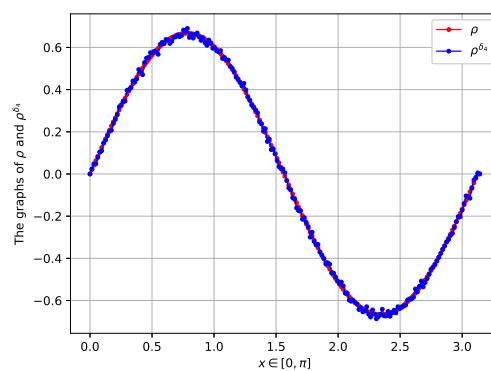
Step 3: The root mean square error $E^\alpha(t)$ as follows:

$$E^\alpha(t) = \frac{\left(\sum_{n=1}^{N_x+1} |\tilde{u}_{\beta,\zeta}^{j,\delta}(x_n, 0) - u(x_n, 0)|^2 \right)^{\frac{1}{2}}}{\left(\sum_{n=1}^{N_x+1} |u(x_n, 0)|^2 \right)^{\frac{1}{2}}}. \quad (5.12)$$

In this work, we study the illustration which is implemented in serial with Python software. Thanks to the Python code `mlf`(α, β, z) to calculate the Mittag-Leffler function, see in [35]. In this subsection, our aim is to test the validity and the advantage of the numerical method. We test the a-priori parameter choice rule, we take $E = 1$ and β according to Theorem 4.3. In our calculations, our ultimate aim is to test the convergence of the regularized solution to the exact solution as the δ tends to zero.

First, with fixed parameters $\alpha = 0.55, \gamma = 0.55, \sigma = 0.5, k = 0.5$ and $\zeta = 0.01452, j = 10$, in Fig 2, with $\delta_1 = 0.4$, we present a drawing describing the exact solution and the normalized solution by the Iterative Fractional Tikhonov method, see the figure on the left, and the exact and regularized solution by the Classical Tikhonov method, see the figure on the right, in the formulas (3.1) and (3.2), for $j = \zeta = 1$, the regularization method is Classical Tikhonov. In Fig 3, 4, 5, we also do this with values $\delta_2 = 0.25, \delta_3 = 0.125$ and $\delta_4 = 0.0125$, respectively. In all these four figures, according to our observations, when $\delta_1 = 0.4$ and $\delta_2 = 0.25$, the volatility of the regularization solution around the exact solution is very strong, but when $\delta_3 = 0.125, \delta = 0.0125$, the oscillation decreases and the convergence of the precision to the exact solution is clearly shown. Furthermore, according to our observations, the iterative fractional Tikhonov method seems to be better than the classical Tikhonov method. In Figure 6, we present error plots with IFT (left figure) and CT (right figure) with $\delta_1 = 0.4, \delta_2 = 0.2$. In Figure 7, we present error plots with IFT (left figure) and CT (right figure) with $\delta_1 = 0.125, \delta_2 = 0.0125$.

Next, with $\alpha = \{0.15, 0.25, 0.35, 0.45, 0.55, 0.65, 0.75, 0.85, 0.95\}$, we check the following results: In Table (1), we show the results by IFT with $N_\Omega = 200$. In Table (2), we show the results by CT with $N_\Omega = 200$. However, in Table (3), in order not to lose generality as well as to ensure objectivity, we investigate one more case whenever $j = 30, \zeta = 0.99354$. And while comparing these numerical results, as δ decreases, the regularized solution is convergent to the exact solution, which is true for all the different α values chosen at above.

(a) $\delta_1 = 0.4$ (b) $\delta_2 = 0.25$ (c) $\delta_3 = 0.125$ (d) $\delta_4 = 0.0125$ **Figure 1:** ρ and ρ_{δ_i} , with $i = \overline{1,4}$

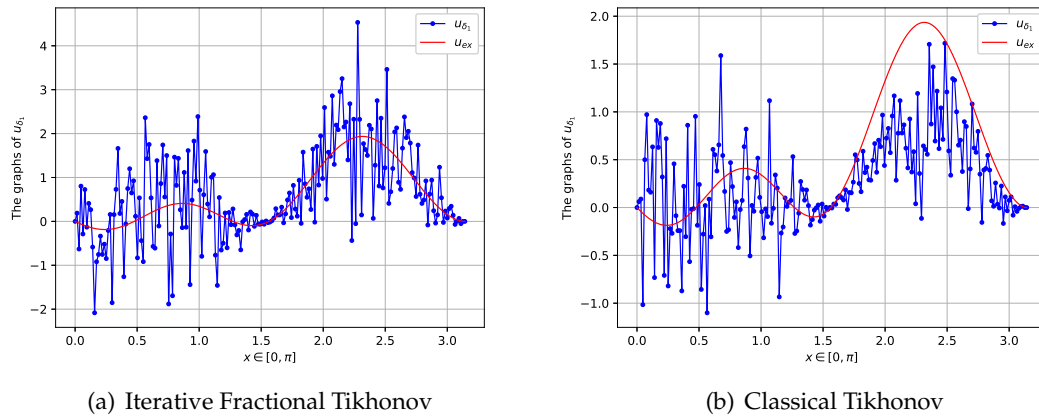


Figure 2: The graph illustrates the sought solution and its approximation at $\delta = 0.4$.

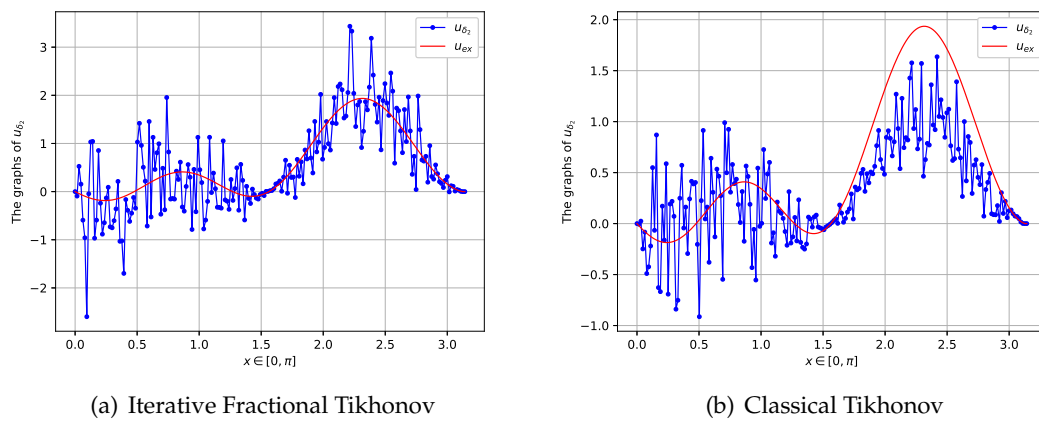


Figure 3: The graph illustrates the sought solution and its approximation at $\delta = 0.25$.

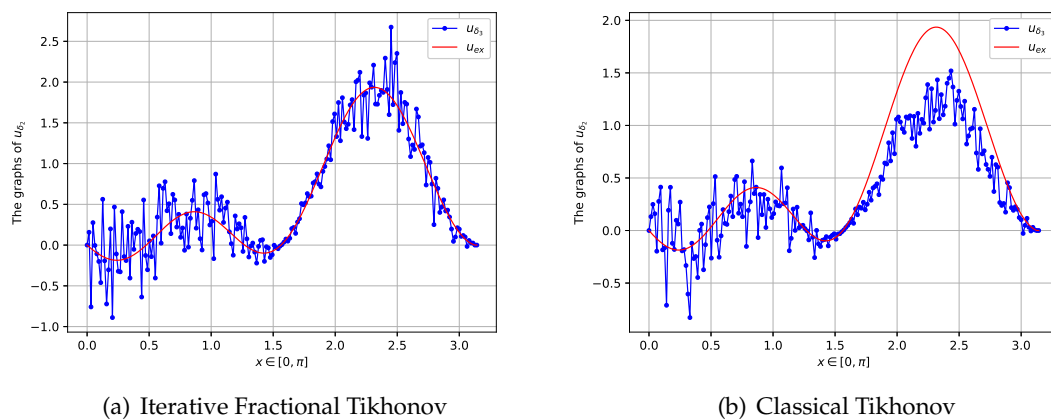
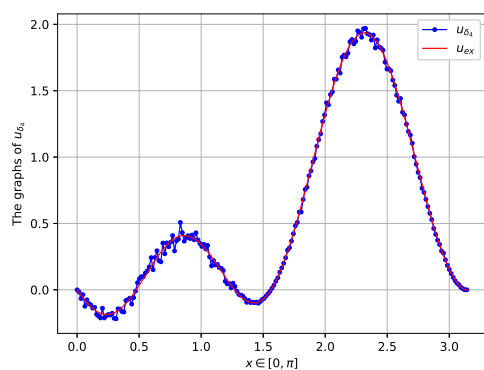
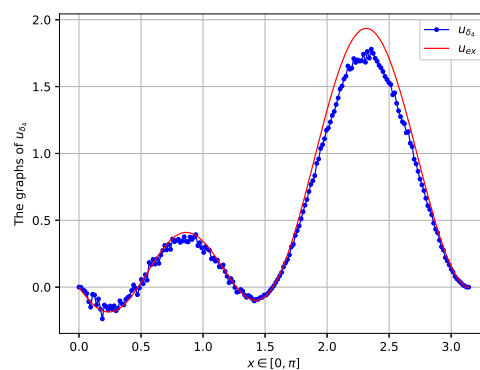


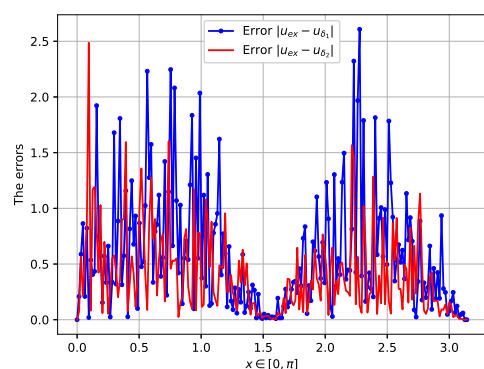
Figure 4: The graph illustrates the sought solution solution and its approximation at $\delta = 0.125$.



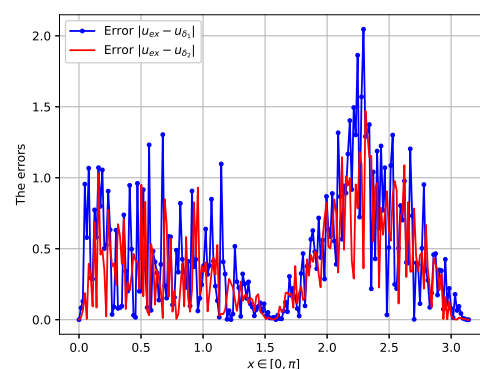
(a) Iterative Fractional Tikhonov



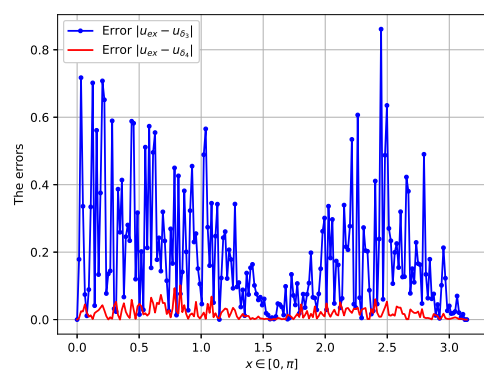
(b) Classical Tikhonov

Figure 5: The graph illustrates the sought solution and its approximation at $\delta = 0.0125$.

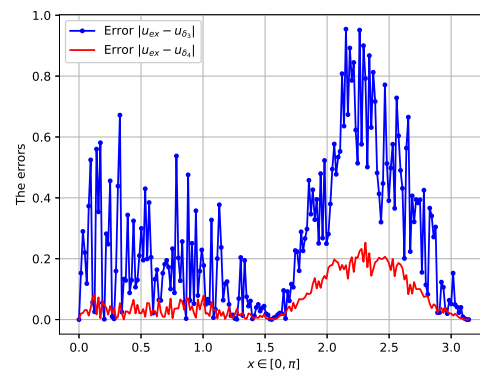
(a) Iterative Fractional Tikhonov



(b) Classical Tikhonov

Figure 6: The error estimate between two methods.

(a) Iterative Fractional Tikhonov



(b) Classical Tikhonov

Figure 7: The error estimate between two methods.

Table 1: Estimate error with $N_\Omega = 200$ by Iterative Fractional Tikhonov method

α	$\delta_1 = 0.4$	$\delta_2 = 0.25$	$\delta_3 = 0.125$	$\delta_3 = 0.0125$
0.15	0.830784737	0.534027096	0.254991446	0.025316185
0.25	0.857751910	0.504904917	0.260685285	0.027822150
0.35	0.804584257	0.551125634	0.232701456	0.026595092
0.45	0.856298856	0.522141059	0.277374881	0.025829387
0.55	0.803380591	0.564224986	0.256799075	0.026375330
0.65	0.812743821	0.533029497	0.264070689	0.023476882
0.75	0.733269088	0.496663041	0.264305573	0.025077295
0.85	0.803595773	0.482508208	0.275593087	0.024045732
0.95	0.786949674	0.482154025	0.254358291	0.027587453

Table 2: Estimate error with $N_\Omega = 200$ by Classical Tikhonov method

α	$\delta_1 = 0.4$	$\delta_2 = 0.25$	$\delta_3 = 0.125$	$\delta_3 = 0.0125$
0.15	0.71617347	0.611802634	0.44174948	0.125431391
0.25	0.688105119	0.530276515	0.418562965	0.118353139
0.35	0.643978051	0.561223527	0.385925229	0.114906974
0.45	0.660188990	0.535043296	0.371122349	0.103127790
0.55	0.613199151	0.496319465	0.35378982	0.095768334
0.65	0.635740781	0.49139739	0.322789475	0.090323164
0.75	0.556250244	0.452133629	0.335976439	0.081892850
0.85	0.556250244	0.452133629	0.335976439	0.081892850
0.95	0.541858220	0.429406811	0.28233584	0.067204429

Table 3: Estimate error with $N_\Omega = 200$ by $\zeta = 0.99354, j = 30$

α	$\delta_1 = 0.4$	$\delta_2 = 0.25$	$\delta_3 = 0.125$	$\delta_3 = 0.0125$
0.15	0.880781526	0.557857313	0.246469173	0.025218831
0.25	0.885101557	0.543878297	0.255122368	0.02421354
0.35	0.843883909	0.472996959	0.242435654	0.027853996
0.45	0.875015838	0.467083717	0.242929699	0.024537286
0.55	0.867965787	0.537539007	0.237189267	0.024976242
0.65	0.865114261	0.482533258	0.255298414	0.022521564
0.75	0.776244072	0.445326495	0.224334396	0.024559725
0.85	0.732329502	0.478656131	0.244231334	0.025484112
0.95	0.811410852	0.461796495	0.252516347	0.025441454

6 Conclusion

Using an iterated fractional Tikhonov regularization approach, we investigated a backward problem for an inhomogeneous time-fractional diffusion equation in this article. The error estimates are derived using the a-priori regularization parameter rule under the usual smoothness source condition. A numerical example is used to demonstrate the method's accuracy. Our convergence results are correct in both theory and numerical calculation examples. However, the number of iterations j is still an open question, and this will still take a long time to investigate.

7 Declarations

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Competing Interests

The authors declare that they have no competing interests.

Ethical Approval

Not applicable.

Authors' Contributions

All authors contributed equally. All the authors read and approved the final manuscript.

Availability Data and Materials

Not applicable.

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